21/11/23

Αθροίσματα - Ασκήσεις

① Άσκηση

$$S=\sum_{s=k}^{\nu}\binom{s}{k}\binom{\nu}{s}=;$$

Λύση

$$S=\sum_{s=k}^{\nu}\frac{s!}{k!(s-k)!}\cdot\frac{\nu!}{s!(\nu-s)!}$$

$$=\frac{\nu!}{k!}\sum_{s=k}^{\nu}\frac{1}{(s-k)!(\nu-s)!}\overset{j=s-k}{=}\frac{\nu!}{k!}\sum_{i=0}^{\nu-k}\frac{1}{i!(\nu-k-i)!}$$

$$=\frac{\nu!}{k!(\nu-k)!}\sum_{i=0}^{\nu-k}\frac{(\nu-k)!}{i!(\nu-k-i)!}=\binom{\nu}{k}\sum_{i=0}^{\nu-k}\binom{\nu-k}{i}$$

$$=\binom{\nu}{k}2^{\nu-k}$$

② Άσκηση (Θέμα 2α Σεπτ 2012)

$$S=\sum_{k=0}^{\nu}(k+2)^2\binom{\nu}{k+1}2^k=;$$

Λύση

$$S\overset{j=k+1}{=}\sum_{j=1}^{\nu+1}\underbrace{(j+1)^2}_{j^2+2j+1=j(j-1)+3j+1}\binom{\nu}{j}2^{j-1}=\frac{1}{2}\sum_{j=1}^{\nu}j(j-1)\binom{\nu}{j}2^j+\frac{1}{2}\sum_{j=1}^{\nu}j\binom{\nu}{j}2^j+\frac{1}{2}\sum_{j=1}^{\nu}\binom{\nu}{j}2^j$$

$$=\frac{1}{2}\sum_{j=2}^{\nu}j(j-1)\frac{\nu(\nu-1)}{j(j-1)}\binom{\nu-2}{j-2}2^j+\frac{3}{2}\sum_{j=1}^{\nu}j\frac{\nu}{j}\binom{\nu-1}{j-1}2^j+\frac{1}{2}\sum_{j=1}^{\nu}\binom{\nu}{j}2^j$$

$$=\frac{\nu(\nu-1)}{2}\cdot 4(2+1)^{\nu-2}+\frac{3\nu}{2}2(2+1)^{\nu-1}+\frac{1}{2}\left[(1+2)^{\nu}-1\right]$$

$$=2\nu(\nu-1)3^{\nu-2}+\nu 3^{\nu}+\frac{3^{\nu}}{2}-\frac{1}{2}$$

③ Άσκηση (Θέμα 2α Μαρ 2012)

$$S=\sum_{k=0}^{\nu} k(\nu-k)\binom{9}{k}\binom{5}{\nu-k}=;$$

Λύση

$$S=\sum_{k=1}^{\nu-1} k(\nu-k)\,\frac{9}{k}\binom{8}{k-1}\frac{5}{\nu-k}\binom{4}{\nu-1-k}$$

$$=45\sum_{k=1}^{\nu-1}\binom{8}{k-1}\binom{4}{\nu-1-k}$$

$$\underset{\substack{\nu-1-k\\=\nu-1-j-1}}{\overset{j=k-1}{=}} 45\sum_{j=0}^{\nu-2}\binom{8}{j}\binom{4}{\nu-2-j} \overset{cauchy}{=} 45\binom{12}{\nu-2}$$

④ Άσκηση (Θέμα 2Β Μαρ 2012)

$$S=\sum_{j=0}^{\nu}\binom{2\nu}{2j}\underbrace{9^{j}}_{3^{2j}}5^{2\nu-2j}$$

Λύση

$$S=5^{2\nu}\sum_{k}^{2\nu}\binom{2\nu}{k}\left(\frac{3}{5}\right)^{k}=S_a$$

$\uparrow$ $k=2j$ άρτιος

Ορίζω $S_n=5^{2\nu}\sum_{k=0}^{2\nu}\binom{2\nu}{k}\left(\frac{3}{5}\right)^{k}$

k περιττός

Τότε

$$S_a+S_n=5^{2\nu}\sum_{k=0}^{2\nu}\binom{2\nu}{k}\left(\frac{3}{5}\right)^{k}=5^{2\nu}\left(1+\frac{3}{5}\right)^{2\nu}=8^{2\nu}$$

$$S_a-S_n=5^{2\nu}\sum_{k=0}^{2\nu}\binom{2\nu}{k}\left(-\frac{3}{5}\right)^{k}=5^{2\nu}\left(1-\frac{3}{5}\right)^{2\nu}=2^{2\nu}$$

