Υπολογισμός Αθροισμάτων

10/4/25

① Βασικές γεννήτριες

$$\sum_{k=0}^{\infty} t^k = (1-t)^{-1} \quad , \quad |t|<1$$

$$\sum_{k=0}^{\infty} \binom{\nu}{k} t^k = (1+t)^{\nu}, \quad \nu \in \mathbb{N}$$

$$\sum_{k=0}^{\infty} \left[{\nu \atop k} \right] t^k = (1-t)^{-\nu}, \quad \nu \in \mathbb{N}$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$$

$$\sum_{k=0}^{\infty} \binom{x}{k} t^k = (1+t)^x, \quad x \in \mathbb{R} \quad |t|<1 \quad \left(\frac{\frac{d^k}{dt^k}(1+t)^k}{k!} \text{ από ανάπτυγμα Taylor} \right)$$

↳ επάγει τις πρώτες 3

② Παράδειγμα

$$S = \sum_{k=0}^{\nu} k(k-1)\binom{\nu}{k} 3^k = \,?$$

Λύση:

$1^{ος}$ τρ. : Παραγώγιση συνν. γενν.

$$\sum_{k=0}^{\nu} \binom{\nu}{k} t^k = (1+t)^{\nu} \overset{d/dt}{\Longrightarrow} \sum_{k=1}^{\nu} k \binom{\nu}{k} t^{k-1} = \nu(1+t)^{\nu-1} \overset{d/dt}{\Longrightarrow} \sum_{k=2}^{\nu} k(k-1)\binom{\nu}{k} t^{k-2} = \nu(\nu-1)(1+t)^{\nu-2}$$

$$\overset{t^2}{\Longrightarrow} \sum_{k=2}^{\nu} k(k-1)\binom{\nu}{k} t^k = \nu(\nu-1)(1-t)^{\nu-2} t^2$$

$$\overset{t=3}{\Longrightarrow} \sum_{k=\cancel{2}\,0}^{\nu} k(k-1)\binom{\nu}{k} 3^k = 9\nu(\nu-1)4^{\nu-2}$$

$2^{ος}$ τρ. : $\binom{\nu}{k} = \frac{\nu!}{(\nu-k)!\,k!} = \frac{\nu}{k}\binom{\nu-1}{k-1}$, $k \geq 1$ $*$

$$S = \sum_{k=0}^{\nu} k(k-1)\binom{\nu}{k} 3^k = \sum_{k=2}^{\nu} k(k-1)\frac{\nu}{k}\cdot\frac{\nu-1}{k-1}\binom{\nu-2}{k-2} 3^k = 3^2 \nu(\nu-1)\sum_{k=2}^{\nu}\binom{\nu-2}{k-2} 3^{k-2}$$

$$\overset{j=k-2}{=} 9\nu(\nu-1)\underbrace{\sum_{j=0}^{\nu-2}\binom{\nu-2}{j} 3^j}_{(1+3)^{\nu-2}}$$

$$= 9\nu(\nu-1)4^{\nu-2}$$

③ Αθροίσματα τύπου $\sum_{k=0}^{v} \underbrace{f(k)}_{\text{πολ. του } k} \binom{v}{k} p^k$

π.χ. $\sum_{k=0}^{v} (3k^2-2k+1)\binom{v}{k}7^k$

Μεθοδολογία

1) Ανάλυση του $f(k)$ σε καθοδικά παραγ. του k:

$f(k) = A_r(k)_r + A_{r-1}(k)_{r-1} + \dots + A_0 \underset{1}{\underset{\shortparallel}{(k)_0}}$

2) Υπολογισμός αθρ. της μορφής $\sum_{k=0}^{v} (k)_r \binom{v}{k} p^k$ (με παραγ. ή με την ταυτότητα $*$)

