EXERCISES

OPERATOR THEORY MODULE, OCTOBER 2025

1. Elementary spectral theory for Banach algebras

Exercise 1.1. Let $(A_\lambda)_{\lambda\in\Lambda}$ denote a family of Banach algebras. The *direct sum*

$$A := \oplus_\lambda A_\lambda$$

is the set of all $(a_\lambda) \in \prod_\lambda A_\lambda$ such that

$$\sup_{\lambda\in\Lambda} \|a_\lambda\| < \infty.$$

Show that this space is a Banach algebra under the pointwise operations:

$$(a_\lambda) + (b_\lambda) = (a_\lambda + b_\lambda),\ \mu(a_\lambda) = (\mu a_\lambda),\ (a_\lambda)(b_\lambda) = (a_\lambda b_\lambda),$$

and the norm given by

$$\|(a_\lambda)\| := \sup_{\lambda\in\Lambda} \|a_\lambda\|.$$

Show that A is unital if and only if every A_λ is unital. Show that A is abelian if and only if every A_λ is abelian.

The *restricted sum*

$$B := \oplus_\lambda^{c_0} A_\lambda$$

is the set of all elements $(a_\lambda) \in A$ such that for each $\varepsilon > 0$ there exists a finite subset F of Λ for which $\|a_\lambda\| < \varepsilon$ if $\lambda \in \Lambda \setminus F$. Show that B is a closed ideal in A.

[Hint: Mimic the case of ℓ^∞ and c_0.]

Exercise 1.2. Let A be a Banach algebra and Ω a non-empty set. Denote by $\ell^\infty(\Omega, A)$ the set of all bounded maps $f\colon \Omega \to A$. Show that $\ell^\infty(\Omega, A)$ is a Banach algebra with the pointwise-defined operations and norm given by

$$\|f\| := \sup_{\omega\in\Omega} \|f(\omega)\|.$$

If Ω is a compact Hausdorff space, show that the set

$$C(\Omega, A) := \{f\colon \Omega \to A \mid\ f \text{ continuous}\}$$

is a closed subalgebra of $\ell^\infty(\Omega, A)$.

[Hint: For the first part compare with the construction in the first exercise. For the second part mimic the case of $C(\Omega)$ in $\ell^\infty(\Omega)$.]

Exercise 1.3. Give an example of a unital non-abelian Banach algebra A in which 0 and A are the only closed ideals.

Exercise 1.4. Let $\mathbb{C}[z]$ denote the single-variable $\mathbb{C}$-valued polynomials equipped with the pointwise operations and norm

$$\|p\| := \sup_{|z|=1} |p(z)|.$$

Is this a Banach algebra?

Exercise 1.5. Let A be a unital Banach algebra. Show the following:

(i) If a is invertible in A show that $\sigma(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$.
(ii) If $p(a) = 0$ for $a \in A$ and $p \in \mathbb{C}[z]$ such that $p(a) = 0$ show that $p(\lambda) = 0$ for all $\lambda \in \sigma(a)$.
(iii) For any $a \in A$ and $n \in \mathbb{N}$ we have $r(a^n) = r(a)^n$.

(iv) If A is abelian show that the Gelfand representation is isometric if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.

Exercise 1.6. Let $H = \ell^2(\mathbb{N})$ be the Hilbert space with inner product

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

Let $S \in \mathcal{B}(\ell^2(\mathbb{N})$ be defined by

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

Find S^* and show that for every $\lambda \in \mathbb{D}$ there exists an $x \in \ell^2(\mathbb{N})$ such that $S^* x = \lambda x$. Deduce that

$$\sigma(S) = \sigma(S^*) = \overline{\mathbb{D}}.$$

[Hint: Compare $\sigma(S)$ with $\sigma(S^*)$.]

Exercise 1.7. Let $H = \ell^2(\mathbb{Z})$ be the Hilbert space with inner product

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}.$$

Let $U \in \mathcal{B}(\ell^2(\mathbb{Z})$ be defined by $U e_n = e_{n+1}$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the orthonormal basis.

Find U^* and show that U (and U^*) is a unitary. Show that

$$\sigma(U) = \sigma_a(U) = \mathbb{T},$$

where $\sigma_a(U)$ is the approximate point spectrum.

[Hint: Compare $\sigma(U)$ with $\sigma(U^*)$. For every $\lambda \in \mathbb{T}$ investigate what happens to

$$\lim_n (\lambda I_H - U) x_n = ? \quad \text{for} \quad x_n = \frac{1}{2n+1} \sum_{k=-n}^{n} \lambda^k e_k$$

where $\{e_n\}_n$ is the orthonormal basis of $\ell^(\mathbb{Z})$. Note that $\lambda^{-1} = \overline{\lambda}$ for $\lambda \in \mathbb{T}$.]