The Stone-Weierstrass Theorem

Lemma: for $m \geq 3$ there is a sequence $(p_n)_n$ of real polynomials such that $0 \leq p_n(t) \leq 1$ $\forall n \geq 1$, $t \in [0,1]$ and:
- $p_n \xrightarrow{n} 0$ uniformly on $[0, \frac{1}{2m}]$,
- $p_n \xrightarrow{n} 0$ uniformly on $[\frac{2}{m}, 1]$.

Proof: for $N \in \mathbb{N}$ we have:

$$1-(1-x)^N \leq Nx \quad \text{and} \quad (1-x)^N \leq \frac{1}{Nx} \quad \forall x \in (0,1)$$

(apply calculus...)

Let $p_n(t) = (1-t^n)^{m^n}$. Then $0 \leq p_n(t) \leq 1$ $\forall n \geq 1, t \in [0,1]$ and we get:

$$\sup_{t \in [0, \frac{1}{2m}]} |p_n(t)| \leq \sup_{t \in [0, \frac{1}{2m}]} (mt)^n \leq \left(\frac{1}{2}\right)^n \xrightarrow{n} 0,$$

$$\sup_{t \in [\frac{2}{m}, 1]} |1-p_n(t)| \leq \sup_{t \in [\frac{2}{m}, 1]} \frac{1}{(mt)^n} \leq \left(\frac{1}{2}\right)^n \xrightarrow{n} 0.$$

$\boxtimes$

Recall that, if X is a compact Hausdorff space then $C(X, \mathbb{R}) = \{ f: X \to \mathbb{R} \mid f \text{ continuous} \}$ equipped with the pointwise operations and the $\|\cdot\|_\infty$-norm becomes a real Banach algebra

Definition. If $A \subseteq C(X, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then we say that A separates the points of X, if for every $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $f \in A$ s.t. $f(x_1) \neq f(x_2)$.

Theorem ($\mathbb{R}$-valued Stone-Weierstrass theorem).
Let X be a compact and Hausdorff space and let $A \subseteq C(X, \mathbb{R})$. If:

(1) A is unital, i.e., $\mathbb{1} \in A$.

(2) A is subalgebra of $C(X, \mathbb{R})$.

(3) A separates the points of X.

then A is uniformly dense in $C(X, \mathbb{R})$.

Proof. For $f, g \in C(X, \mathbb{R})$ we write $f \leq g$ for when $f(x) \leq g(x)$ $\forall x \in X$. For $K \subseteq X$ we write $f \leq g$ on K when $f(x) \leq g(x)$ $\forall x \in K$.

Claim 1. If $x, y \in X$ with $x \neq y$ then $\exists f \in A$ with $f \geq 0$ and $f(x) = 0$, $f(y) > 0$.

proof. For $x \neq y$ there is $g \in A$ (by (3)) s.t. $g(x) \neq g(y)$. Let $f := (g - g(x)\mathbb{1})^2$. By (1)+(2) we get $f \in A$ and moreover: $f \geq 0$,

$f(x) = (g(x) - g(x))^2 = 0$ $\quad f(y) = (g(y) - g(x))^2 > 0$. $\square$

Claim 2. If $L \subseteq X$ is compact and $x \notin L$ then there is $f \in A$ s.t. $\begin{cases} 0 \le f \le 1 \\ f(x) = 0, \; f > 0 \text{ on } L. \end{cases}$

proof. For each $y \in L$ $\exists f_y \in A$ s.t. $f_y \ge 0$, $f_y(x) = 0$ and $f_y(y) > 0$. Since f_y is continuous $\exists U_y$ open n'hood of y s.t. $f_y > 0$ in U_y. The family $\{U_y\}_{y \in L}$ covers L, and by compactness take $\{U_{y_i}\}_{i=1}^{k}$ a finite subcover. Set:

$$0 \ne f_0 := f_{y_1} + \dots + f_{y_k} \geq 0$$

and take $f := \frac{1}{\|f_0\|_a} f_0$. Then f does the trick. $\square$

Claim 3. If $K, L \subseteq X$ are compact and $K \cap L = \phi$ then there is $f \in A$ with $0 \le f \le 1$ such that $f \le 1/4$ on K and $f \geq 3/4$ on L.

proof. Let $x \in K$. By Claim (2) there is $g_x \in A$ such that $0 \le g_x \le 1$, $g_x(x) = 0$ and $g_x > 0$ on L. Since L is compact, $g_x(L)$ is compact and thus a subset of some $[s,t]$ with $s > 0$. Hence there is an $m \in \mathbb{N}$ (depending on x) such that $g_x \geq 2/m$ on L.

For $x \in K$ let $U_x = \{z \in X : g_x(z) < \frac{1}{2m}\}$.

Then $x \in U_x$ and $\{U_x\}_{x \in K}$ is an open cover of K.

