Type, cotype and K-convexity

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1 The pre-history of type and cotype, as I remember it

At the end of the sixties, Pietsch [Pi] promoted the notion of p-summing operators between Banach spaces, which extends to all values of $p \in [1, +\infty)$ the study of some classes of operators introduced by Grothendieck [Gro], under different names, for the special values $p = 1, 2$. In an important paper devoted to p-summing operators, Lindenstrauss and Pełczyński [LP] gave a second birth to what we know in Banach space theory as *the* Grothendieck theorem; one formulation of it states that *every operator from* ℓ_1 *to* ℓ_2 *is* 1*-summing;* another formulation is the famous Grothendieck's inequality. Around 1969, L. Schwartz introduced *radonifying maps*, a notion that turned out to be closely related to p-summing maps. A special case of this notion deals with the Wiener measure and with linear maps from a Hilbert space H to a Banach space X, that transform the canonical cylindrical Gaussian measure of H into a true Radon probability measure on X (see L. Gross [Gr1,Gr2] for another viewpoint on this subject). L. Schwartz organized a seminar at the Ecole Polytechnique in Paris ([Sem], 1969–70) about these topics. This is one of the reasons why Paris, and especially the Ecole Polytechnique, became one of the places where the subject of type and cotype was developed.

Type and cotype conditions appeared first in the framework of p-summing operators, or more precisely in connection with the factorization through L_p, $p > 1$, of operators with values in L_1 (in this paper, *operator* means bounded linear operator). In the spring of 1972 I saw the preprint of the paper [Ro] by H. Rosenthal; this paper played an essential role for me; it contains several ideas that I later used and developed in [Ma2]. Two of these ideas taken from [Ro] are the *factorization conditions* and the notion of *stable type* p. By Pietsch's factorization theorem, which extends some factorization results due to Grothendieck [Gro], every q-summing operator from $C(K)$ to a Banach space factors through the natural injection $C(K) \to L_q(K, \mu)$, for some probability measure μ on K. Rosenthal dualizes this fact, and shows that given $T : X \to L_1$ linear such that T^* is q-summing, then T factors through a multiplication operator $M_f : L_p \to L_1$ by a function $f \in L_q$ ($1/p + 1/q = 1$; let us simply write L_r for $L_r(K, \mu)$, $0 < r \le +\infty$); we have thus $T = M_f \circ T_1$, where $T_1 : X \to L_p$ is bounded and linear. One can give direct conditions on T that guarantee this factorization, with no need to further reference to q-summing maps: if an operator $T : X \to L_1$ is such that

$$\int \Big(\sum_i |T(x_i)|^p\Big)^{1/p} \, d\mu \le C \,\Big(\sum_i \|x_i\|^p\Big)^{1/p},$$

for some C and every finite sequence $(x_i) \subset X$, then T factors as $T = M_f \circ T_1$ for some $f \in L_q$. The proof of the factorization theorem is just an application of the Hahn-Banach separation theorem, either directly as in [Ma1], or by going back to Pietsch's factorization as in [Ro]. One gets in this way a function $f \in L_q$ such that $\|f\|_q \leq 1$ and $\int |T(x)/f|^p \, d\mu \leq C^p \, \|x\|^p$ for every $x \in X$. The above operator T_1 is then defined by $T_1(x) = T(x)/f \in L_p$ for every $x \in X$. Next, it is shown in [Ro] that a simple norm condition on X, that happens to be true for $X = L_s$ when $2 \geq s > p > 1$, easily implies the above factorization condition, as soon as $T : X \to L_1$ is bounded (and linear). This condition on a Banach space X is of the form

$$\int \|\sum_i f_i(t) x_i\| \, dt \leq K \, (\sum_i \|x_i\|^p)^{1/p},$$

where K is a constant depending only upon X, (f_i) is a sequence of L_1-normalized p-stable variables, and (x_i) an arbitrary sequence in X. This condition was called *stable type* p in [Ma1,Ma2]; it was used in [Ro] (without this name) for the injection of $X \subset L_1$ to L_1, and in the general case in [Ma2]. For example, since a Hilbert space has type 2, we obtain in this way that every bounded linear map from a Hilbert space to L_1 factors through a multiplication $M_f : L_2 \to L_1$, a statement dual to one of the results of [Gro]: *every operator from a $C(K)$-space to a Hilbert space is* 2*-summing.* By trace duality, this yields that every operator from ℓ_1 to ℓ_2 is 2-summing; we may call this *the easy Grothendieck theorem.* The same proof shows that every operator from a $C(K)$-space, to a space X such that the dual X^* has type 2, is 2-summing: this result appeared for the first time in a paper by Dubinsky, Pełczyński and Rosenthal [DPR].

It is obvious to generalize to operators from X to L_r the condition that gives a factorization through a multiplication operator $L_p \to L_r$ ($0 < r < p$, see [Ma1,Ma2]). In particular, some of the results obtained for $0 < r < 1$ are parallel to results obtained earlier by Nikišin [N1,N2]: since every Banach space X has stable type $1 - \varepsilon$ for every $\varepsilon > 0$, every operator from X to L_r, $0 < r < 1$, factors through $L_{1-\varepsilon}$ when $1 - \varepsilon \geq r$.

A first relation between these topics and finite dimensional geometry comes from the paper [Ro]; there, a delicate quantitative Lemma (Lemma 6 from [Ro]) shows that when the injection from a subspace $X \subset L_1$ to L_1 does not factor through any L_p, $p > 1$, then X must contain complemented almost isometric copies of ℓ_1^n for every $n \geq 1$, proving thus that every reflexive subspace of L_1 embed in some L_p, $p > 1$ (the main result of [Ro]). This Lemma was extended in [Ma2] to a general Banach space X as follows: when there exists an operator $T : X \to L_p$ that does not factor through any $L_{p+\varepsilon}$, $\varepsilon > 0$, then the injections $\ell_1^n \to \ell_p^n$, $n \in \mathbb{N}$, uniformly factor through X. In particular, when there exists an operator $T : X \to L_1$ that does not factor through any $L_{1+\varepsilon}$, $\varepsilon > 0$, then X contains uniformly isomorphic and complemented copies of ℓ_1^n, for every $n \geq 1$. This gives a new (bizarre) proof of Grothendieck's theorem: since ℓ_1^n is not uniformly complemented in c_0, the preceding statement implies that every bounded linear map from c_0 to L_1 factors through $L_{1+\varepsilon}$, and it reduces Grothendieck's theorem to a much easier variant. It is a model for a list of *reduction* results, for example this sort of extension of the Grothendieck theorem: *every operator from a cotype* 2 *space* X *to any Banach space, which is* 2*-summing, is already* 1*-summing* (see [Ma2]; as we have just said, when $X = L_1$, this is the information that one needs in order to pass from the easy Grothendieck theorem to the real one). This line of results displayed interesting connections between some simple finite dimensional phenomenons and analytic facts about Banach spaces.

In the same years, Hoffman-Jørgensen [HJ1] proved general results about series of vector valued independent random variables, that are in the spirit of Kahane's inequalities for vector valued Rademacher series; he also defined Rademacher type-p and showed connections to the law of large numbers in [HJ2]. The notion of type 2 (with a different name) appeared first in [DPR], and it was shown in this article that stable type 2 and Rademacher type 2 are identical. The results from [HJ1] imply that stable type p and Rademacher type p are closely related for every $p \in (1, 2]$: stable type p implies Rademacher type p, and Rademacher type p implies stable type $p-\varepsilon$ for every $\varepsilon > 0$. Later on, it has been universally admitted that Rademacher type is easier to work with, and the notion of stable type p essentially disappeared, except for $p = 2$, because 2-stable type and cotype express interesting properties of Gaussian probability measures on a Banach space. With Rademacher type p (we say simply type p in what follows), several points are simplified; it is obvious that type p implies type r for $r \le p$, and the opposite for cotype; the results for L_r spaces are easier to formulate, and simple to prove using Khintchine's inequality: L_r has type r and cotype 2 when $1 \le r \le 2$ and type 2 and cotype r when $2 \le r < +\infty$. Clearly, L_r does not have type $r+\varepsilon$, $\varepsilon > 0$ when $1 \le r \le 2$, and does not have cotype $r - \varepsilon$ when $2 \le r \le +\infty$. This suggested that one could possibly read some geometrical information about X from the limit values of p and q that give type p or cotype q for X.

The first attempts to relate type, cotype to the fact that X contains almost isometric copies of some classical spaces concerned ℓ_∞^n and ℓ_1^n. The first result [MP1] gave the equivalence between non-trivial cotype for X and the fact that X does not contain ℓ_∞^n uniformly; today, the proof in [MP1] looks a bit ridiculous by its complication. It was presented at the Conference at Oberwolfach, October 73; at the same meeting, James presented a much deeper result, namely his solution of the "reflexive *vs* B-convex" problem (see below). This was perhaps the beginning of what was later called "Local theory". For the relation between the absence of ℓ_1^ns in a Banach space X and other properties of this X, the first steps are due to Beck, Giesy and James, several years before this story [Be,G1,J1]; Beck showed the relevance to the law of large numbers in Banach spaces of the fact that X does not contain copies of ℓ_1^ns. Beck and Giesy defined *B-convex Banach spaces* as follows: the Banach space X is B-convex if for some $n > 1$ and $\varepsilon > 0$, and for all norm one vectors $(x_i)_{i=1}^n$ in X, at least one choice of signs gives $\| \sum_{i=1}^{n} \pm x_i \| \leq n\,(1-\varepsilon)$. Giesy proved several Banach space flavoured results about B-convexity, for example that X^* and X^{**} are B-convex when X is B-convex. James [J1] also worked on this class, which he called *uniformly non* ℓ_1^n; in this paper [J1], he conjectured that B-convex spaces can be renormed to be uniformly convex, and must therefore be reflexive (and he disproved this conjecture in 1973, as we have said above).

Shortly after the result for cotype and ℓ_∞^n, Pisier proved the type and ℓ_1^n case [P1]; he developed the submultiplicativity method for the type constants, which was important for the following paper [MP2]. Pisier's result showed that the class of B-convex spaces coincides with the class of spaces X that have type p for some $p > 1$. Then Pisier and I started to work on the relations between the limit values for the type or cotype of X, and the existence of subspaces of X that look somewhat like ℓ_p^n. Our first approach to the results of [MP2] was to strengthen the Dvoretzky-Rogers factorization [DR] for a Banach space X, using information on the limits of type and cotype; it just happened that the beautiful result of Krivine [Kr2] (see section 4) appeared during the preparation of [MP2] and allowed us to prove a much more satisfactory result. In the first version of [MP2], we proved that when X has type $p-\varepsilon$ but not $p+\varepsilon$ for every $\varepsilon > 0$, then the injections $\ell_1^n \rightarrow \ell_p^n$ factor almost isometrically through a subspace of X for all $n \geq 1$, which means that we can find norm one vectors $x_1, \ldots, x_n$ in X such that

$$(\sum_{i=1}^{n} |a_i|^p)^{1/p} \leq (1+\varepsilon)\, \|\sum_{i=1}^{n} a_i x_i\| \leq (1+\varepsilon)\, \sum_{i=1}^{n} |a_i|$$

for all scalars (a_i); the second inequality is of course obvious. When $p < 2$, this is a strengthening of the Dvoretzky-Rogers Lemma which says that the above statement holds in every Banach space when $p = 2$. Krivine's theorem appeared shortly after the first version of [MP2] was written; fortunately, Studia Math was so slow to publish at that time that we were able to modify our article in the form which is known as Maurey-Pisier or Maurey-Pisier-Krivine theorem. I will call it here MP+K theorem, to emphasize the fact that these three persons did not work together on this particular paper.

