# Tychonoff's Theorem

- Thm: Let $\{X_i : i \in I\}$ a family of top. spaces. Then $X_i$ is compact $\forall i \in I \iff \prod_{i \in I} X_i$ is compact (wrt the cartesian topology).

- The cartesian topology is defined as follows: the basis consists of finite tubes:
$$X_i \times \dots \times X_i \times U_{i_1} \times \dots \times U_{i_n} \times X_i \times \dots$$

- Pf: 1) Suppose that $X$ is not compact. Then the set
$$S = \{C : C \text{ is an open cover of } X \text{ without finite subcover}\}$$
is non-empty.
If $\{C_\lambda\}_{\lambda \in \Lambda}$ is a chain then $C = \bigcup_\lambda C_\lambda$ is an upper bound in $S$. Indeed it is an upper bound. If it had finite sub-cover say $\{U_1 \dots U_n\}$ then $\exists \lambda_0$ s.t. $\{U_1 \dots U_n\} \subset C_{\lambda_0}$. But then $C_{\lambda_0}$ has finite sub-cover. Contradiction.
So let $C_0$ be ~~the~~ a maximal in $S$.

2) Rem: If $V \subset X$ open and $V \notin C_0$ then there are $U_1 \dots U_n \in C_0$ s.t. $V \cup U_1 \cup \dots \cup U_n = X$.
Indeed $\{V\} \cup C_0$ is an open cover of $X$.
- if $\{V\} \cup C_0 \in S$ then $\{V\} \cup C_0 = C_0 \Rightarrow V \in C_0$ contra.
- if $\{V\} \cup C_0 \notin S \Rightarrow$ it has finite sub-cover.

3) For every $i$ define the projections
$$p_i : X = \prod X_i \longrightarrow X_i.$$
then $p_i$ are continuous.

4) <u>Rem</u>: for every $i \in I$ we have:

$$\bigcup \{ W \subset X_i : W \text{ open with } p_i^{-1}(W) \in G_0 \} \neq X_i .$$

<u>pf</u>: if they are equal and since $X_i$ compact then we have a finite subcover. So

$$X_i = W_1 \cup \dots \cup W_n \quad \text{with} \quad p_i^{-1}(W_j) \in G_0$$

But then $X = p_i^{-1}(X_i) = p_i^{-1}(W_1) \cup \dots \cup p_i^{-1}(W_n)$ so $G_0$ has a finite sub-cover. contra.

5) (Axiom of choice) for every $i \in I$ we choose $y_i$ s.t.

$$y_i \in X_i \setminus \bigcup \{ W \subset X_i : W \text{ open}, \ p_i^{-1}(W) \in G_0 \}.$$

let $y = (y_i)_{i \in I} \in X$.

6) Since $G_0$ covers $X$, there is a $U \in G_0$ (open) s.t. $y \in U$. Let a basic set of the form

$$V = p_{i_1}^{-1}(W_1) \cap \dots \cap p_{i_k}^{-1}(W_k)$$

for $W_1, \dots W_k$ open in $X_{i_1} \dots X_{i_k}$, s.t. $x \in V \subseteq U$.

7) By the choice of $y$ and since $y \in p_{i_1}^{-1}(W_1), \dots, p_{i_k}^{-1}(W_k)$ we get that $p_{i_1}^{-1}(W_1), \dots, p_{i_k}^{-1}(W_k) \notin G_0$.
So by <u>Rem (3)</u> there are $U_{1,m}, \dots, U_{k_m,m}$ s.t.

$$X = (U_{1,1} \cup \dots \cup U_{k_1,1}) \cup p_{i_1}^{-1}(W_1) =$$
$$= \dots = (U_{1,m} \cup \dots \cup U_{k_m,1}) \cup p_{i_m}^{-1}(W_m) .$$

8) But then $\{U_{1,1}, \dots, U_{k_1,1}, U_{2,1}, \dots U_{k_2,1}, \dots U_{k,1}, \dots U_{k,k_k}\} \cup \{U\}$ is in $G_0$ and a finite sub-cover. contra.