Take $\{U_{x_i}\}_{i=1}^{k}$ a finite subcover.
By Lemma for each $i=1,\dots,k$ there is a polynomial p_i for which, by setting $f_i = p_i(g_{x_i})$ we get: $0 \le f_i \le 1$, $f_i \le 1/4$ on U_{x_i} and $f_i \geq \left(\frac{3}{4}\right)^{1/k}$ on L. Since p_i is a polynomial and $g_{x_i} \in A$ we have $f_i \in A$.
Then the function $f := f_1 \cdots f_k$ is in A and does the trick. $\square$

<u>Claim 4</u>. If $f \in C(X, \mathbb{R})$ then there is $g \in A$ such that $\|f - g\|_\infty \le \frac{3}{4}\|f\|_\infty$.

$-1 \le f(x) \le 1 \quad \forall x \in X$

<u>proof</u>. Wlog we may assume $\|f\|_\infty = 1$ (replace f with $f/\|f\|_\infty$). Let $h \in A$ be the function from Claim 3 for the sets:

$$K := \{x \in X \mid f(x) \le -1/4\},\quad L := \{x \in X \mid f(x) \geq 1/4\}$$

and set $g := h - \frac{1}{2}$. Since $\frac{1}{4} \le g \le \frac{1}{2}$ on L then $|f - g| \le \frac{3}{4}$ on L. Likewise we have $|f - g| \le \frac{3}{4}$ on K. Since $-\frac{1}{2} \le g \le \frac{1}{2}$ and $-\frac{1}{4} \le f \le \frac{1}{4}$ on $(K \cup L)^c$ we have $|f - g| \le \frac{3}{4}$ on $(K \cup L)^c$. Thus $\|f - g\|_\infty \le \frac{3}{4}$. $\square$

Claim 5. A is uniformly dense in $C(X,\mathbb{R})$.

proof. let $f \in C(X,\mathbb{R})$. By Claim (4) there is $g_1 \in A$ s.t. $\|f - g_1\|_\infty \le \frac{3}{4}\|f\|_\infty$. Applying the same claim for $f - g_1$ we obtain $g_2 \in A$ such that

$$\|(f - g_1) - g_2\|_\infty \le \frac{3}{4}\|f - g_1\|_\infty \le \left(\frac{3}{4}\right)^2 \|f\|_\infty.$$

Continuing inductively we have a sequence (g_n) in A s.t.

$$\left\|f - \sum_{i=1}^{k} g_i\right\| \le \left(\frac{3}{4}\right)^k \|f\|_\infty.$$

taking the limit $k \to \infty$ we have that

$$f = \lim_k \sum_{i=1}^{k} g_i \quad \text{and} \quad \sum_{i=1}^{k} g_i \in A. \quad \boxtimes$$

Theorem ($\mathbb{C}$-valued Stone-Weierstrass Theorem).
Let X be a compact Hausdorff space and $A \subseteq C(X,\mathbb{C})$

If:
(1) A is unital, i.e. $\mathbb{1} \in A$.
(2) A is subalgebra of $C(X,\mathbb{C})$.
(3) $f^* \in A$ $\forall f \in A$
(4) A separates the points of X

then A is uniformly dense in $C(X,\mathbb{C})$.

proof. For $f \in C(X) \equiv C(X,\mathbb{C})$ we write:

$$Re(f) = \frac{f + f^*}{2} \quad \text{and} \quad Im(f) = \frac{f - f^*}{2i}$$

Then $f = Re(f) + i\,Im(f)$ and $Re(f), Im(f) \in C(X, \mathbb{R})$.

If $f \in A$ then $Re(f), Im(f) \in A$.

Set: $A_1 = \{ Re(f) \mid f \in A \}$.

Claim: A_1 separates the points of X.

proof. For $x \neq y$ in X take $f \in A$ s.t. $f(x) \neq f(y)$.
Then $Re(f)(x) \neq Re(f)(y)$ or $Im(f)(x) \neq Im(f)(y)$.
If the first holds then $Re(f) \in A_1$ separates the points in X.
If the second holds then $g := if = Im(f) - i\,Re(f)$ is in A and $Im(f) = Re(g)$. Then $Re(g) \in A_1$ and separates the points in X. □

Moreover A_1 is unital subalgebra of $C(X, \mathbb{R})$.
Hence by $\mathbb{R}$-valued S-W Theorem A_1 is dense in $C(X, \mathbb{R})$.

For $f \in C(X, \mathbb{R})$ take $g_1, g_2 \in A_1$ s.t.

$$\| Re(f) - g_1 \|_\infty < \varepsilon/2 \text{ and } \| Im(f) - g_2 \|_\infty < \varepsilon/2$$

Then $\| f - (g_1 + i g_2) \|_\infty < \varepsilon$ and $g_1 + i g_2 \in A$.

$\blacksquare$

Disclaimer: This presentation is from the notes of R. Levene (University College Dublin) available on line.