Kwapień was visiting Paris in 1971 and 72, just before all this started, and he played a significant role in the mathematical education of some of the young French; he gave several seminar talks that had a serious impact on us; he read and found the mistakes in several false "new proofs" that I had for the Grothendieck theorem, and he was the first person who checked the eventually correct proof of that I gave in [Ma2]. His result in [Kw] had a great influence on the subject of type and cotype; it appeared actually before the definitions of type and cotype were given, but it is nice to formulate it as follows: *if X has both type 2 and cotype 2, then X is isomorphic to a Hilbert space.* This is one of the first *isomorphic* characterizations of the Hilbert space. Some time later, I used in [Ma3] a small modification of Kwapień's argument and showed that every bounded linear operator from a subspace X_0 of a type 2 space X to a cotype 2 space Y factors through a Hilbert space, and extends to an operator from the whole space X to Y. In particular, every cotype 2 subspace X_0 of a type 2 space X is Hilbertian and complemented in X. This was a generalization of a well known result due to Kadec and Pełczyński [KP], that *Hilbertian subspaces of* L_p*,* $2 \le p < +\infty$*, are complemented.*

Super-properties appeared in the work of James on super-reflexivity (see [J2] and [J3], and section 2 below); ultraproduct methods [DK] give more insight on super-properties: a property is a super-property when it passes to ultrapowers. Super-reflexivity is obviously a super-property, and B-convexity is another super-property; James showed that super-reflexive spaces are B-convex. Deciding whether B-convex and super-reflexive spaces are the same class, as was conjectured by James in [J1], remained a difficult problem for some time, and was finally solved by James, who constructed a non-reflexive B-convex space ([J4], improved in [J5]); before this, Brunel and Sucheston [BS1,BS2] had tried to prove that B-convex spaces were reflexive, and a part of their attempt introduced an important concept, that of *spreading model*, which will be used here in sections 4 and 5. From this point on, there were two clearly distinct settings: super-reflexive spaces are those that can be renormed to be uniformly convex (Enflo [En]); they have *martingale* type p (the basis for Pisier's renorming theorem [P2]), and the class of B-convex or type-p spaces, $p > 1$, is strictly larger. However, contrary to the general case, type and uniform convexity are strongly related for lattices (see Johnson [Jo], and [LT, 1.f]). In a lattice X with non-trivial cotype, it is possible to prove Khintchine-type inequalities. Given $(x_i)_{i=1}^n$ in X, these inequalities permit to replace the estimate of a Rademacher average $\sum_{i=1}^n \varepsilon_i(t) x_i$ in $L_2(X)$ by an estimate of the square function $\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ in X. This kind of "functional calculus" for lattices was developed by Krivine in [Kr1], where he obtained interesting formulations of the Grothendieck theorem, relating operators between lattices and the square function (see also [LT, 1.f.14]).

Early signs of a tendency to move from abstract Banach spaces to the study of C^*-algebras and operator spaces also came in this framework. N. Tomczak [To] proved that the Schatten classes have the same type or cotype properties than the L_p spaces. Pisier [P3] generalized Grothendieck's theorem to C^*-algebras; the result was revisited by Haagerup [Ha] and was the start for many further exchanges between them. Several other factorization results related to Grothendieck's theorem were proved in those years, see [P8].

The first really striking application of cotype as a classification tool appears in the results of Figiel, Lindenstrauss and Milman [FLM]. They showed that Dvoretzky's theorem takes a very strong form in cotype 2 spaces: *if* X *has cotype* 2*, there exists a constant* $c > 0$ *such that for every integer* n*, every* n*-dimensional subspace of* X *contains a further subspace* X_0 *such that* $\dim X_0 =$

$m \geq c\, n$ *and* $d(X_0, \ell_2^m) \leq 2$. This result makes use of a certain fundamental formula

$$k = [\eta(\tau)\, n\, M_r^2 / b^2]$$

proved in [FLM, Theorem 2.6], relating the dimension k of $(1+\tau)$-spherical sections of an n-dimensional normed space to some integral invariant M_r. This formula appears already –with a different normalization– as *equation (14)* in [Mi1]. It gives the spectacular consequence above when using cotype 2 in an appropriate way; actually, [FLM] quantify the dimension of spherical sections in terms of the cotype q property, for every $q \geq 2$, and the previous result for cotype 2 is a special case. Another approach to the problem of spherical sections, the notion of *volume ratio* developed by Szarek and Tomczak [ST], also singles out the special behaviour of cotype 2 spaces. This approach is based on the work of Szarek [Sz], who introduced volume arguments in a new proof of the results of Kašin [Kš] about ℓ_1^n; of course Szarek need not mention cotype 2 when working with the explicit norm of ℓ_1^n! The fact that cotype 2 spaces have a uniformly bounded volume ratio was proved later by Bourgain and Milman [BMi], and this motivated the introduction of *weak cotype* 2 by Milman and Pisier ([MP], see also Chapter 10 of Pisier's book [P9]).

Type is a nice tool for estimating the behaviour of the entropy of a convex hull; a simple observation of mine, written in [P5], was used in entropy problems by Carl [Ca]. This observation states that in a Banach space X with type $p > 1$, every point x from the convex hull of a subset A of the unit ball B_X can be approximated by a convex combination of n points of A, with an error of order $n^{-1/q}$ (with q conjugate to p). Lemma 9 below is in the spirit of this result.

Type and cotype have some simple stability properties; for example, the dual of a type p space has cotype q for the conjugate exponent, but the converse is false as shown by the pair (ℓ_1, ℓ_∞). The two young and ignorant authors of [MP2] left open a nice intriguing conjecture: is it possible to dualize cotype when we have some non-trivial type? It is clear that what is needed is the boundedness of the Rademacher projection on $L_2(X)$. Spaces such that the Rademacher projection is bounded were called K-convex in [MP2] (was it because K was the first available letter after J for J-convex, a notion due to James and named by Brunel and Sucheston [BS2], or to acknowledge the importance of Kwapień's work on Rademacher averages?) It was conjectured in [MP2] that every space with type $r > 1$ should be K-convex, which would imply that the dual X^* of a space X with cotype q and some non-trivial type should be of type p, with $1/p + 1/q = 1$. Six years later, Pisier proved what I consider the most beautiful result in this area, making use of Kato's theorem on holomorphic semi-groups (see [P6] and section 6 of this article): *every B-convex space is K-convex.*

Although very beautiful, the preceding theorem is not the one that has been most useful for local theory. The most useful is another result obtained earlier by Pisier [P4], on the way to the general theorem above. This result asserts that the K-convexity constant of X is bounded by $C\,(1+\ln d_X)$, where d_X is the distance from X to the Hilbert space of the appropriate dimension (see Theorem 13 below). In particular, the K-convexity constant is bounded by $C\,(1+\ln n)$ for any n-dimensional normed space. The quantitative finite-dimensional K-convexity, together with the notion of ℓ-norm, leads to a powerful tool for geometric estimates (Theorem 3.11 in [P9]; this theorem appeared first in [FT]). These results play an important role in the QS-theorem of Milman ([Mi2], see also [P9]).

2 Super-properties

Several of the properties P that are defined for a Banach space X are expressed in the following way: suppose that a number $N_P(E)$ is associated to every finite dimensional normed space E, in such a way that $N_P(F)$ tends to $N_P(E)$ when the Banach-Mazur distance $d(F,E)$ between F and E tends to 1; the most common such dependence is when $N_P(F) \le d(F,E)\,N_P(E)$. We then say that the Banach space X *satisfies property* P when $N_P(X) = \sup_E N_P(E) < +\infty$, where the supremum is extended to all finite dimensional subspaces E of X.

Clearly, the fact that such a property P holds for X only depends upon the family $\mathcal{F}(X)$ of all finite dimensional normed spaces F such that for every $\varepsilon > 0$, there exists $E \subset X$ for which $d(F,E) < 1+\varepsilon$. After James [J2], we say that Y is *finitely representable* in X when $\mathcal{F}(Y) \subset \mathcal{F}(X)$; for instance, L_p is finitely representable in ℓ_p, and it is known that X^{**} is finitely representable in X for every Banach space X (local reflexivity). A property P of Banach spaces is called *super-property* if we know that whenever a Banach space X has P, then every Banach space Y finitely representable in X has P. Clearly, every property P expressed by $N_P(X) < +\infty$ as above is a super-property. Super-properties were defined by James in [J2].

Type and cotype are such properties. Let us recall a few definitions and facts that are developed in [JL]. Let $(\varepsilon_i)_{i=1}^{+\infty}$ denote the sequence of Rademacher functions on $[0,1]$, or any independent sequence of centered Bernoulli random variables. Let $p \in [1,+\infty)$. We say that X has type p when there exists a constant T such that

$$\Big(\int_0^1 \|\sum_{i=1}^n \varepsilon_i(t)\,x_i\|^2\,dt\Big)^{1/2} \le T\,(\sum_{i=1}^n \|x_i\|^p)^{1/p},$$

for every $n \ge 1$ and every sequence $(x_i)_{i=1}^n \subset X$; we denote by $T_p(X)$ the smallest constant T with this property; obviously, every normed space X has type 1 with $T_1(X) = 1$. On the other hand, it follows from Khintchine's inequalities that no non-zero normed space has type p when $p > 2$. Saying that X has type p is obviously equivalent to the fact that the family of finite dimensional subspaces E of X satisfies $\sup_E T_p(E) < +\infty$, thus having type p is a super-property. We say that X has cotype q when there exists a constant $C_q(X)$ such that

$$(\sum_{i=1}^{n} \|x_i\|^q)^{1/q} \le C_q(X) \Bigl(\int_0^1 \|\sum_{i=1}^{n} \varepsilon_i(t)\, x_i\|^2 \,dt\Bigr)^{1/2}$$

for every $n \ge 1$ and every sequence $(x_i)_{i=1}^n \subset X$; again, this is equivalent to the fact that $\sup_E C_q(E) < +\infty$, and cotype is therefore another super-property. In both definitions of type and cotype, the choice of the L_2 norm for the Rademacher averages is irrelevant (except for the exact value of the constants); this follows from Kahane's inequalities (see [Ka, Chapter II, Th. 4]), which state that for every $q < \infty$, there exists a constant K_q such that

$$\Bigl(\int_0^1 \|\sum_{i=1}^{n} \varepsilon_i(t)\, x_i\|^q \,dt\Bigr)^{1/q} \le K_q \int_0^1 \|\sum_{i=1}^{n} \varepsilon_i(t)\, x_i\| \,dt$$

for every $n \ge 1$ and every family $(x_i)_{i=1}^n$ of vectors in a Banach space.

It is easy to show that when X has type or cotype, then the same holds for the space $L_2(X)$ of X-valued square integrable functions. This fact is used below in section 5 and section 6.

3 Ultrapowers and some operator lemmas

In the next section about Krivine's theorem, we use a classical fact for operators on a complex Banach space X: if λ is a boundary point of the spectrum $\mathrm{Sp}(T)$ of $T \in \mathcal{L}(X)$, then λ is an *approximate eigenvalue* for T, which means that there exists a sequence $(x_n) \subset X$ of norm one vectors such that $\lim_n (T(x_n) - \lambda x_n) = 0$. We shall need a slightly less classical fact about commuting operators, which is very easy to obtain using the notion of ultrapower (Lemma 1 below). We shall first recall a few facts about ultrapower techniques. These techniques became popular in Banach space theory after the paper by Dacunha-Castelle and Krivine [DK]; approximately at the same time, similar objects were introduced for C^*-algebras [Ja]. The limit spaces used by James [J2] in his study of super-reflexivity, the spreading models of Brunel-Sucheston [BS1], belong to the same family of tools which make possible to construct an abstract space from different pieces taken at different places.

Suppose that $\mathcal{U}$ is a non-trivial ultrafilter on $\mathbb{N}$. If X is a Banach space, we consider in $X_\infty := \ell_\infty(X)$ the closed subspace $K_\mathcal{U}$ of all sequences $\mathbf{y} = (y_n) \in X_\infty$ such that $\lim_{n\to\mathcal{U}} \|y_n\| = 0$, and we let $X_\mathcal{U}$ be the quotient space $X_\infty/K_\mathcal{U}$. Let $\pi_\mathcal{U}$ denote the quotient map from X_∞ to $X_\mathcal{U}$. If $\mathbf{x} = (x_n)$ and $\xi = \pi_\mathcal{U}(\mathbf{x})$, then $\|\xi\| = \lim_{n\to\mathcal{U}} \|x_n\|$. We have a canonical isometry $i_{X,\mathcal{U}}$ from X to $X_\mathcal{U}$ that sends $x \in X$ to the class of the constant sequence $\mathbf{x} = (x_n)$ where $x_n = x$ for every n. Using this isometric embedding we shall consider that $X \subset X_\mathcal{U}$.