④ Παράδειγμα

$S = \sum_{k=0}^{v} (3k^2-2k+1)\binom{v}{k}7^k$

$3k^2-2k+1 = A_2(k)_2 + A_1(k)_1 + A_0(k)_0 = A_2k(k-1) + A_1k + A_0$

$= A_2k^2 + (A_1-A_2)k + A_0$

$\Leftrightarrow \begin{cases} A_2 = 3 \\ A_1 - A_2 = 2 \\ A_0 = 1 \end{cases}$

$\Leftrightarrow \quad A_2 = 3$ $\qquad$ Επομένως $3k^2-2k+1 = 3(k)_2 + (k)_1 + (k)_0$

$A_0 = 1$

$A_1 = 1$

Άρα $S = 3\underbrace{\sum_{k=0}^{v}(k)_2\binom{v}{k}7^k}_{(v)_2(1+7)^{v-2}7^2} + \underbrace{\sum_{k=0}^{v}(k)_1\binom{v}{k}7^k}_{v(1+7)^{v-1}7^1} + \underbrace{\sum_{k=0}^{v}(k)_0\binom{v}{k}7^k}_{(1+7)^v}$

Υπολογίζω το $\sum_{k=0}^{v}(k)_r\binom{v}{k}p^k$ είτε με παραγ. είτε με την $*$

1ος τρ.: $\sum_{k=0}^{v}\binom{v}{k}t^k = (1+t)^v \xrightarrow{d^r/dt^r} \sum_{k=0}^{v} \underbrace{k(k-1)\dots(k-r+1)}_{(k)_r}\binom{v}{k}t^{k-r}$

$= v(v-1)\dots(v-r+1)(1+t)^{v-r} \longrightarrow (v)_r$

$\xRightarrow{t^r} \sum_{k=0}^{v}(k)_r\binom{v}{k}t^k = (v)_r(1+t)^{v-r} \xRightarrow{t=p} \sum_{k=0}^{v}(k)_r\binom{v}{k}p^k = (v)_r(1+p)^{v-r}p^r$

Τελικά, $S = 3\cdot 49v(v-1)8^{v-2} + 7v8^{v-1} + 8^v$

⑤ Παράδειγμα

$S = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k}4^k = ?$

Λύση:

$1^{ος}$ τρ.: ολοκλήρωση

$$\sum_{k=0}^{\nu}\binom{\nu}{k}t^k = (1+t)^\nu \overset{\int_0^u dt}{\Longrightarrow} \sum_{k=0}^{\nu}\binom{\nu}{k}\int_0^u t^k dt = \int_0^u (1+t)^\nu dt$$

$$\Rightarrow \sum_{k=0}^{\nu}\binom{\nu}{k}\frac{u^{k+1}}{k+1} = \frac{(1+u)^{\nu+1}-1}{\nu+1} \overset{1/u}{\Longrightarrow} \sum_{k=0}^{\nu}\binom{\nu}{k}\frac{1}{k+1}u^k = \frac{(1+u)^{\nu+1}-1}{u(\nu+1)}$$

$$\overset{u=4}{\Longrightarrow} \sum_{k=0}^{\nu}\frac{1}{k+1}\binom{\nu}{k}4^k = \frac{5^{\nu+1}-1}{4(\nu+1)}$$

$2^{ος}$ τρ.: με την *

$$\sum_{k=0}^{\nu}\frac{1}{k+1}\binom{\nu}{k}4^k = \frac{1}{\nu+1}\sum_{k=0}^{\nu}\frac{\nu+1}{k+1}\binom{\nu}{k}4^k = \frac{1}{4(\nu+1)}\sum_{k=0}^{\nu}\binom{\nu+1}{k+1}4^{\overset{k+1}{\underset{j}{\shortparallel}}}$$

$$\underset{j=k+1}{=} \frac{1}{4(\nu+1)}\underbrace{\sum_{j=1}^{\nu+1}\binom{\nu+1}{j}4^j}_{\sum_{j=1}^{\nu+1}\binom{\nu+1}{j}4^j - \binom{\nu+1}{0}4^0} = \frac{1}{4(\nu+1)}\left((1+4)^{\nu+1}+1\right) = \frac{5^{\nu+1}-1}{4(\nu+1)}$$