Άρα $S_a = \frac{8^{2\nu}+2^{2\nu}}{2}$

⑤ Άσκηση (Δύσκολη)

$$S = \sum_{k=0}^{\nu} \binom{2k}{k}\binom{2(\nu-k)}{\nu-k} = ;$$

Λύση

Λήμμα

$$\binom{2k}{k} = \dots \binom{\text{σταθερό}}{k}$$

$$\binom{2k}{k} = \frac{2k(2k-1)\cdot(2k-2)(2k-3)(2k-4)(2k-5)\dots 5\cdot 4\cdot 3\cdot 2\cdot 1}{k!\,k!}$$

$$= 2^k \cdot \frac{k!\,\overbrace{(2k-1)(2k-3)(2k-5)\cdots 5\cdot 3\cdot 1}^{k \text{ το πλήθος}}}{k!\ \ k!}$$

$$= 2^{2k}\, \frac{(k-\frac{1}{2})(k-\frac{3}{2})(k-\frac{5}{2})(k-\frac{7}{2})\cdots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}{k!} \Big\} (k-\tfrac{1}{2})_k$$

$$= 2^{2k}\binom{k-\frac{1}{2}}{k} = 2^{2k}\binom{\frac{1}{2}+k-1}{k} = 2^{2k}(-1)^k\binom{-\frac{1}{2}}{k}$$

Άρα $S = \sum_{k=0}^{\nu} 2^{2k}(-1)^k\binom{-\frac{1}{2}}{k} 2^{2(\nu-k)}(-1)^{\nu-k}\binom{-\frac{1}{2}}{\nu-k}$

$$= 2^{2\nu}(-1)^{\nu}\sum_{k=0}^{\nu}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{\nu-k}$$

$$= 2^{2\nu}(-1)^{\nu}\binom{-\frac{1}{2}-\frac{1}{2}}{\nu} = 2^{2\nu}(-1)^{\nu}\binom{-1}{\nu} = 2^{2\nu}\left[\begin{matrix}1\\ \nu\end{matrix}\right] = 2^{2\nu}\binom{\nu+1-1}{\nu}$$

$$= 2^{2\nu}$$

⑥ Άσκηση (Θέμα 2Β Φεβ 2011 - Ομ Α)

$$S=\sum_{k=0}^{\nu}\binom{\nu}{k}\left(\nu k+\binom{\nu}{k}\right)=;$$

Λύση

$$S=\sum_{k=0}^{\nu}\binom{\nu}{k}\nu k+\sum_{k=0}^{\nu}\binom{\nu}{k}\binom{\nu}{k}$$

$$=\nu\sum_{k=1}^{\nu}k\frac{\nu}{k}\binom{\nu-1}{k-1}+\sum_{k=0}^{\nu}\binom{\nu}{k}\binom{\nu}{\nu-k}$$

$$\overset{i=k-1}{=}\nu\sum_{i=0}^{\nu-1}\binom{\nu-1}{i}+\binom{2\nu}{\nu}=\nu^2 2^{\nu-1}+\binom{2\nu}{\nu}$$

Cauchy

⑦ Άσκηση (Θέμα 3α Σεπτ 2010 - Β)

$$S=\sum_{j=m}^{\infty}\binom{j}{m}3^{m-j}$$

Λύση

$$S\underset{i=j-m}{=}\sum_{i=0}^{\infty}\binom{i+m}{m}\left(\frac{1}{3}\right)^i=\sum_{i=0}^{\infty}\binom{i+m}{i}\left(\frac{1}{3}\right)^i$$

$$=\sum_{i=0}^{\infty}\underbrace{\binom{m+1+i-1}{i}}_{\left[\frac{m+1}{i}\right]\overset{!!}{}(-1)^i\binom{-m-1}{i}}\left(\frac{1}{3}\right)^i=\sum_{i=0}^{\infty}\binom{-m-1}{i}\left(-\frac{1}{3}\right)^i$$

$$=\left(1-\frac{1}{3}\right)^{-m-1}=\left(\frac{3}{2}\right)^{m+1}$$

$$(1+t)^x=\sum_{i=0}^{\infty}\binom{x}{i}t^i,\ |t|<1$$

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Ⓑ Άσκηση (Θέμα 3Β Σεπτ 2010-Β)

$$S = \sum_{\substack{j=0 \\ j \text{ περιττό}}}^{m} (2j+1)\binom{m}{j} 3^j$$

Λύση

Έστω $S_a = \sum_{\substack{j=0 \\ j \text{ άρτιος}}}^{m} (2j+1)\binom{m}{j} 3^j$

S_n το ζητούμενο

$$S_a + S_n = \sum_{j=0}^{m} (2j+1)\binom{m}{j} 3^j$$

$$S_a - S_n = \sum_{j=0}^{m} (2j+1)\binom{m}{j} 3^j (-1)^j$$

Έχω $\sum_{j=0}^{m} (2j+1)\binom{m}{j} t^j = 2\underbrace{\sum_{j=0}^{m} j\binom{m}{j} t^j}_{2m\sum_{j=1}^{m}\binom{m-1}{j-1} t^j} + \underbrace{\sum_{j=0}^{m}\binom{m}{j} t^j}_{(1+t)^m}$

$= 2mt(1+t)^{m-1} + (1+t)^m$

Άρα $\left.\begin{aligned} S_a + S_n &= 6m4^{m-1} + 4^m \\ S_a - S_n &= -6m(-2)^{m-1} + (-2)^m \end{aligned}\right\}$

$$\Rightarrow S_n = \frac{6m4^{m-1} + 4^m + 6m(-2)^{m-1} - (-2)^m}{2}$$

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