The crucial fact is here: suppose that $\eta_1, \ldots, \eta_\ell \in X_\mathcal{U}$ are represented by sequences $\mathbf{y}_j = (y_{j,n})_{n\ge 0} \in X_\infty$, for $j = 1, \ldots, \ell$, and that we have a finite number of inequality relations

$$(R) \qquad a_i < \|x_i + \sum_{j=1}^{\ell} b_{i,j}\, \eta_j\| < c_i, \quad i = 1, \ldots, k,$$

where $a_i, c_i \in \mathbb{R}$, $x_i \in X$, $(b_{i,j})$ is a matrix of scalars. Let us say that a property depending upon $n \in \mathbb{N}$ is true *when n is $\mathcal{U}$-large* if the set $A \subset \mathbb{N}$ of those n for which the property holds belongs to $\mathcal{U}$; then we can say that when n is $\mathcal{U}$-large, we have in X

$$(R_n) \qquad a_i < \|x_i + \sum_{j=1}^{\ell} b_{i,j}\, y_{j,n}\| < c_i, \quad i = 1, \ldots, k.$$

This implies that $X_\mathcal{U}$ is finitely representable in X (and slightly more: if E is any finite dimensional subspace of $X_\mathcal{U}$, we can find a $(1+\varepsilon)$-isomorphism T from E into X such that $T(x) = x$ for every $x \in E \cap X$). We see that F belongs to $\mathcal{F}(X)$ if and only if F is isometric to a subspace of $X_\mathcal{U}$. Every super-property of X passes to $X_\mathcal{U}$, for example type or cotype.

Suppose now that T is a bounded linear operator on X. We define T_∞ on X_∞ in the obvious way,

$$T_\infty(\mathbf{x}) = (T(x_n)),$$

whenever $\mathbf{x} = (x_n) \in X_\infty$. It is clear that $K_\mathcal{U}$ is stable under T_∞, so that T_∞ induces a bounded linear map $T_\mathcal{U}$ on $X_\mathcal{U}$. It is easy to check that $T \to T_\mathcal{U}$ is an isometric homomorphism of unital Banach algebras from $\mathcal{L}(X)$ to $\mathcal{L}(X_\mathcal{U})$.

Using the above principle $(R) \Rightarrow (R_n)$, we see that if $\mathbf{x} = (x_n) \in X_\infty$ and if $\xi = \pi(\mathbf{x}) \in X_\mathcal{U}$, then this vector ξ satisfies $T_\mathcal{U}(\xi) = \lambda\xi$ if and only if $\lim_{n\to\mathcal{U}} (T(x_n) - \lambda x_n) = 0$; in particular, if X is complex, for every boundary point λ of the spectrum of T we can find a sequence $(x_n) \subset X$ of norm one vectors such that $\lim_n (T(x_n) - \lambda x_n) = 0$, which shows that the eigenspace $\ker(T_\mathcal{U} - \lambda I)$ is not trivial.

Lemma 1 *Suppose that X is a complex Banach space, and that S, T are commuting bounded linear operators on X. If $(x_n) \subset X$ is a sequence of norm one vectors such that $T(x_n) - \lambda x_n$ tends to 0, we can find $\mu \in \mathbb{C}$ and a norm one vector $x \in X$ such that $T(x) \sim \lambda x$ and $S(x) \sim \mu x$.*

PROOF. We know that $X_\lambda = \ker(T_{\mathcal{U}} - \lambda I)$ is not $\{0\}$, and $S_{\mathcal{U}}$ commutes with $T_{\mathcal{U}}$, therefore X_λ is stable under $S_{\mathcal{U}}$. If μ is a boundary point of the spectrum of the restriction of $S_{\mathcal{U}}$ to X_λ, we can find a norm one vector ξ in X_λ such that $S_{\mathcal{U}}(\xi) \sim \mu\xi$. Bringing back ξ to X —using $(R) \Rightarrow (R_n)$, with $\eta_1 = \xi$, $\eta_2 = T_{\mathcal{U}}(\xi)$ and $\eta_3 = S_{\mathcal{U}}(\xi)$— we obtain for every $\varepsilon > 0$ a norm one vector $x \in X$ such that $\|T(x) - \lambda x\| < \varepsilon$ and $\|S(x) - \mu x\| < \varepsilon$.

Let X be a complex Banach space, and let T be an into isomorphism from X into X, with $\|x\| \le C\,\|T(x)\|$ for every $x \in X$. For every integer $n \ge 1$, we may define K_n as the smallest constant for which

$$\|x\| \le K_n\,\|T^n(x)\|$$

for every $x \in X$. It is clear that $K_{m+n} \le K_m\,K_n$, so that $r = \lim_n K_n^{1/n}$ exists by a standard lemma. Also, $K_n \le C^n$ and $K_n\,\|T^n(x)\| \le K_n\,\|T\|^n \|x\|$ yield that $0 < \|T\|^{-1} \le r \le C$.

Lemma 2 *There exists $\lambda \in \mathbb{C}$ with $|\lambda| = r$ and a sequence (x_n) of norm one vectors in X such that $\lim_n (T(x_n) - \lambda^{-1}x_n) = 0$.*

PROOF. We introduce an operator S of which r will be the spectral radius; this S acts as a sort of inverse for $T_{\mathcal{U}}$. For every $x \in X$, let $N(x)$ denote the supremum of k such that x belongs to the range of T^k (this value $N(x)$ may be $+\infty$). Let Z_0 be the subspace of $X_{\mathcal{U}}$ consisting of all ξ that have a representative $\mathbf{x} = (x_n)$ such that $\lim_{\mathcal{U}} N(x_n) = +\infty$. It is obvious that Z_0 is stable under $T_{\mathcal{U}}$; let Z be the closure in $X_{\mathcal{U}}$ of Z_0, and let T_Z denote the restriction of $T_{\mathcal{U}}$ to Z.

When $\xi \in Z_0$, we see that $\xi = T_Z(\eta)$ for some (unique) η: indeed, if $\mathbf{x} = (x_n)$ belongs to the class of ξ and $\lim_{\mathcal{U}} N(x_n) = +\infty$, we have that $N(x_n) \ge 1$ when n is $\mathcal{U}$-large, which means that $A = \{n : N(x_n) \ge 1\} \in \mathcal{U}$; hence for every $n \in A$ we have $x_n = T(y_n)$ for some $y_n \in X$; if we let $y_n = 0$ for $n \notin A$, then $\mathbf{y} = (y_n)$ satisfies $\lim_{n \to \mathcal{U}} N(y_n) = +\infty$ (because $N(y_n) \ge N(x_n) - 1$); if $\eta = \pi(\mathbf{y})$, then η belongs to Z and $T_Z(\eta) = \xi$; clearly $\|\eta\| \le C\,\|\xi\|$. This shows that T_Z is invertible in $\mathcal{L}(Z)$.

Let $S = T_Z^{-1}$. It is quite clear that $\|S^n\| \le K_n$, so that the spectral radius $\rho(S) = \lim_k \|S^k\|^{1/k}$ of S satisfies $\rho(S) \le r$; we shall see that $r = \rho(S)$. Let us fix $k \geq 2$ and $\varepsilon > 0$. For n large, we know that $K_{nk} > (r-\varepsilon)^{nk}$, thus we can find a vector $x_n \in X$ such that $\|x_n\| > (r-\varepsilon)^{nk}\|T^{nk}(x_n)\|$. Let h be a large integer, but small compared to n, say $h - 1 < \sqrt{n} \leq h$ for example. If we had

$$\|T^{jk}(x_n)\| \leq (r-2\varepsilon)^k \, \|T^{jk+k}(x_n)\| = (r-2\varepsilon)^k \, \|T^k(T^{jk}(x_n))\|$$

for every $j = h, \ldots, n-1$, it would follow that

$$(r-\varepsilon)^{nk}\|T^{nk}(x_n)\| < \|x_n\| \leq C^{hk} \, (r-2\varepsilon)^{nk-hk} \, \|T^{nk}(x_n)\|,$$

which is impossible when n is large. For every $n \geq n_0$, and for some j such that $\sqrt{n} \leq j < n$, we may thus find a vector $y_n = \alpha \, T^{jk}(x_n)$ such that $1 = \|y_n\| > (r-2\varepsilon)^k \, \|T^k(y_n)\|$, and this vector satisfies $N(y_n) \geq k\sqrt{n}$. If $\mathbf{y} = (y_n)$ and $\eta = \pi(\mathbf{y})$ we get $\eta \in Z$ and $\|S^k(\eta)\| > (r-2\varepsilon)^k \, \|\eta\|$. It follows that the spectral radius of S is larger than $r - 2\varepsilon$, hence equal to r.

Let $\lambda \in \mathrm{Sp}(S)$ be such that $|\lambda| = r$. It follows from the “boundary of the spectrum lemma” that we can find a norm one vector $\xi \in Z$ such that $S(\xi) \sim \lambda\,\xi$, or $T_Z(\xi) \sim \lambda^{-1}\xi$; bringing back ξ to X in the usual way gives a norm one vector x for which $T(x) \sim \lambda^{-1}x$, as was to be proved.

4 Krivine’s theorem

See [MS, Chapter 12] or [BLi, Chapter 12] for a more precise presentation of the results of this section. I prefer here to tell a pleasant story, rather than being too technical. Roughly speaking, Krivine’s theorem says that every Banach space X contains $(1+\varepsilon)$-isomorphs of ℓ_p^n, for some $p \in [1, +\infty]$ and every $n \geq 1$, or in other words it says that some ℓ_p (or c_0, when $p = +\infty$) is finitely representable in X. More precise statements tell us that, given a basic sequence in X, or simply a sequence (x_n) with no Cauchy subsequence, there exists $p \in [1, +\infty]$ such that for every $n \geq 1$ and $\varepsilon > 0$, we can find blocks of the given sequence that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^n. It is sometimes useful to be more specific, and to predict what values of p can be realized, starting from some norm invariants of the sequence (x_n). This will be the case in the next section about type, cotype and the MP+K theorem.

The proofs of Krivine’s theorem are usually divided into two steps: the first step replaces the given sequence by one that has some minimal regularity; this step uses only subsequences, or just differences of two vectors from the original sequence (as opposed to the second step, that requires clever long blockings). The argument is due to Brunel and Sucheston: given a sequence with no Cauchy subsequence, and using Ramsey’s theorem, we may find a subsequence which is asymptotically *invariant under spreading*, see [BS1], and also [Go]; alternatively, this can be achieved by general abstract arguments involving iterated ultrapowers, usual in model theory where a somewhat parent notion of *indiscernible sequence* is defined. Given a Banach space X and a space Y of scalar sequences, we say that Y is a *spreading model* for X if there exists a normalized sequence $(x_n) \subset X$, with no Cauchy subsequence, such that

$$\| \sum_{j=1}^{k} a_j \, \mathbf{e}_j \|_Y = \lim \| \sum_{j=1}^{k} a_j \, x_{n_j} \|_X$$

for every $k \geq 1$ and all scalars $(a_j)_{j=1}^{k}$; the limit is taken when $n_1 \to \infty$ and $n_1 < n_2 < \ldots < n_k$, and $(\mathbf{e}_j)$ denotes the standard unit vector basis for the space of scalar sequences.

The second part of this first step, also due to Brunel and Sucheston, is to observe that the differences $(\mathbf{e}_{2j+1} - \mathbf{e}_{2j})$ are *suppression-unconditional* in Y (see below for a definition); further, the differences are bounded away from zero because the sequence (x_n) had no Cauchy subsequence; this implies that we can find 2-unconditional finite sequences $(z_i)_{i=1}^{k}$ in X, with k as large as we wish, whose vectors z_i are differences $z_i = x_{n_{2i}} - x_{n_{2i-1}}$ of two suitable vectors from the given sequence (x_n). The spreading model Y is finitely representable in X, in a special way: any finite sequence (y_k) of blocks of the basis in Y can be sent to blocks from the sequence (x_n) in X. We shall therefore present the rest of the proof of Krivine's theorem assuming that we start from this situation, replacing the original space X by a spreading model X', which is (block) finitely representable in X and has a nice basis. The real thing is to prove Krivine's theorem for X'.

Let X be a Banach space with a basis $(e_n)_{n \geq 0}$; we say that this basis is a *suppression-unconditional basis* when for every $x \in X$, the norm does not increase if we replace one of the non-zero coordinates of x by 0; this yields that the basis is unconditional, with unconditionality constant ≤ 2 (in the real case). Let X be a Banach space with a suppression-unconditional basis $(e_n)_{n \geq 0}$; we say that the norm is *invariant under spreading* if for every integer $k \geq 0$ and all $n_0 < n_1 < \ldots < n_k$,

$$\|\sum_{j=0}^{k} a_j \, e_{n_j}\| = \|\sum_{j=0}^{k} a_j \, e_j\|$$

for all scalars (a_j). Let $x = \sum_{j=0}^{k} a_j \, e_j$ be a vector with finite support in X; we say that y is a *copy* of x if $y = \sum_{j=0}^{k} a_j \, e_{n_j}$ for some $n_0 < n_1 < \ldots < n_k$. If $x = \sum a_j e_j$ and $y = \sum b_j e_j$, we write $x < y$ when all non-zero coordinates of x appear before those of y, that is $\max\{j : a_j \neq 0\} < \min\{j : b_j \neq 0\}$. We say that $x_1, \ldots, x_n$ are *successive vectors* if $x_1 < x_2 < \ldots < x_n$.