⑥ Αθροίσματα τύπου $\sum_{k=0}^{\nu}\frac{f(k) \to \text{πολ. του } k}{(k+1)(k+2)\dots(k+r)}\binom{\nu}{k}p^k$

π.χ. $\sum_{k=0}^{\nu}\frac{k^3+3k^2-2k+5}{(k+1)(k+2)}\binom{\nu}{k}7^k = ?$

Μεθοδολογία

1) Αναπτύσσουμε το $\frac{f(k)}{(k+1)\dots(k+r)}$ σε καθ. παρ. $= A_{-r}(k)_{-r} + A_{-r+1}(k)_{-r+1} + \dots + A_{-1}(k)_{-1}$
$+ A_0(k)_0 + \dots + A_j(k)_j \to$ βαθμ. $f(k)-r$

2) Υπολ. ξεχωριστά αθρ. τύπου $\sum_{k=0}(k)_{i}\binom{\nu}{k}p^k$, $i \in \mathbb{Z}$

Εναλλακτικά $= \sum_{k=0}^{\nu}\frac{f(k)}{(k+1)\dots(k+r)}\binom{\nu}{k}p^k = \frac{1}{(\nu+1)\dots(\nu+r)}\sum_{k=0}^{\nu}f(k)\binom{\nu+r}{k+r}p^k$

και μετά όπως με το προηγ. είδος αθρ.

(7) Παράδειγμα

$$S = \sum_{k=0}^{v} \frac{k^3+3k^2-2k+5}{(k+1)(k+2)} \binom{v}{k} 7^k = ;$$

Λύση: $\dfrac{k^3+3k^2-2k+5}{(k+1)(k+2)} = A_{-2}(k)_{-2} + A_{-1}(k)_{-1} + A_0 (k)_0 + A_1 (k)_1$

$$= \frac{A_{-2}}{(k+1)(k+2)} + \frac{A_{-1}}{k+1} + A_0 + A_1 k = \frac{A_{-2} + A_1(k+2) + A_0(k+1)(k+2) + A_1 k (k+1)(k+2)}{(k+1)(k+2)}$$

$$\Leftrightarrow \left\{ \begin{array}{l} 1 = A_1 \\ 3 = A_0 + 3A_1 \\ -2 = A_{-1} + 3A_0 + 2A_1 \\ 5 = A_{-2} + 2A_{-1} + 2A_0 \end{array} \right\} \Leftrightarrow \begin{array}{l} A_1 = 1 \\ A_0 = 0 \\ A_{-1} = -4 \\ A_{-2} = 13 \end{array}$$

Άρα $S = 13 \sum_{k=0}^{v} (k)_{-2} \binom{v}{k} 7^k - 4 \sum_{k=0}^{v} (k)_{-1} \binom{v}{k} 7^k + \sum_{k=0}^{v} (k) \binom{v}{k} 7^k$

Έχουμε $\sum_{k=0}^{v} (k)_{-2} \binom{v}{k} 7^k = \sum_{k=0}^{v} \frac{1}{(k+1)(k+2)} \binom{v}{k} 7^k = \frac{1}{7(k+1)(k+2)} \sum_{k=0}^{v} \binom{v+2}{\underbrace{k+2}_{=j}} 7^{k+2}$

$$= \frac{\sum_{j=2}^{v+2} \binom{v+2}{j} 7^j}{49(v+1)(v+2)} = \frac{(1+7)^{v+2} - \binom{v+2}{0} 7^0 - \binom{v+2}{1} 7^1}{49(v+1)(v+2)}$$

Τελικά, $S = \dfrac{13 \cdot 18^{v+2} - 1 - 7(v+2)}{49(v+1)(v+2)} - 4 \cdot \dfrac{8^{v+1} - 1}{7(v+1)} + 7v \cdot 8^{v-1}$