9) Indeed, if $x \in X \setminus \bigcup_{m=1}^{k} \bigcup_{l=1}^{k_m} U_{l,m}$ then by (7) $x \in p_{i_m}^{-1}(W_m)$ for all $m = 1 \dots k$.
So $x \in \bigcap p_{i_m}^{-1}(W_m) = V \subseteq U$. ∎

# Consequences of Tychonoff's Thm

1. Thm (Banach-Alaoglu) Let $(X, \|\cdot\|)$. Then the unit ball of $(X^*, w^*)$ is a compact Hausdorff space and the mapping:
$$T : (X, \|\cdot\|) \longrightarrow (X^{**}, \|\cdot\|_\infty) : x \longmapsto T(x), \quad T(x)x^* = x^*(x).$$
is an isometric embedding.

Pf: 1) Let $K = \prod_{x \in X} \underbrace{\cancel{[-\|x\|, \|x\|]}}_{B(0,\|x\|) \subseteq \mathbb{C}}$. Then by Tychonoff it is compact
It is Hausdorff as product of Hausdorff spaces.
2) Let $\varphi : (X^*, w^*)_1 \longrightarrow K : x^* \longmapsto (x^*(x))_{x \in X}$.
It is well defined since $|x^*(x)| \leq \|x^*\| \cdot \|x\| = \|x\|$.

3) $\varphi$ is a homeomorphism:
$$x_i^* \xrightarrow{w^*} x^* \iff x_i^*(x) \xrightarrow{\|\cdot\|_X} x^*(x) \ \forall x \in X \iff \varphi(x_i) \xrightarrow{\text{cort.}} \varphi(x).$$

4) $\varphi((X^*, w^*)_1)$ is a closed subspace of $K$:
Let $(x_i^*) \in (X^*, w^*)_1$ s.t. $\varphi(x_i^*) \xrightarrow{\text{cort}} g = (g_x)_{x \in X} \in K$.
Then $x_i^*(x) \xrightarrow{\|\cdot\|} g_x$ for all $x \in X$.
So $g_{x+y} = \lim x_i^*(x+y) = \lim x_i^*(x) + x_i^*(y) \overset{\exists}{=}$
$= \lim x_i^*(x) + \lim x_i^*(y) = \cancel{g_x} + g_y$
Similarly $g_{\lambda x} = \lambda g_x$.
Also for $\|x\| \leq 1$ then ~~$|g_x| = \lim |x_i^*(x)| \leq$~~ $|g_x| \leq 1$
since $g_x \in B(0, \|x\|) = B(0, 1)$.
So $g_x \in X^*$.

5) finally $\|T(x)\| = \sup\{ |T(x)x^*| : \|x^*\| \leq 1\} =$
$= \sup\{|x^*(x)| : \|x^*\| \leq 1\} = \|x\|$

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2. Thm (Goldstine). Let $(X, \|\cdot\|)$ embedded in $X^{**}$. Then $B_1(X, \|\cdot\|)$ is $w^*$-dense in $B_1(X^{**}, \|\cdot\|_\infty)$

Pf: Since $(X, \|\cdot\|) \subseteq (X^{**}, \|\cdot\|_\infty) \Rightarrow B_1(X, \|\cdot\|) \subseteq B_1(X^{**}, \|\cdot\|_\infty)$

$\Rightarrow \overline{B_1(X, \|\cdot\|)}^{w^*} \subseteq B_1(X^{**}, \|\cdot\|_\infty)$.

(Recall $\|x\| = \sup\{|x^*(x)| : \|x^*\| \le 1\}$).

If $\exists\ x^{**} \in B_1(X^{**}) \setminus \overline{B_1(X)}^{w^*}$ then by Hahn-Banach, there is $x^*$ with $\|x^*\| = 1$ and $\lambda > 0$ s.t.

$|x^{**}(x^*)| > \lambda$ and $|\hat{x}(x^*)| \le \lambda$ for $x \in B_1(X)$

$\hat{x}(x^*) = x^*(x)$

Then:

$1 = \|x^*\| = \sup\{|x^*(x)| : \|x\| \in B_1(X)\} \le \lambda$
$< |x^{**}(x^*)| \le \|x^{**}\| \cdot \|x^*\| = 1 \cdot 1 = 1$, contra.

$\blacksquare$