After the preliminary work has been done, the heart of Krivine's result is the following Theorem 3. The arguments of Brunel-Sucheston imply that for every Banach space X, we can find a space X_0 with a suppression-unconditional basis, invariant under spreading, such that X_0 is finitely representable in X; if X_0 contains ℓ_p^k, then X will also. We shall therefore assume that X is a Banach space with a suppression-unconditional basis $(e_n)_{n\geq 0}$, and a norm invariant under spreading. For every integer $n \geq 1$, let R_n be the smallest constant and S_n be the largest constant such that for every $x \in X$, we have

$$S_n \, \|x\| \leq \|\sum_{i=1}^{n} x_i\| \leq R_n \, \|x\|$$

whenever $x_1 < x_2 < \ldots < x_n$ are successive copies of x.

Theorem 3 *Let X be a Banach space with a suppression-unconditional basis $(e_n)_{n\geq 0}$, and a norm invariant under spreading; suppose that $p \geq 1$ is defined by the equation*

$$(\mathbf{a}): \;\; 2^{1/p} = \limsup_n (R_{2^n})^{1/n} \quad \textit{or} \quad (\mathbf{b}): \;\; 2^{1/p} = \liminf_n (S_{2^n})^{1/n}.$$

For every $k \geq 1$ and $\varepsilon > 0$ it is possible to find k successive blocks $x_1 < \ldots < x_k$ in X that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^k, and that are copies of some norm one vector $x \in X$.

PROOF. Let I be the set of rational numbers r such that $0 \leq r < 1$, let $(f_r)_{r\in I}$ be the standard unit vector basis for $\mathbb{R}^{(I)}$, and let us define a norm on the linear span Y_0 of $(f_r)_{r\in I}$ as follows: if $r_0 < r_1 < \ldots < r_k$, let

$$\|\sum_{j=0}^{k} a_j \, f_{r_j}\|_Y = \|\sum_{j=0}^{k} a_j \, e_j\|_X$$

for all scalar coefficients (a_j). If Y_0 is real, we complexify it in any reasonable way, for example

$$\|x + iy\| = \sup_\theta \, \|\sin(\theta)\, x + \cos(\theta)\, y\|,$$

which preserves invariance under spreading and unconditionality. Let Y be the completion of Y_0; it is clear that (f_r) is a suppression-unconditional basis for Y, invariant under spreading in the new context. We say that $y' \in Y$ is a copy of $y = \sum_{r\in I} a_r \, f_r$ if $y' = \sum_{r\in I} a_r \, f_{\phi(r)}$ for some increasing map ϕ from I into itself. What we mean by successive copies of a given vector in

Y is clear. It is also clear that Y is finitely representable in X, and a finite-dimensional subspace of Y generated by successive copies of some vector in Y can be approximated by a subspace of X, generated by successive copies of some vector in X.

We can now relate the behaviour of sums of copies of vectors in X to the properties of some linear operators defined on this space Y. Indeed, we may define a *doubling operator* D on Y by the formula

$$\forall y \in Y, \quad D(y) = \sum_{0 \le r < 1/2} y(2r)\, f_r + \sum_{1/2 \le r < 1} y(2r-1)\, f_r,$$

or $D(y)(r) = y(2r \bmod 1)$, considering y as a function $I \to \mathbb{C}$. For every $y \in Y_0$, the vector $D(y)$ is the sum of two copies $y_1 < y_2$ of y, hence $\|y\| \le \|D(y)\| \le 2\,\|y\|$. It is clear that the constant R_{2^n} for the initial space X is equal to the norm of D^n, therefore in case **(a)**, we see that $2^{1/p}$ is the spectral radius of D. We may thus find $\lambda \in \mathbb{C}$ with $|\lambda| = 2^{1/p}$ and a norm one vector $z \in Y_0$ such that $D(z) \sim \lambda z$. In case **(b)**, the constant S_{2^n} appears to be the reciprocal of the constant K_n associated to the into isomorphism D (see before Lemma 2), therefore if $2^{1/p} = \lim_n S_{2^n}^{1/n}$, we know by Lemma 2 that we can again find $\lambda \in \mathbb{C}$ with $|\lambda| = 2^{1/p}$ and a norm one vector $z \in Y_0$ such that $D(z) \sim \lambda z$. Using unconditionality, we get $D(|z|) \sim |\lambda|\,|z|$. In both cases **(a)** and **(b)** we found a norm one vector $y = \alpha|z| \in Y_0$ (with $1/2 \le \alpha \le 2$) such that $D(y) \sim 2^{1/p}\, y$. Reproducing y in X gives a norm one vector $x \in X$ such that, when $x_1 < x_2$ are copies of x, then $x_1 + x_2$ is very close to some copy x' of $2^{1/p}\, x$. I like to call such a vector x a *Krivine vector*. Suppose that $x_1 < x_2 < \ldots < x_k$ are copies of this vector x. If $n \ge 1$ is given and if $D(y) - 2^{1/p}\, y$ has norm smaller than some $\varepsilon_n > 0$, we deduce that

$$(K) \qquad \|\sum_{j=1}^{k} a_j x_j\|^p \sim \sum_{j=1}^{k} a_j^p,$$

provided all coefficients are of the form $a_j = 2^{-k_j/p}$, for some integer k_j such that $0 \le k_j \le n$, and $\sum_j a_j^p = 1$ (if $K = \max k_j$, replace each $a_j x_j$ by 2^{K-k_j} copies of $2^{-K/p}\, x$; this gives 2^K copies of $2^{-K/p}\, x$, which we may group two by two again, obtaining after K steps a single copy of the vector x).

This is not quite enough, and we also introduce an operator T on Y which reproduces three times every vector $y \in Y$,

$$T(y) = \sum_{0 \le r < 1/3} y(3r)\, f_r + \sum_{1/3 \le r < 2/3} y(3r-1)\, f_r + \sum_{2/3 \le r < 1} y(3r-2)\, f_r.$$

It is clear that $DT = TD$ is the operator that replaces every vector x by six copies of x; the commutation property and Lemma 1 enable us to find a norm one vector z such that $D(z) \sim 2^{1/p} z$ and $T(z) \sim \mu z$; then $T(|z|) \sim |\mu|\,|z|$, so that we may assume that z and μ are real and ≥ 0. Some simple lattice arguments (involving comparisons of the norms of sums of respectively 2^h, 3^i and 2^j copies of z when $2^h < 3^i < 2^j$) show that necessarily $\mu = 3^{1/p}$.

If $D(z) - 2^{1/p}\, z$ and $T(z) - 3^{1/p}\, z$ are small enough, and if $z_1 < z_2 < \ldots < z_k$ are copies of this vector z, we may try to extend relation (K) to coefficients (a_j) such that $a_j = 2^{\ell_j} 3^{m_j}$ for some $\ell_j, m_j \in \mathbb{Z}$; since these values are dense in $[0, \infty)$, we are in a good position. However, dealing with the error terms is painful, and we may instead pass to the ultrapower $Y_{\mathcal{U}}$, which is still a lattice, with a linear ordering defined in this way: we say that $\xi < \eta$ if ξ and η have representatives (x_n) and (y_n) with $x_n < y_n$ for every n, and we say that η is a copy of ξ if ξ and η have representatives (x_n) and (y_n) such that y_n is a copy of x_n for every n; in $Y_{\mathcal{U}}$ we can find a norm one vector η such that $D_{\mathcal{U}}(\eta) = 2^{1/p}\eta$ and $T_{\mathcal{U}}(\eta) = 3^{1/p}\eta$; to get this, we take for η the class of a normalized sequence (z_n) in Y with $D(z_n) - 2^{1/p} z_n \to 0$ and $T(z_n) - 3^{1/p} z_n \to 0$. In this framework where we have equalities, it is easy to prove that when $\eta_1, \ldots, \eta_k$ are successive copies of this vector η and when the coefficients (a_j) satisfy $a_j = 2^{\ell_j} 3^{m_j}$, with $\ell_j, m_j \in \mathbb{Z}$, then $\| \sum_{j=1}^k a_j \eta_j \|^p = \sum_{j=1}^k a_j^p$; next, we extend this by density to all non-negative scalars. Going back to X, and using the special form of the vectors $\eta_1, \ldots, \eta_k$, we can find successive copies $x_1, \ldots, x_k$ of some norm one vector x in X such that

$$(K') \qquad (1+\varepsilon)^{-p/2} \sum_{j=1}^k a_j^p \leq \|\sum_{j=1}^k a_j x_j\|^p \leq (1+\varepsilon)^{p/2} \sum_{j=1}^k a_j^p,$$

for all non-negative scalars (a_j). Everything would be fine if the basis in X was 1-unconditional, but it is not so: what we get so far is a sequence $x_1, \ldots, x_k$ which is $2(1+\varepsilon)$-equivalent to the ℓ_p^k-basis in the real case, and $4(1+\varepsilon)$ in the complex case, for every $k \geq 2$: if $v = \sum a_j x_j$, the ℓ_p-norm of the coefficients is dominated by $(\|v^+\|^p + \|v^-\|^p)^{1/p} \leq 2^{1/p}\, \|v\|$, using first (K') then suppression unconditionality; in the other direction use $\|v\| \leq 2^{1-1/p}\, (\|v^+\|^p + \|v^-\|^p)^{1/p}$.

Suppose $p < \infty$ for simplicity; if $k = m^2$ and if we form new blocks $y_1, \ldots, y_m$ in X of the form $y_i = m^{-1/p} \sum_{j=1}^m (-1)^j x_{m(i-1)+j}$, then $(y_1, \ldots, y_m)$ is still a sequence of successive copies of some $y \in X$, hence invariant under spreading, 5-equivalent to the ℓ_p^m basis (say), but the unconditional constant is improved to something arbitrarily close to 1 as m grows (in the complex case, a similar trick using a primitive root of unity does the required job). We build a limit space X_1 from the sequence $([y_1^{(m)}, \ldots, y_m^{(m)}])_m$, by setting

$$\|\sum_{j=1}^n a_j\, e_j\|_{X_1} = \lim_{m \to \mathcal{U}} \|\sum_{j=1}^{n \wedge m} a_j\, y_j^{(m)}\|$$

for every $n \geq 1$ and all scalars (a_j). This space X_1 is finitely representable in X, with a 1-unconditional basis, invariant under spreading, and 5-equivalent to the ℓ_p basis. In X_1 we have clearly $2^{1/p} = \limsup_n (R_{2^n})^{1/n}$. Applying the above construction to X_1 gives new blocks $x_1, \ldots, x_k$ that satisfy (K') in X_1: this finishes the proof, since the basis in X_1 is 1-unconditional.

The proof above is due to Lemberg [Le], who was Krivine's PhD student in the years '80. The fundamental facts are still the same as in the original paper [Kr2], but the details in [Kr2] are harder to follow. Combining the arguments of Brunel-Sucheston and the preceding Theorem, we obtain one of the usual forms of Krivine's theorem.

Corollary 4 *Suppose that X is a real or complex Banach space, and (x_n) a bounded sequence in X with no Cauchy subsequence. For some $p \in [1, +\infty]$, for every $k \geq 1$ and $\varepsilon > 0$ it is possible to find k successive blocks of the given sequence that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^k.*

Our next Corollary is expressed in a slightly unnatural way, but suitable for the next section.

Corollary 5 *Suppose $r, s \geq 1$ are given. If for some $\kappa > 0$ and for every $n \geq 2$, a Banach space X contains a normalized suppression-unconditional sequence $\mathbf{y}^{(n)} = (y_1^{(n)}, \ldots, y_n^{(n)})$ such that*

$$\|\sum_{i \in C} y_i^{(n)}\| \geq \kappa\, |C|^{1/r}$$

for every subset $C \subset \{1, \ldots, n\}$, or such that

$$\|\sum_{i \in C} y_i^{(n)}\| \leq \kappa\, |C|^{1/s}$$

for every subset $C \subset \{1, \ldots, n\}$, then for some $p \leq r$ (or $p \geq s$) and for every $k \geq 1$, $\varepsilon > 0$, it is possible when $n \geq N(k, \varepsilon)$ to form k successive blocks of the given sequence $\mathbf{y}^{(n)}$ that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^k.

