EXERCISES

OPERATOR THEORY MODULE, NOVEMBER 2025

2. C*-algebras and Hilbert space operators

Exercise 2.1. Show the following for a C*-algebra A.

(i) If $u \in A$ is a unitary then $\sigma(u) \subseteq \mathbb{T}$. Does the converse hold?
(ii) If $a \in A$ is normal then $r(a) = \|a\|$.

Exercise 2.2. Show the following for a C*-algebra A.

(i) If (L, R) is a double centraliser on a C*-algebra A then $\|L\| = \|R\|$.
(ii) $M(A)$ is a C*-algebra (with the operations and norm we defined in class).

Exercise 2.3. Let A be a C*-subalgebra of $\mathcal{B}(H)$ for a Hilbert space H. Show that the unitisation of A is canonically isomorphic to the C*-algebra generated by

$$\{\begin{bmatrix} a + \lambda I_H & 0 \\ 0 & \lambda I_{\mathbb{C}} \end{bmatrix} \mid a \in A, \lambda \in \mathbb{C}\}$$

inside $\mathcal{B}(H \oplus \mathbb{C})$.

Show that if $I_H \notin A$ then the unitisation of A is canonically isomorphic to the C*-algebra

$$A + \mathbb{C}I_H = \{a + \lambda I_H \mid a \in A, \lambda \in \mathbb{C}\}$$

inside $\mathcal{B}(H)$.

Exercise 2.4. Let $\phi\colon A \to B$ be a $*$-morphism between C*-algebras A and B. Show that if ϕ is one-to-one then it is an isometric map.

Exercise 2.5. Let A be a unital C*-algebra. Let $a \in A_{sa}$ and $0 < \varepsilon < 1/4$. Suppose $\sigma(a) \subseteq [0, \varepsilon] \cup [1 - \varepsilon, 1]$. Show that there is a projection $p \in A$ with $\|p - a\| \leq \varepsilon$.

Exercise 2.6. Let A be a unital C*-algebra, and let $a, b \in A$ with b normal and $f \in C(\sigma(b))$. Show that if a commutes with b then a commutes with $f(b)$.

Exercise 2.7. Let A be a unital C*-algebra and let $a \in A_{sa}$. Suppose that $\|a\| \leq 1$. Show that $1 - a^2 \geq 0$.

Exercise 2.8. Let A be a unital C*-algebra, $a \in \mathrm{Inv}(A)$ and $p \in A$ a projection. Show that if a commutes with p then a is invertible in the corner pAp. If $a \in A$ is invertible in pAp, does it follow that $a \in \mathrm{Inv}(A)$?

Exercise 2.9. Compute the operator norm of the following matrix (seen as a bounded linear operator on $\mathbb{C}^2$):

$$x = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Exercise 2.10. Let A be a unital C*-algebra. Let $p_1, \dots, p_n \in A$ be commuting projections and let B be the C*-algebra generated by $\{p_1, \dots, p_n\}$. Show that for every $b \in B$ there exists a set $F \subseteq \{1, \dots, n\}$ such that

$$\|b\| = \|Q_F b\| \quad \text{for} \quad Q_F := \prod_{i \in F} p_i \prod_{j \notin F} (1 - p_j).$$