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Υπολογισμοί Αθροισμάτων

(1) Τύπος Cauchy - Γενίκευση

Θεωρία

$x, y \in \mathbb{R}$, $\nu \geq 0$, $\nu \in \mathbb{Z}$

$$\binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{k} \binom{y}{\nu-k}$$

Απόδειξη:

Γενικό Διωνυμικό Θεώρημα: $(1+t)^x = \sum_{k=0}^{\infty} \binom{x}{k} t^k$, $|t|<1$.

Έχουμε για t με $|t|<1$:

$$\sum_{\nu=0}^{\infty} \underbrace{\binom{x+y}{\nu}}_{\gamma_\nu} t^\nu = (1+t)^{x+y} = (1+t)^x (1+t)^y =$$

$$= \left(\sum_{k=0}^{\infty} \underbrace{\binom{x}{k}}_{a_k} t^k \right) \left(\sum_{k=0}^{\infty} \underbrace{\binom{y}{k}}_{b_k} t^k \right) \Rightarrow$$

$$\Rightarrow \gamma_\nu = a_0 b_\nu + a_1 b_{\nu-1} + \dots + a_\nu b_0 =$$

$$= \sum_{k=0}^{\nu} a_k b_{\nu-k} \Rightarrow \binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{k} \binom{y}{\nu-k}$$

② Παράδειγμα

$$s = \sum_{k=0,1}^{\nu} k \binom{8}{k} \binom{17}{\nu-k} = ;$$

Λύση:

$$s = \sum_{k=1}^{\nu} \cancel{k} \frac{8}{\cancel{k}} \binom{7}{k-1} \binom{17}{\nu-k} =$$

$\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}$

$$= 8 \sum_{k=1}^{\nu} \binom{7}{k-1} \binom{17}{\nu-k} \overset{j=k-1}{=}$$

$$= 8 \sum_{j=0}^{\nu-1} \binom{7}{j} \binom{17}{\nu-1-j} =$$

$$\overset{\text{Cauchy}}{=} 8 \binom{7+17}{\nu-1}$$

$$= 8 \binom{24}{\nu-1}$$

③ Αθροίσματα τύπου $\sum_{k=0}^{N} f(k) \binom{r}{k} \binom{s}{\nu-k}$, $r,s \in \mathbb{R}$

πολυώνυμο του k

1) Ανάπτυξη του $f(k)$ σε

$A_0 k_0 + A_1 (k)_1 + \ldots + A_\rho (k)_\rho$

βαθμός πολυωνύμου.

2) Υπολογίζω $\displaystyle\sum_{k=0}^{\nu} \underbrace{(k)_i}_{k(k-1)\cdot\ldots\cdot(k-i+1)} \binom{r}{k}\binom{s}{\nu-k} =$

$$= \sum_{k=i}^{\nu} \cancel{k(k-1)\cdot\ldots\cdot(k-i+1)}\ \overbrace{\frac{r}{k}\,\frac{r-1}{k-1}\cdot\ldots\cdot\frac{r-i+1}{k-i+1}}^{(r)_i}\binom{r-i}{k-i}\binom{s}{\nu-k}$$

$$= (r)_i \sum_{k=i}^{\nu}\binom{r-i}{k-i}\binom{s}{\nu-k} \ \underline{\underline{j=k-i}}$$

$$= (r)_i \sum_{j=0}^{\nu-i}\binom{r-i}{j}\binom{s}{\nu-i-j} =$$

$$= (r)_i \binom{r-i+s}{\nu-i}$$

3) Παράδειγμα

$$s=\sum_{k=0}^{\nu}(k^2+5k-8)\binom{12}{k}\binom{17}{\nu-k}$$

Λύση: $k^2+5k-8 = k(k-1)+6k-8 = (k)_2+6(k)_1-8(k)_0$

Άρα,

$$s=(12)_2\binom{12+17-2}{\nu-2}+6(12)_1\binom{12+17-1}{\nu-1} - 8\binom{12+17}{\nu}$$

4) Υπενθύμιση

$$\underbrace{\left[{x \atop k}\right]}_{\frac{x(x+1)\cdot\ldots\cdot(x+k-1)}{k!}} = \binom{x+k-1}{k} = (-1)^k\binom{-x}{k},\ x\in\mathbb{R} \Rightarrow$$

$$\begin{bmatrix} x \\ k \end{bmatrix} = \binom{x+k-1}{k} = (-1)^k \binom{-x}{k}, \quad x \in \mathbb{R}.$$

$$\binom{x}{k} = \begin{bmatrix} x-k+1 \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -x \\ k \end{bmatrix}, \quad x \in \mathbb{R}$$