PROOF. We construct as we did before a limit space X' from the long sequences as follows. Using Brunel-Sucheston principle, we may select from our long sequences $(y_i^{(n)})$ some (finite) subsequences $z_1^{(n)}, \ldots, z_{k_n}^{(n)}$ that are almost indiscernible, and have a length k_n tending to ∞ with n; then we define a norm on c_{00} (the space of finitely supported scalar sequences) by

$$\|\sum_{i=1}^{m} c_i \mathbf{e}_i\|_{X'} = \lim_{n \to \mathcal{U}} \|\sum_{i=1}^{m \wedge k_n} c_i z_i^{(n)}\|$$

where $(\mathbf{e}_i)_{i\ge 0}$ denotes the unit vector basis of c_{00}. Notice that when n is $\mathcal{U}$-large, the length k_n exceeds m; this yields that $(\mathbf{e}_i)$ is normalized in X'. We obtain a space X' with a normalized suppression-unconditional basis and a norm invariant under spreading. In the first case, we get for every $n \ge 1$

$$\kappa\, n^{1/r} \le \|\sum_{i=1}^{n} \mathbf{e}_i\|_{X'} \le R_n \, \|\mathbf{e}_1\|_{X'} = R_n,$$

and similarly in the second case we obtain that $S_n \le \kappa\, n^{1/s}$. We know from Theorem 3 that we may get ℓ_p^k in X', with p such that $2^{1/p} = \lim_n (R_{2^n})^{1/n}$ or $2^{1/p} = \lim_n (S_{2^n})^{1/n}$, thus $p \le r$ in the first case and $p \ge s$ in the second.

5 Type, cotype and ℓ_p^ns. The MP+K theorem

Let X be a Banach space. We denote by p_X the supremum of all p such that X has type p, and by q_X we denote the infimum of all q such that X has cotype q. It is clear using Khintchine's inequality that $p_X \le 2 \le q_X$, already when $X = \mathbb{R}$.

Theorem 6 *Let X be an infinite dimensional Banach space; for every integer $k \ge 1$ and $\varepsilon > 0$, the space X contains $(1+\varepsilon)$-isomorphs of $\ell_{p_X}^k$ and of $\ell_{q_X}^k$.*

For the type case and $1 < p < 2$, there exists a quantitative estimate due to Pisier [P7], see also [MS, Theorem 13.12]. The dimension k of a good isomorph in X of ℓ_p^k is given there as a function of the stable type p constant $ST_p(X)$ of the normed space X.

PROOF. If $p_X = 2$ we may use Dvoretzky's theorem [D2]. Assume $p_X < 2$ and choose r such that $p_X < r < 2$. For each $n \ge 1$, let $\varphi(n)$ denote the smallest constant such that

$$\int_0^1 \|\sum_{i=1}^{n} \varepsilon_i(t) x_i\|^r \, dt \le \varphi(n)^r \sum_{i=1}^{n} \|x_i\|^r$$

for every family $x_1, \ldots, x_n$ of n vectors in X. It is clear that φ is non-decreasing, and tends to $+\infty$ since X does not have type r. Suppose that $x_1, \ldots, x_n$ are chosen in X so that $\sum_{i=1}^n \|x_i\|^r = 1$ and

$$\int_0^1 \|\sum_{i=1}^{n} \varepsilon_i(t) x_i\|^r \, dt > \frac{1999}{2000}\, \varphi(n)^r.$$

We shall use an exhaustion argument inspired by Nikišin's paper [N2]. Let $(B_\alpha)_{\alpha\in I}$ be a maximal family of disjoint subsets of $\{1,\ldots,n\}$ such that

$$\int_0^1 \|\sum_{i\in B_\alpha} \varepsilon_i(t)x_i\|^r\,dt < \frac{1}{2000}\sum_{i\in B_\alpha}\|x_i\|^r.$$

If B denotes the union of these sets B_α, and m denotes the cardinality of I (notice that $m<n$ because $|B_\alpha|>1$), we get

$$\begin{aligned}\int_0^1 \|\sum_{i\in B}\varepsilon_i(t)x_i\|^r\,dt &= \int \|\sum_{\alpha\in I}\varepsilon_\alpha(s)(\sum_{i\in B_\alpha}\varepsilon_i(t)x_i)\|^r\,ds\,dt\\ &\le \varphi(m)^r\sum_{\alpha\in I}\int_0^1\|\sum_{i\in B_\alpha}\varepsilon_i(t)x_i\|^r\,dt \le \frac{\varphi(m)^r}{2000}\sum_{\alpha\in I}\sum_{i\in B_\alpha}\|x_i\|^r\\ &\le \frac{\varphi(n)^r}{2000}\sum_{i=1}^n\|x_i\|^r = \frac{\varphi(n)^r}{2000}.\end{aligned}$$

Let A denote the complement of B and for every $j\ge 0$ let

$$A_j = \{k\in A : 2^{-j-1} < \|x_k\| \le 2^{-j}\}.$$

Observe that $\|x_k\|\le 1$ for every k because $\sum_{i=1}^r\|x_i\|^r = 1$, so that the sets $(A_j)_{j\ge0}$ cover the set A. Let $N=\max_j|A_j|$ denote the maximal cardinality of the sets $(A_j)_{j\ge0}$. Then

$$\Big(\int_0^1\|\sum_{i\in A}\varepsilon_i(t)x_i\|^r\,dt\Big)^{1/r} \le \sum_{j=0}^{+\infty}\Big(\int_0^1\|\sum_{i\in A_j}\varepsilon_i(t)x_i\|^r\,dt\Big)^{1/r} \le N\sum_{j=0}^{+\infty}2^{-j} = 2\,N.$$

We obtain

$$\Big(\frac{1999}{2000}\Big)^{1/r}\varphi(n) < \Big(\int_0^1\|\sum_{i=1}^n\varepsilon_i(t)x_i\|^r\,dt\Big)^{1/r} \le \frac{\varphi(n)}{2000^{1/r}} + 2\,N$$

which shows that N is big when $\varphi(n)$ is big. Let j_0 be such that $|A_{j_0}|=N$. By maximality of B we obtain for every non-empty subset C of A_{j_0}

$$\int_0^1\|\sum_{i\in C}\varepsilon_i(t)x_i\|^r\,dt \ge \frac{1}{2000}\sum_{i\in C}\|x_i\|^r \ge \frac{2^{-(j_0+1)r}}{2000}\,|C|.$$

Replacing the vectors $(x_i)_{i\in A_{j_0}}$ by normalized vectors (y_i), we obtain a normalized sequence $(y_1, y_2, \ldots, y_m)$, as long as we wish, such that

$$(\int_0^1 \|\sum_{i\in C} \varepsilon_i(t)\, y_i\|^r\, dt)^{1/r} \geq \kappa\, |C|^{1/r}$$

for every subset C of $\{1, \ldots, m\}$ (with $\kappa = \frac{1}{2}\, 2000^{-1/r}$). This inequality remains true if we replace the $L_r(X)$ norm by the norm of $L_1(X)$ and κ by some $\kappa' > 0$ (use Kahane's inequalities). For every $n \geq 1$, we may thus find an unconditional normalized sequence in $L_1(X)$, of the form $(\varepsilon_j y_j^{(n)})_{j=1}^n$, with the above property, and since $r < 2$ it implies that for some $c = c(r, \kappa') > 0$, we have $\| \sum_{j=1}^n a_j \varepsilon_j y_j^{(n)} \|_{L_1(X)} \geq c\, (\sum_{j=1}^n |a_j|^2)^{1/2}$ for all scalars. From Corollary 5 follows that for every integer m, we can, when n is large enough, get blocks $z_1, \ldots, z_m \in L_1(X)$ of $(\varepsilon_j y_j^{(n)})_{j=1}^n$ that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^m for some $p \leq r$, and the ℓ_2-norm of the coefficients in each block z_i is bounded by $c(r, \kappa')^{-1}$. By Kahane's inequalities again, all $L_s(X)$ norms are equivalent on the span of $(\varepsilon_j y_j^{(n)})_{j=1}^n$, hence the sequence $(z_1, \ldots, z_m)$ considered in $L_2(X)$ is uniformly equivalent to the unit vector basis of ℓ_p^m; since $L_2(X)$ has type s whenever X has type s, we have for every $s < p_X$ and for some constants K, K_s

$$K^{-1} m^{1/p} \leq \int_0^1 \| \sum_{i=1}^m \varepsilon_i(t) z_i \|_{L_2(X)}\, dt \leq K_s\, m^{1/s}$$

for every $m \geq 1$. This yields that $s \leq p$, for every $s < p_X$, hence $p_X \leq p$. Starting with a long enough sequence $(z_i)_{i=1}^m$ and blocking again in the ℓ_p-sense we may find three blocks $b_1, b_2, b_3 \in L_1(X)$ of some sequence $(\varepsilon_j y_j)$, supported on three disjoint intervals J_1, J_2, J_3 and such that, letting $\omega = (\varepsilon_j)$ and

$$b_i(\omega) = \sum_{j\in J_i} a_j\, \varepsilon_j\, y_j, \quad i = 1, 2, 3,$$

then the three functions b_1, b_2, b_3 are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^3 in the norm of $L_1(X)$, and the coefficients satisfy $\sum_{j\in J_i} |a_j|^2 < \tau^2/12$ for $i = 1, 2, 3$ and a small $\tau > 0$ (use $p < 2$). For every fixed triple (c_1, c_2, c_3) of scalars, this implies by Azuma's inequality (see [MS, 7.4]) a strong concentration for the set of ω such that

$$\|c_1\, b_1(\omega) + c_2\, b_2(\omega) + c_3\, b_3(\omega)\| \sim (|c_1|^p + |c_2|^p + |c_3|^p)^{1/p},$$

and allows us to select a choice of $\omega = (\varepsilon_j)$ that works for all $(c_i)_{i=1}^3$, by a standard δ-net argument on the unit sphere of ℓ_p^3; this shows that for most of the choices ω of signs, the vectors $b_1(\omega), b_2(\omega), b_3(\omega)$ in X form a nice copy of the unit vector basis of ℓ_p^3. We may choose r close enough to p_X so that $\ell_{p_X}^3$ is almost isometric to ℓ_p^3, since $p_X \le p \le r$, and this ends the proof in this case $k = 3$. The reader will easily pass from 3 to an arbitrary integer k.

Let us be more specific about the use of Azuma's inequality. On the space $\Omega = \{-1, 1\}^n$, we define for every c in the unit sphere of ℓ_p^3 the function

$$f_c(\omega) = f_c(\varepsilon_1, \ldots, \varepsilon_n) = \|\sum_{i=1}^{3} c_i \,(\sum_{j \in J_i} a_j \varepsilon_j \, y_j)\|$$

and we consider the finite martingale

$$M_j(\varepsilon_1, \ldots, \varepsilon_j) = \int f_c(\varepsilon_1, \ldots, \varepsilon_n) \, d\varepsilon_{j+1} \ldots d\varepsilon_n,$$

for $j = 0, \ldots, n$. The differences $(d_j)_{j=0}^n$ of this martingale satisfy $|d_{j+1}| = |M_{j+1} - M_j| \le |a_{j+1}|$, hence $S^2 = \sum_{j=1}^n |d_j|^2 \le \tau^2/4$. Azuma's inequality gives

$$P(\{\omega \in \Omega : |f_c(\omega) - M_0| \ge t\}) \le 2\exp(-t^2/(4S^2)) \le 2\exp(-t^2/\tau^2)$$

for every $t > 0$, where $M_0 = M_0(c)$ is equal to the norm of $c_1 b_1 + c_2 b_2 + c_3 b_3$ in $L_1(X)$, which is $(1+\varepsilon)$-equivalent to the ℓ_p^3-norm of c, namely 1. If Λ is a δ-net on the unit sphere of ℓ_p^3 and if τ was so small that $2\,|\Lambda| < \exp(\delta^2/\tau^2)$, we may find ω such that

$$\Big| \|c_1 b_1(\omega) + c_2 b_2(\omega) + c_3 b_3(\omega)\| - M_0(c) \Big| \le \delta$$

for every $c \in \Lambda$, from which the result follows.

Let us pass to the cotype case. If $q_X = 2$ we may use Dvoretzky's theorem. Assume $q_X > 2$. We choose s such that $q_X > s > 2$. Let $\psi(n)$ denote the smallest constant such that

$$\sum_{i=1}^n \|x_i\|^s \le \psi(n)^s \int_0^1 \|\sum_{i=1}^n \varepsilon_i(t) x_i\|^s \, dt$$

for every family $x_1, \ldots, x_n$ of n vectors in X. It is clear that ψ is non-decreasing, and tends to $+\infty$ since X does not have cotype s. Suppose that $x_1, \ldots, x_n$ are chosen in X so that $\sum_{i=1}^n \|x_i\|^s = 1$ and

$$1 > \frac{1999}{2000} \psi(n)^s \int_0^1 \|\sum_{i=1}^n \varepsilon_i(t) x_i\|^s \, dt.$$

Let $(B_\alpha)_{\alpha\in I}$ be a maximal family of mutually disjoint non-empty subsets of $\{1,\dots,n\}$ such that

$$\sum_{i\in B_\alpha} \|x_i\|^s \le \frac{1}{2000} \int_0^1 \|\sum_{i\in B_\alpha} \varepsilon_i(t)x_i\|^s \, dt.$$

If B denotes the union of these sets B_α, and $m < n$ denotes the cardinality of the index set I, we get

$$\begin{aligned}
\sum_{i\in B} \|x_i\|^s &= \sum_{\alpha\in I} \sum_{i\in B_\alpha} \|x_i\|^s \le \sum_{\alpha\in I} \frac{1}{2000} \int_0^1 \|\sum_{i\in B_\alpha} \varepsilon_i(t)x_i\|^s \, dt \\
&\le \frac{\psi(m)^s}{2000} \int \|\sum_{\alpha\in I} \varepsilon_\alpha(s)(\sum_{i\in B_\alpha} \varepsilon_i(t)x_i)\|^s \, ds \, dt \\
&\le \frac{\psi(n)^s}{2000} \int_0^1 \|\sum_{i=1}^n \varepsilon_i(t)x_i\|^s \, dt.
\end{aligned}$$

Let A denote the complement of B and for every $j \ge 0$ let

$$A_j = \{k \in A : 2^{-j-1} < \|x_k\| \le 2^{-j}\}.$$

We have

$$\sum_{i\in A} \|x_i\|^s > \frac{1998}{2000} \psi(n)^s \int_0^1 \|\sum_{i=1}^n \varepsilon_i(t)x_i\|^s \, dt.$$

Let j_1 be the smallest $j \ge 0$ such that A_j is not empty. If $N = |A_{j_0}|$ is the largest cardinality of the sets A_j, then

$$N \sum_{j=j_1}^{+\infty} 2^{-js} \ge \frac{1998}{2000} \psi(n)^s \, 2^{-j_1 s - s}$$

which shows that N is large when $\psi(n)$ is large. By maximality of B,

$$\sum_{i\in C} \|x_i\|^s > \frac{1}{2000} \int_0^1 \|\sum_{i\in C} \varepsilon_i(t)x_i\|^s \, dt$$

for every non-empty subset $C \subset A_{j_0}$. We change the $(x_i)_{i\in A_{j_0}}$ to normalized vectors, and go to a limit space X', finitely representable in X and containing a normalized sequence $(y_i)_{i\ge 0}$ such that for some κ_0,

$$\kappa_0 \, |C|^{1/s} \ge (\int_0^1 \|\sum_{i\in C} \varepsilon_i(t)y_i\|^s \, dt)^{1/s}$$

for every finite subset C. But this sequence $(y_i)_{i\geq 0}$ can't have any Cauchy subsequence, or else the above property would be true with $y_i \sim y$, in other words, true in a one dimensional setting; in this case, Khintchine's inequality tells us that the integral is larger than $|C|^{1/2} > \kappa_0\,|C|^{1/s}$, which is impossible when $|C|$ is large. By Brunel-Sucheston, we can pass to differences $(y_m - y_n)$ in order to get a suppression-unconditional sequence invariant under spreading (with a poor normalization). We have

$$(\int_0^1 \|\sum_{i\in C} \varepsilon_i(t)(y_{2i+1} - y_{2i})\|^s \, dt)^{1/s} \leq 2\kappa_0\,|C|^{1/s}$$

for every finite subset C, but we may now get rid of the signs $(\varepsilon_i(t))$ since the sequence of differences is 2-unconditional. We obtain therefore in X' a normalized suppression-unconditional sequence (x_i) such that

$$\|\sum_{i\in C} x_i\| \leq \kappa'\,|C|^{1/s}$$

for every finite subset C. We end by applying the second case of Corollary 5.

6 K-convexity and Pisier's theorem

When X is a type p space, then the dual X^* has cotype q for the conjugate exponent $(1/p+1/q=1)$; this is very easy: if $(x_i^*)_{i=1}^n$ is given in X^*, we can find $(x_i)_{i=1}^n \subset X$ such that $\sum_{i=1}^n x_i^*(x_i) > (\sum_{i=1}^n \|x_i^*\|^q)^{1/q} - \varepsilon$ and $\sum_{i=1}^n \|x_i\|^p = 1$; then, by orthogonality of the functions (ε_i)

$$\begin{aligned}(\sum_{i=1}^n \|x_i^*\|^q)^{1/q} - \varepsilon < \sum_{i=1}^n x_i^*(x_i) &= \int_0^1 (\sum_{i=1}^n \varepsilon_i(t)x_i^*)(\sum_{j=1}^n \varepsilon_j(t)x_j)\, dt \\ &\leq \Big(\int_0^1 \|\sum_{j=1}^n \varepsilon_j(t)x_j\|^2\, dt\Big)^{1/2}\Big(\int_0^1 \|\sum_{i=1}^n \varepsilon_i(t)x_i^*\|^2\, dt\Big)^{1/2} \\ &\leq T_p(X)\,\Big(\int_0^1 \|\sum_{i=1}^n \varepsilon_i(t)x_i^*\|^2\, dt\Big)^{1/2},\end{aligned}$$

therefore $C_q(X^*) \leq T_p(X)$. Obviously the converse is false since ℓ_1, dual of c_0, has cotype 2, while c_0 has no non-trivial type. However, this does not happen when X^* has cotype q *and* non-trivial type: then, X has type p. This fact was conjectured in [MP2] (although the authors had little evidence that supported this conjecture at the time), and proved by Pisier six years later

[P6]. Using local reflexivity, and since type and cotype are super-properties, the preceding claim is equivalent to saying that *when a Banach space Y has non-trivial type and cotype q, then the dual Y^* has type p with* $1/p+1/q=1$. This will follow from the easy Lemma 7 below and from the main result of this section, Theorem 12.

Let us consider the group $G=\{1,-1\}$ and let μ denote the invariant probability measure on G^m, that gives measure 2^{-m} to every atom. On G^m, let ε_i, $i=1,\ldots,m$ denote the ith coordinate function, $\varepsilon_i(g_1,\ldots,g_m)=g_i$. If $\alpha\subset\{1,\ldots,m\}$ let $w_\alpha=\prod_{i\in\alpha}\varepsilon_i$; using the standard convention, we get the constant function 1 on G^m when $\alpha=\emptyset$. Let $|\alpha|$ denote the cardinality of the set α. This family of functions (w_α) is *the Walsh system*; it is the family of characters of the abelian group G^m. Every function f from G^m to a Banach space X can be expressed as

$$\forall\omega\in G^m,\quad f(\omega)=\sum_\alpha w_\alpha(\omega)x_\alpha,$$

for some family $(x_\alpha)\subset X$. Given a function $f=\sum_\alpha w_\alpha x_\alpha$, the part of the expansion corresponding to sets α with $|\alpha|=1$ is the Rademacher projection $R_X(f)=\sum_{|\alpha|=1}w_\alpha x_\alpha$ of the function f (we have $w_\alpha=\varepsilon_i$ when $\alpha=\{i\}$).

Lemma 7 *If the Rademacher projection R_X is bounded on $L_2(G^m,\mu,X)$ by some constant K, uniformly in $m\geq 1$, then the cotype q property of X dualizes to the type p property of X^*, and*

$$T_p(X^*)\leq K\,C_q(X)\qquad(1/p+1/q=1).$$

PROOF. Suppose that $f\in L_2(G^m,X)$; the Rademacher projection of f is of the form $R_Xf=\sum_{i=1}^m\varepsilon_ix_i$, where $x_i=\int\varepsilon_i(\omega)f(\omega)\,d\mu(\omega)$. The cotype q property and the boundedness of R_X imply that the map $f\to(x_i)_{i=1}^m$ is bounded from $L_2(G^m,X)$ to $\ell_q^m(X)$. It follows that the adjoint map is bounded from $\ell_p^m(X^*)$ to $L_2(G^m,X^*)$, and this adjoint map is the map that sends $(x_i^*)_{i=1}^m$ to $\sum_{i=1}^m\varepsilon_ix_i^*$. We get therefore

$$\Big(\int_0^1\|\sum_{i=1}^m\varepsilon_i(t)x_i^*\|^2\,dt\Big)^{1/2}\leq\|R_X\|\,C_q(X)\,\Big(\sum_{i=1}^m\|x_i^*\|^p\Big)^{1/p}.$$

Definition 8 *We say that X is K-*convex *if there exists a constant K such that for every $m\geq 1$ and every function $f\in L_2(G^m,X)$, expressed as $f=\sum_\alpha w_\alpha x_\alpha$, we have*

$$\|R_Xf\|_{L_2}=\|\sum_{|\alpha|=1}w_\alpha x_\alpha\|_{L_2}\leq K\,\|f\|_{L_2},$$

which means that $\|R_X\|_{\mathcal{L}(L_2(X))} \le K$. The smallest possible constant K is the K-convexity constant of X. It is equal to the supremum of $\|R_X\|$, when the number m of Rademacher functions tends to infinity.

When this supremum is finite, we may directly define R_X on the infinite product $G^{\mathbb{N}}$, and the K-convexity constant is the norm of R_X on $L_2(G^{\mathbb{N}}, X)$. It is clear that K-convexity is a super-property, and it passes to the dual X^* with the same constant. It follows from Kahane's inequality that the projection is also bounded in $L_q(X)$ for $2 \le q < +\infty$, and using duality we see that R_X is then bounded in $L_p(X)$ for all p such that $1 < p < +\infty$.

It follows from Lemma 7 that the K-convexity constant of $L_1(G^m)$ tends to infinity with m (because L_1 has cotype 2 while its dual L_∞ does not have type 2), but it is instructive to give a concrete estimate. Let $\widehat{g} = (\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_m) \in G^m$ be fixed; the function $f_{\widehat{g}}$, equal to 2^m at $\widehat{g}$ and to 0 elsewhere, has norm one in $L_1(G^m)$, and its expansion is

$$f_{\widehat{g}} = \sum_\alpha w_\alpha(\widehat{g})\, w_\alpha.$$

It follows that the function f from G^m to $L_1(G^m)$ defined by

$$f(g, g') = \sum_\alpha w_\alpha(g)\, w_\alpha(g')$$

has norm one in $L_2(G^m, L_1(G^m))$, but its Rademacher projection $(Rf)(g) = \sum_{j=1}^m \varepsilon_j(g)\, \varepsilon_j$ has norm $\ge \sqrt{m/2}$ by Khintchine's inequality. Observe that we get $K_X \ge c\sqrt{\log \dim X}$, with $X = L_1(G^m)$. It is known that for any Banach lattice X we have $K_X \le C\sqrt{1 + \log \dim X}$ (see [P4]), and the preceding simple example shows that this result is precise for lattices. For general Banach spaces, see Theorem 13 below.

Let us describe the semi-group approach: on the multiplicative group $G = \{1, -1\}$ we consider for $-1 \le c \le 1$ the probability measure $\mu_c^{(1)}$ defined by

$$\mu_c^{(1)} = \frac{1+c}{2}\, \delta_1 + \frac{1-c}{2}\, \delta_{-1},$$

where δ_g denotes the unit mass at $g \in G$. Using $\delta_{-1} * \varepsilon_1 = -\varepsilon_1$ we get that $\mu_c^{(1)} * \varepsilon_1 = c\, \varepsilon_1$. Also $\mu_b^{(1)} * \mu_c^{(1)} = \mu_{bc}^{(1)}$. On G^m we consider the m-fold tensor product $\mu_c = \mu_c^{(m)}$ of m copies of $\mu_c^{(1)}$. We see that $\mu_c * w_\alpha = c^{|\alpha|} w_\alpha$ and $\mu_b * \mu_c = \mu_{bc}$. Given a function $f \in L_2(G^m, X)$, expressed as $\sum_\alpha w_\alpha x_\alpha$, we see that $\mu_c * f = \sum_\alpha c^{|\alpha|} w_\alpha x_\alpha$. Since μ_c is a probability measure, convolution with μ_c is a norm 1 operator on $L_2(G^m, X)$, for every real or complex Banach space X and every $c \in [-1, 1]$.