(5) Παράδειγμα

$$S = \sum_{k=0}^{\nu} \binom{r+k}{k} \binom{s-k}{\nu-k} = ;$$

Λύση:

$$S = \sum_{k=0}^{\nu} \binom{(r+1)+k-1}{k} \binom{(s-\nu+1)+(\nu-k)-1}{\nu-k} =$$

$$= \sum_{k=0}^{\nu} (-1)^k \binom{-(r+1)}{k} (-1)^{\nu-k} \binom{-(s-\nu+1)}{\nu-k} =$$

$$= (-1)^{\nu} \sum_{k=0}^{\nu} \binom{-r-1}{k} \binom{-s+\nu-1}{\nu-k} \overset{\text{Cauchy}}{=}$$

$$= (-1)^{\nu} \binom{-(r+1+s-\nu+1)}{\nu} =$$

$$= \binom{r+s-\cancel{\nu}+2+\cancel{\nu}-1}{\nu} = \binom{r+s+1}{\nu}$$

⑥ Παράδειγμα:

$$S=\sum_{k=0}^{\nu} \frac{1}{(k+1)(k+2)} \binom{15}{k}\binom{35}{\nu-k}$$

Λύση:

$$s=\frac{1}{16\cdot 17}\cdot\sum_{k=0}^{\nu} \frac{16\cdot 17}{(k+1)(k+2)} \binom{15}{k}\cdot\binom{35}{\nu-k}=$$

$\frac{\nu+1}{k+1}\binom{\nu}{k}=\binom{\nu+1}{k+1}$

$$=\frac{1}{16\cdot 17}\sum_{k=0}^{\nu}\binom{17}{k+2}\binom{35}{\nu-k} \qquad j=k+2,\ k=j-2$$

$$=\frac{1}{16\cdot 17}\sum_{j=2}^{\nu+2}\binom{17}{j}\binom{35}{\nu+2-j}=$$

$$=\frac{1}{16\cdot 17}\left(\sum_{j=0}^{\nu+2}\binom{17}{j}\binom{35}{\nu+2-j}-\binom{17}{0}\binom{35}{\nu+2}-\binom{17}{1}\binom{35}{\nu+1}\right)$$

$$\underset{\text{Cauchy}}{=}\frac{1}{16\cdot 17}\left(\binom{17+35}{\nu+2}-\binom{35}{\nu+2}-17\binom{35}{\nu+1}\right).$$

⑦ Αθροίσματα $\displaystyle\sum_{k=0}^{\nu}\frac{f(k)}{(k+1)(k+2)\cdot\ldots\cdot(k+i)}\binom{r}{k}\binom{s}{\nu-k}$ πολυώνυμο

1) Ανάπτυξη του $f(k)/(k+1)\cdot\ldots\cdot(k+i)$

σε $A_{-i}(k)_{-i}+A_{-i+1}(k)_{-i+1}+\ldots+A_{\ell}(k)_{\ell}$.

2) Υπολογίζω κάθε άθροισμα όπως στο παράδειγμα.

(8) Αθροίσματα σε περιττούς ή άρτιους δείκτες

πχ.

$$S_a = \sum_{\substack{k=0 \\ k \text{ άρτιος}}}^{\nu} \frac{1}{k+1}\binom{\nu}{k} =$$

$$= \frac{1}{1}\binom{\nu}{0} + \frac{1}{3}\binom{\nu}{2} + \frac{1}{5}\binom{\nu}{4} + \dots + \frac{1}{2\lfloor \frac{\nu}{2} \rfloor + 1}\binom{\nu}{2\lfloor \frac{\nu}{2} \rfloor}$$

ο τελευταίος άρτιος $\leq \nu$

Αν ν φυσικός:

τελευταίος άρτιος $\leq \nu = 2\lfloor \frac{\nu}{2} \rfloor$

τελευταίος περιττός $\leq \nu$ = (τελευτ. άρτιοι $\leq \nu-1$) + 1 =

$$2\left\lfloor \frac{\nu-1}{2} \right\rfloor + 1$$

Η τεχνική είναι:

Εισάγω το συμπληρωματικό άθροισμα

$$S_\pi = \sum_{k=0,\ k \text{ περριτός}}^{\nu} \frac{1}{k+1}\binom{\nu}{k}$$

Έχουμε:

$$S_a + S_\pi = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} \Longrightarrow$$

$$S_\alpha - S_\pi = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k}(-1)^k.$$

- Έχω

$$\sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} t^k = \frac{1}{\nu+1}\sum_{k=0}^{\nu}\frac{\nu+1}{k+1}\binom{\nu}{k} t^k =$$

$$= \frac{1}{\nu+1}\sum_{k=0}^{\nu}\binom{\nu+1}{k+1} t^k \overset{j=k+1}{=\!=\!=}$$

$$= \frac{1}{\nu+1}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j} t^{j-1} = \frac{1}{(\nu+1)t}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j} t^j =$$

$$= \frac{1}{(\nu+1)t}\left(\sum_{j=0}^{\nu+1}\binom{\nu+1}{j} t^j - \binom{\nu+1}{0} t^0\right) =$$

$$= \frac{1}{(\nu+1)t}\left((1+t)^{\nu+1} - 1\right) = \frac{(1+t)^{\nu+1}-1}{(\nu+1)t}$$

Άρα,

$$\left.\begin{aligned} \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} &= \frac{2^{\nu+1}-1}{\nu+1} \\ \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k}(-1)^k &= \frac{-1}{-(\nu+1)} = \frac{1}{\nu+1} \end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \left.\begin{aligned} S_\alpha + S_\pi &= \frac{2^{\nu+1}-1}{\nu+1} \\ S_\alpha - S_\pi &= \frac{1}{\nu+1} \end{aligned}\right\} \Rightarrow S_\alpha = \frac{2^{\nu+1}}{2(\nu+1)} \Rightarrow$$

$$\boxed{S_\alpha = \frac{2^\nu}{\nu+1}}$$

9) Αθροίσματα με χρήση συμμετρίας

$$\binom{\nu}{\kappa} = \binom{\nu}{\nu-\kappa}.$$

Πχ.

$$S = \binom{2\nu}{0} + \binom{2\nu}{1} + \binom{2\nu}{2} + \dots + \binom{2\nu}{\nu} = ;$$

Λύση:

$$2S = \binom{2\nu}{0} + \binom{2\nu}{1} + \dots + \binom{2\nu}{\nu-1} + \binom{2\nu}{\nu}$$
$$+ \binom{2\nu}{2\nu} + \binom{2\nu}{2\nu-1} + \dots + \binom{2\nu}{\nu+1} + \binom{2\nu}{\nu} =$$

$$= \sum_{\kappa=0}^{2\nu} \binom{2\nu}{\kappa} + \binom{2\nu}{\nu} = 2^{2\nu} + \binom{2\nu}{\nu}$$

Άρα, $S = \dfrac{2^{2\nu} + \binom{2\nu}{\nu}}{2}$.

10) Παράδειγμα

$$S = \sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa}^2 = ;$$

Λύση:

$$S = \sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa}\binom{\nu}{\nu-\kappa} = \binom{2\nu}{\nu}$$

Cauchy.

11 Παράδειγμα

$$\sum_{k=0}^{\nu} (k+1)\binom{\nu}{k}^2 = \sum_{k=0}^{\nu} k\binom{\nu}{k}^2 + \sum_{k=0}^{\nu}\binom{\nu}{k}^2 =$$

$$= \sum_{k=\cancel{0}1}^{\nu} k\binom{\nu}{k}\binom{\nu}{\nu-k} + \binom{2\nu}{\nu} =$$

$$= \sum_{k=1}^{\nu} \cancel{k}\frac{\nu}{\cancel{k}}\binom{\nu-1}{k-1}\binom{\nu}{\nu-k} + \binom{2\nu}{\nu} =$$

$$= \nu \sum_{k=1}^{\nu}\binom{\nu-1}{k-1}\binom{\nu}{\nu-k} + \binom{2\nu}{\nu} \overset{j=k-1}{=\!=}$$

$$= \sum_{j=0}^{\nu-1}\binom{\nu-1}{j}\binom{\nu}{\nu-1-j} + \binom{2\nu}{\nu} \overset{\text{Cauchy}}{=\!=}$$

$$= \binom{2\nu-1}{\nu-1} + \binom{2\nu}{\nu}$$