In order to pass to the classical semi-group setting, we shall perform the following change of variable. For $t \geq 0$, let $\nu_t = \mu_{\mathrm{e}^{-t}}$. We get that $\nu_t * \nu_s = \nu_{s+t}$. Given a function $f = \sum_\alpha w_\alpha x_\alpha \in L_2(G^m, X)$ we set

$$(W) \qquad T_t f = \nu_t * f = \sum_\alpha \mathrm{e}^{-|\alpha| t}\, w_\alpha x_\alpha$$

and we call $(T_t)_{t \geq 0}$ the *Walsh semi-group*. We noticed that each T_t is a contraction on $L_2(G^m, X)$. Let P_i, $i = 1, \ldots, m$ denote the projection on $L_2(G^m, X)$ defined by

$$(P_i f)(\varepsilon_1, \ldots, \varepsilon_m) = \int f(\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon, \varepsilon_{i+1}, \ldots, \varepsilon_m)\, d\varepsilon.$$

It is clear that P_i is a norm one projection, and $P_i P_j = P_j P_i$ for all $i, j = 1, \ldots, m$. Let $Q_i = I - P_i$. We have $P_i \varepsilon_i = 0$, $P_i \varepsilon_j = \varepsilon_j$ for $j \neq i$. For every $\alpha \subset \{1, \ldots, m\}$ let $P^\alpha = \prod_{i \in \alpha} P_i$. We see by checking the action on every w_α that

$$T_t = \prod_{i=1}^m (P_i + \mathrm{e}^{-t}\, Q_i) = \prod_{i=1}^m ((1 - \mathrm{e}^{-t}) P_i + \mathrm{e}^{-t}\, I).$$

It follows, by expanding the last product, that T_t is a convex combination of commuting norm one projections of the form P^α.

For the next lemma it is natural to quantify the type-p property of a Banach space X in a way close to the definition of B-convex Banach space. We let $N(X)$ denote the smallest integer $n \geq 1$ such that

$$\int_0^1 \| \sum_{i=1}^n \varepsilon_i(t) x_i \| \, dt \leq n/16$$

for every family $x_1, \ldots, x_n$ of vectors in X such that $\|x_i\| \leq 1$ for each i. Of course, if X has type $p > 1$, then we have $N(X) \leq (16\, T_p(X))^q$, where $q < +\infty$ is the number conjugate to $p > 1$. We let $N(X) = +\infty$ when X is not B-convex.

Lemma 9 *Suppose that X is a B-convex Banach space, and assume that M is a convex combination of contractive commuting projections on X. Then*

$$\|M^{n+1} - M^n\| \leq 1/4$$

when $n \geq \max(N(X), 256)$.

PROOF. Let

$$M = \sum_\alpha c_\alpha \, P_\alpha$$

where $c_\alpha \geq 0$, $\sum_\alpha c_\alpha = 1$, and where the (P_α)s are commuting projections on X, such that $\|P_\alpha\| \leq 1$; we get in particular that $\|M\| \leq 1$. Let ξ be a random variable on some probability space Ω, with values in the space of operators on X and with $P(\xi = P_\alpha) = c_\alpha$ for every α. Then $E\xi = M$, and if ξ_1, ξ_2 are two independent copies of ξ, then $E\xi_1\xi_2 = M^2$. Let $\xi_1, \dots, \xi_n$ be independent copies of ξ, with $n \geq \max(N(X), 256)$. Suppose that $x \in X$, $\|x\| = 1$ and let us consider, for a fixed choice ε of $\varepsilon_i = \pm 1$, the random variable Z_ε on Ω defined by $Z_\varepsilon(\omega) = \|\sum_{i=1}^n \varepsilon_i \xi_i(\omega) x\|$.

Let $B = \{i : \varepsilon_i = 1\}$ and $C = \{i : \varepsilon_i = -1\}$, $k = |B|$ and $\ell = |C|$. Assume that $k \leq \ell$. Then, letting ξ^B denote the (random) operator equal to $\prod_{j \in B} \xi_j$, and noting that $\|\xi^B(\omega)\| \leq 1$ for every ω,

$$\|\sum_{i=1}^n \varepsilon_i \xi_i x\| \geq \|\xi^B(\sum_{i=1}^n \varepsilon_i \xi_i x)\| = \|\sum_{i \in B} \xi^B x - \sum_{i \in C} \xi^B \xi_i x\|.$$

Taking expectation on Ω,

$$\begin{aligned} E\,\|\sum_{i=1}^n \varepsilon_i \xi_i x\| &\geq \|\sum_{i \in B} E\,\xi^B x - \sum_{i \in C} E\,\xi^B \xi_i x\| \\ &= \|kM^k x - \ell M^{k+1} x\| \geq \ell\,\|M^k x - M^{k+1} x\| - |\ell - k| \\ &\geq \frac{n}{2}\,\|M^n x - M^{n+1} x\| - |\ell - k|. \end{aligned}$$

(if $\ell \leq k$, we replace ξ^B by ξ^C). On the other hand we get, taking the expectation E' over all signs, noting that $|\ell - k| = |\sum_{i=1}^n \varepsilon_i|$ and since $n \geq N(X)$

$$\frac{n}{2}\,\|M^n x - M^{n+1} x\| - \sqrt{n} \leq EE'\,\|\sum_{i=1}^n \varepsilon_i \xi_i x\| \leq n/16.$$

so that, using $n \geq 256$

$$\|M^n x - M^{n+1} x\| \leq 1/8 + 2n^{-1/2} \leq 1/4.$$

Remark 10 *Suppose that X is a type-p Banach space, with $p > 1$ and type-p constant T_p. Assume that M is a convex combination of contractive commuting projections on X. Then*

$$\|x + M(x)\| \geq (4T_p)^{-q} \|x\|$$

for every $x \in X$ (q is the exponent conjugate to p). It follows that $I + M$ is invertible and that $\|(I+M)^{-1}\| \le (4T_p)^q$.

It is well known to experts that the uniform invertibility of $I + T_t$ is precisely what is needed in Kato's theorem for proving that a semi-group $(T_t)_{t \ge 0}$ is holomorphic. The proof of the remark is a slight modification of the proof of the preceding lemma. Suppose that $\|x\| = 1$ and $\|x + Mx\| < \varepsilon$. It follows that $\|M^k x + M^{k+1} x\| < \varepsilon$ for every $k \ge 0$ since $\|M\| \le 1$, and $\|M^k x\| \ge \|x\| - k\varepsilon = 1 - k\varepsilon$ by the triangle inequality. Taking expectations as before,

$$E \,\|\sum_{i=1}^n \varepsilon_i \xi_i x\| \ge \|nM^k x\| - \ell\varepsilon \ge n(1 - k\varepsilon) - \ell\varepsilon \ge n - (n+1)^2 \varepsilon/2.$$

Taking the expectation E' over all signs

$$n - n^2 \varepsilon \le EE' \,\|\sum_{i=1}^n \varepsilon_i \xi_i x\| \le T_p \, n^{1/p}.$$

If we choose n such that $1/4 < n\varepsilon \le 1/2$, then $n/2 \le T_p \, n^{1/p}$, thus $4n \le (4T_p)^q$ since $q \ge 2$ and $\varepsilon \ge (4T_p)^{-q}$.

If we want to see why things can go wrong when X contains ℓ_1^ns, we may modify the example showing that the K-convexity constant of $L_1(G^m)$ is large. We shall only sketch the idea. Let us consider the function f_0 from $[0,1]^m$ to the space of measures on $[0,1]^m$ such that $f_0(x)$ is the Dirac mass at x for every $x \in [0,1]^m$ (this function is not Bochner measurable; a genuine example should correct this fact). If P_i is defined for every $g \in L_2([0,1]^m, X)$ by

$$(P_i g)(x_1, \ldots, x_m) = \int_0^1 g(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m) \, dy$$

for $i = 1, \ldots, m$, then the (P_i) are commuting norm one projections. For every $\alpha \subset \{1, \ldots, m\}$, the vector value $(P^\alpha f_0)(x)$ is the Lebesgue measure on some $|\alpha|$-dimensional unit cube. When x varies, these probability measures are pairwise disjoint, and this is the source of all the problems. The corresponding semi-group $S_t = \prod_{i=1}^m ((1 - \mathrm{e}^{-t}) P_i + \mathrm{e}^{-t} I)$ behaves very badly. In particular, the inequality $\|I - S_t\| \ge 2\,(1 - \mathrm{e}^{-mt})$ shows that the hypothesis for Kato's theorem is not satisfied uniformly in m in this example, where $X = M$ (the space of measures).

We are ready to begin the proof that B-convexity implies K-convexity, using the Walsh semi-group $(T_t)_{t \ge 0}$ defined by relation (W). Recall that each operator T_t is a convex combination of commuting norm one projections on $L_2(X)$. If X is B-convex, then $L_2(X)$ is also B-convex; it follows from Lemma 9 that

$$\|T_{nt} - T_{(n+1)t}\| \le 1/4$$

for every $t > 0$, when $n \geq \max(N(L_2(X)), 256)$. For the rest of the paper we assume that X is a complex Banach space.

We strongly recommend reading [MS, chapter 14] (and appendix IV about Kato's theorem for semi-groups). For the lazy reader who does not want to hear about general semi-groups, we shall sketch a proof of Kato's theorem in the simplified setting which is needed here. We consider m Rademacher functions $\varepsilon_1, \ldots, \varepsilon_m$ and the corresponding 2^m Walsh functions (w_α) that are defined by the formula $w_\alpha = \prod_{i \in \alpha} \varepsilon_i$, where α ranges over the 2^m subsets of $\{1, \ldots, m\}$; next we fix 2^m vectors (y_α) in X, and we let E be the 2^m-dimensional complex subspace of $L_2(G^m, X)$ generated by the algebraic basis $(w_\alpha y_\alpha)$. Our operators $(T_t)_{t \geq 0}$ act diagonally on this basis of E, since $T_t(w_\alpha y_\alpha) = \mathrm{e}^{-t|\alpha|}\, w_\alpha y_\alpha$ for every α. Defining the complex extension T_z of T_t on E is straightforward: we simply say that T_z acts on E by $T_z(w_\alpha y_\alpha) = \mathrm{e}^{-z|\alpha|}\, w_\alpha y_\alpha$ for every α, but of course the problem is to find bounds for the norm of T_z, independent of the particular subspace $E \subset L_2(X)$. We see that $T_t = \mathrm{e}^{-tA}$, where A is represented in the basis $(w_\alpha\, y_\alpha)$ by a diagonal matrix with entries in $\{0, 1, \ldots, m\}$, namely $A(w_\alpha\, y_\alpha) = |\alpha|\, w_\alpha\, y_\alpha$. The Rademacher projection corresponds to the matrix B obtained by replacing in A all diagonal entries $\neq 1$ by zero entries.

For the proof of Theorem 12 below, we shall keep m and the 2^m-dimensional subspace $E \subset L_2(G^m, X)$ fixed. Our aim is to find a bound for the norm of the matrix B, acting on this subspace E by $B(w_\alpha y_\alpha) = w_\alpha y_\alpha$ if $|\alpha| = 1$ and $B(w_\alpha y_\alpha) = 0$ otherwise; we are looking for a bound K independent of m and of the particular subspace E. From the nature of the problem it is clear that such a bound K will be a bound for the norm of the Rademacher projection R_X acting on $L_2(G^{\mathbb{N}}, X)$, that is to say a bound for the K-convexity constant of X.

The control of the complex extension of the semi-group begins with a standard exercise in functions of one complex variable. Consider $\eta = v + i\pi$, $v > 0$, and the two conjugate rays $R = \mathbb{R}_+ \eta$ and $\overline{R} = \mathbb{R}_+ \overline{\eta}$, symmetric with respect to the real axis, contained in the half plane $\Re z > 0$. Let $\xi = \pi + iu$, with $|u| < v$, and consider the holomorphic function $f(z) = \mathrm{e}^{-\xi z}$. Then for every real $a \geq 0$, we have

$$\mathrm{e}^{-\xi a} = \frac{1}{2i\pi} \int_\Gamma \mathrm{e}^{-\xi z} (z - a)^{-1}\, dz$$

where Γ is essentially the path given by these two rays, except for a little detour to avoid $z = 0$ (this is needed in the case $a = 0$; see the figure in [MS, appendix IV]). We have

$$|\, \mathrm{e}^{-\xi z}\, | \leq \mathrm{e}^{-\pi(1 - |u|/v)\, \Re z}$$

for every z in the convex cone limited by R and $\overline{R}$, therefore the integral is convergent since $|u| < v$. It is a standard exercise to show that the integral over Γ is indeed equal to $\mathrm{e}^{-\xi a}$ (approximate $\int_\Gamma$ by the integral over a bounded closed contour that uses part of the two rays and part of a large circle centered at 0, and apply Cauchy's formula).

In our (finite-dimensional) vector situation, the generator A of the semi-group is expressed by a diagonal matrix with non-negative real diagonal, so that the next equation is by no means harder to prove than the scalar case,

$$\mathrm{e}^{-\xi A} = \frac{1}{2i\pi} \int_\Gamma \mathrm{e}^{-\xi z} (zI - A)^{-1} \, dz.$$

This can be done not only for $\xi = \pi + iu$, $|u| < v$, but as well for any $\xi = \alpha + i\beta$ with $\alpha > 0$ and $\pi \, |\beta| < v \, \alpha$, in other words for every ξ in a sector of angle θ around the positive real axis, where $\pi \tan \theta = v$. The above formula, extended to these values of ξ, defines the complex extension of the semi-group. It is clear (and standard) that we can bound the complex extension of the semi-group, acting on the fixed finite-dimensional subspace E, if we have a suitable bound for the norm of the resolvent $(zI - A)^{-1}$ on the two rays R and $\overline{R}$ (again, this norm is understood as norm of an operator from E to E).

Lemma 11 *Let $E \subset L_2(G^m, X)$ be as above. Assume that X is a B-convex Banach space, let $n \ge \max(N(L_2(X)), 256)$ and let v be such that $0 < v \le 1/n$. For every complex number z belonging to the ray $R = \mathbb{R}_+(v + i\pi)$ or to the conjugate ray $\overline{R}$, we have*

$$\|(zI - A)^{-1}\| \le 36 \, \pi \, n / |\Im z|.$$

PROOF. Let $\lambda = v \pm i\pi$, and suppose that $\varepsilon > 0$ is chosen in such a way that $\|(A - \lambda I)^{-1}\| > 1/\varepsilon$; we can find a norm one vector $x \in E$ such that $\|Ax - \lambda x\| < \varepsilon$. The function $\varphi(t) = T_t(x) = \mathrm{e}^{-tA} \, x$ satisfies the differential equation $\varphi'(t) = -A \, \varphi(t) = -T_t(Ax)$. Since T_t is a contraction semi-group, we deduce that for every $t > 0$

$$\|\varphi'(t) + \lambda \varphi(t)\| = \|T_t(Ax - \lambda x)\| \le \varepsilon.$$

If we write this as $\varphi'(t) + \lambda \varphi(t) = g(t)$ with $\|g(t)\| \le \varepsilon$ and solve the differential equation, we get

$$\varphi(t) = \mathrm{e}^{-\lambda t} (x + \int_0^t \mathrm{e}^{\lambda s} \, g(s) \, ds),$$

which implies that $\|\varphi(t) - \mathrm{e}^{-\lambda t} x\| \le \varepsilon t$. Let $n \ge \max(N(L_2(X)), 256)$. By Lemma 9, we know that for every $s > 0$, we have $\|T_{(n+1)s} - T_{ns}\| \le 1/4$, since T_s is a convex combination of commuting norm one projections. We shall use this fact with $s = 1$; when $s = 1$, we get $\|\varphi(n+1) - \varphi(n)\| \le 1/4$ and $\mathrm{e}^{-\lambda s} = \mathrm{e}^{-\lambda} = -\mathrm{e}^{-v}$ since $\mathrm{e}^{i\pi} = -1$. We have

$$1/3 < \mathrm{e}^{-1} < \mathrm{e}^{-vn}(1 + \mathrm{e}^{-v}) = \|\,\mathrm{e}^{-\lambda(n+1)} x - \mathrm{e}^{-\lambda n} x\|.$$

By the triangle inequality,

$$\begin{aligned} 1/3 &< \|\,\mathrm{e}^{-\lambda(n+1)} x - \mathrm{e}^{-\lambda n} x\| \\ &\le \|\varphi(n+1) - \mathrm{e}^{-\lambda(n+1)} x\| + \|\varphi(n) - \mathrm{e}^{-\lambda n} x\| + 1/4, \end{aligned}$$

hence

$$1/12 \le \|\varphi(n+1) - \mathrm{e}^{-\lambda(n+1)} x\| + \|\varphi(n) - \mathrm{e}^{-\lambda n} x\| \le (2n+1)\varepsilon \le 3n\,\varepsilon.$$

It follows that $\|(A - \lambda I)^{-1}\| \le 36\,n$. We may apply the same proof to the generator $A_s = s^{-1}A$, for every $s > 0$; obviously, this A_s also generates a semi-group consisting of convex combinations of commuting contractive projections, and this implies as above that

$$\|(\lambda I - A_s)^{-1}\| \le 36\,n \quad \text{or} \quad \|(s\lambda I - A)^{-1}\| \le 36\,n/s$$

hence

$$\|(zI - A)^{-1}\| \le 36\,\pi\,n/|\Im z|$$

when z belongs to the rays $R = \mathbb{R}_+(v + i\pi)$ or $\overline{R}$.

Theorem 12 *Let X be a B-convex Banach space. Then the Rademacher projection R_X is bounded on $L_2(G^{\mathbb{N}}, X)$, and*

$$\|R_X\| \le \mathrm{e}^{\kappa \max(N(L_2(X)),\, 256)}$$

for some universal constant κ.

PROOF. Let us consider again the 2^m-dimensional subspace E of $L_2(G^m, X)$. We may deduce from Lemma 11 that the semi-group (T_t) acting on E has a nicely bounded complex extension to the sector mentioned before, but since we are mainly interested in the Rademacher projection we are going to take a shortcut. Let $N = \max(N(L_2(X)), 256)$ and $v = 1/N$; consider the path Γ consisting of the ray $R = \mathbb{R}_+ (v + i\pi)$ and its conjugate $\overline{R}$. Provided that the integral makes sense,

$$\frac{1}{2i\pi} \int\limits_{\Gamma} \varphi(z)\, (zI - A)^{-1}\, dz$$

represents, when φ is holomorphic on $\mathbb{C}$, the diagonal matrix where each diagonal entry k of A is replaced by $\varphi(k)$. Recall that the diagonal entries of A are integers. In order to get the matrix B of the Rademacher projection, we naturally introduce $\varphi_1(z) = \sin(\pi z)/(1 - z)$ that kills all entries $\neq 1$ in A. We need to multiply this φ_1 by a suitable exponential that guarantees that the integral converges and that the Cauchy formula applies to the unbounded contour Γ. Let us consider the matrix

$$C = \frac{1}{2i\pi} \int\limits_{\Gamma} \frac{\sin(\pi z)}{\pi(1 - z)}\ \mathrm{e}^{-\pi^2 z/v}\, (zI - A)^{-1}\, dz.$$

The sin function eliminates the problem at 0. Also, one can check that the integral is absolutely convergent. It is easy to see that this matrix C is a multiple of the Rademacher projection B, namely $C = \mathrm{e}^{-\pi^2/v}\, B$ and using the bound from Lemma 11 we can show that

$$\|C\| \leq \kappa_1 N,$$

where κ_1 is an universal constant. It follows that $\|B\| \leq \kappa_1 N\, \mathrm{e}^{\pi^2 N} \leq \mathrm{e}^{\kappa\, N}$.

Let us detail the preceding computation. We have $N = \max(N(L_2(X)), 256)$ and $v = 1/N$, thus $0 < v < 1/2$. Let $z_0 = v \pm i\pi$. If $z = s\, z_0$, $s > 0$, then $|\sin(\pi z)| \leq \mathrm{e}^{\pi^2 s}$ and $|\cos(\pi z)| \leq \mathrm{e}^{\pi^2 s}$, therefore $|\sin(\pi z)\, \mathrm{e}^{-\pi^2 z/v}\,| \leq 1$. We also have $|1 - z| \geq \pi s$ and $|1 - z| \geq 1 - sv \geq 1/2$ when $0 < s < 1$. Next, we use

$$|\sin(\pi z)| \leq \pi |z| \max_{0<u<1} |\cos(u\pi z)| \leq \pi s\, |z_0|\, \mathrm{e}^{\pi^2 s}$$

for $0 < s < 1$, so that

$$\begin{aligned} \|C\| &\leq \frac{1}{\pi} \int\limits_0^1 \frac{2\pi s |z_0|}{\pi}\, \frac{36N}{s}\, |z_0|\, ds + \frac{1}{\pi} \int\limits_1^\infty \frac{1}{\pi^2 s}\, \frac{36\, N}{s}\, |z_0|\, ds \\ &\leq \frac{72\, N\, |z_0|^2}{\pi} + \frac{36\, N\, |z_0|}{\pi^3} \leq 500\, N. \end{aligned}$$

We finish with another result of Pisier, that has been very useful for local theory. We did not try to optimize the constant, but to give an argument as simple as possible (essentially identical to Pisier's proof).

Theorem 13 *Let d_X denote the Banach-Mazur distance from X to the Hilbert space of the same dimension. Then the norm of the Rademacher projection in $L_2(X)$ is bounded by $4 \ln d_X$ when $d_X \geq \mathrm{e}$ (and by d_X in any case).*

PROOF. The proof is comparatively simple. Let $f = \sum_\alpha w_\alpha x_\alpha$ be a function from G^m to X. Assume that $\|f\|_{L_2(X)} = 1$. For $z \in \mathbb{C}$, let

$$P(z) = \sum_\alpha z^{|\alpha|} w_\alpha x_\alpha.$$

This defines a holomorphic function (a polynomial) from $\mathbb{C}$ to $L_2(X)$. The Rademacher projection $R_X(f)$ of f is the derivative $P'(0)$ of P at $z = 0$. When $X = H$ is a Hilbert space, we get by orthogonality, for every z in the closed unit disc D in $\mathbb{C}$

$$\|P(z)\|^2_{L_2(H)} = \sum_\alpha |z|^{2|\alpha|} \|x_\alpha\|^2 \leq \sum_\alpha \|x_\alpha\|^2 = \|f\|^2_{L_2(H)},$$

thus $\|P(z)\| \leq 1$ for every $z \in D$ in this case. If the distance from X to some Hilbert space is $\leq d$, then clearly $\|P(z)\| \leq d$, therefore $\|P(z)\| \leq d_X$ for every $z \in D$. On the other hand, we have seen that $\|P(x)\|_{L_2(X)} \leq \|f\|_{L_2(X)} = 1$ when x is real and $|x| \leq 1$, because $P(x) = \mu_x * f$ in that case, with μ_x a probability measure. For convenience, we transfer the problem to the closed strip $S = \{z : |\Im z| \leq 1\}$: the mapping $\varphi(z) = \tanh(\pi z/4)$ maps S to the closed unit disc, and sends the line $\Im z = 0$ to the segment $-1 \leq x \leq 1$. The $L_2(X)$-valued function $q(z) = P(\varphi(z))$ is bounded on S, holomorphic on the open strip S_0, bounded by d_X on S and by 1 on the line $\Im z = 0$. The result follows then from $q'(0) = \pi P'(0)/4$ and from the following lemma. When $d_X \geq \mathrm{e}$ we get $|P'(0)| = 4\,\pi^{-1}\,|g'(0)| \leq 4\,\pi^{-1}\,\mathrm{e}\,\ln d_X < 4\,\ln d_X$.

Lemma 14 *Let g be a bounded and continuous function, defined on the closed strip $S = \{z : |\Im z| \leq 1\}$, holomorphic on the open strip S_0, with values in a Banach space Y; assume that g is bounded by $C \geq \mathrm{e}$ on S and bounded by 1 on the line $\Im z = 0$. Then $|g'(0)| \leq \mathrm{e}\ln C$.*

PROOF. Let $g_1(z) = (g(z) - g(-z))/2$; then g_1 obeys the same bounds as does g, and $g_1'(0) = g'(0)$; furthermore, $g_1(0) = 0$. Let $0 < \theta \leq 1$. Since g_1 is bounded, $|g_1| \leq 1$ on the line $\Im z = 0$ and $|g_1| \leq C$ on the line $\Im z = 1$, the three lines Lemma implies that $|g_1|$ is bounded by $1^{1-\theta}C^\theta = C^\theta$ on the line

$\Im z = \theta$; the same argument applies to the line $\Im z = -\theta$; now $k(z) = g_1(z)/z$ (with $k(0) = g_1'(0)$) is bounded on the strip S, and bounded by $\theta^{-1}C^\theta$ on the two lines $\Im z = \pm\theta$, therefore $|g'(0)| = |g_1'(0)| = |k(0)| \le \theta^{-1}C^\theta$. The optimal choice of θ in $(0,1]$ is $\theta = (\ln C)^{-1}$, which is licit because $\ln C \ge 1$.

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