(Positive) functionals on $M_n(\mathbb{C})$

Recall that $M_n(\mathbb{C})$ can be seen as $B(\mathbb{C}^n)$ with the induced operator C^*-norm. Recall the definition of the trace map:

$$Tr : M_n(\mathbb{C}) \to \mathbb{C}, \quad Tr(x) = \sum_{i=1}^{n} x_{ii}.$$

It is easy to show that $Tr(xy) = Tr(yx)$. Therefore if U is a unitary in $M_n(\mathbb{C})$ then

$$Tr(U^* x U) = Tr(x U U^*) = Tr(x).$$

It follows that the trace is independent of the choice of the basis. Moreover if x is positive then $x = U^* D U$ for a unitary U and $D = diag\{\lambda_1, \dots, \lambda_n\}$ with λ_i being the positive eigenvalues of x. In this case:

$$Tr(x) = Tr(D) = \sum_{i=1}^{n} \lambda_i \geq 0$$

Note that if $Tr(x) \geq 0$ then it does not follow that $x \geq 0$ for example take

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Claim 1. Let $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ be a linear functional. Then φ is bounded.

proof. This essentially follows because $M_n(\mathbb{C})$ is finitely generated by $\{E_{ij} : i,j=1,\dots n\}$. Indeed we have:

$$|\varphi(x)| = \left|\sum_{i,j=1}^{n} \varphi(x_{ij} E_{ij})\right| = \left|\sum_{i,j=1}^{n} x_{ij}\, \varphi(E_{ij})\right| \leq$$

$$\leq \sum_{i,j=1}^{n} |x_{ij}| \cdot |\varphi(E_{ij})| \leq \left(\max_{i,j=1\dots n} |\varphi(E_{ij})|\right) \cdot \sum_{i,j=1}^{n} |x_{ij}|$$

$$\leq \left(\max_{i,j=1\dots n} |\varphi(E_{ij})|\right) \cdot n^2 \cdot \|x\| \qquad \forall\, x \in M_n(\mathbb{C}).$$

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Claim 2. Let $\varphi \in M_n(\mathbb{C})^d$. Then there exists a unique $A \in M_n(\mathbb{C})$ such that:

$$\varphi(x) = \mathrm{Tr}(A^* x) \quad \text{and} \quad \|\varphi\| = \mathrm{Tr}(|A|).$$

proof. Define $A \in M_n(\mathbb{C})$ by $A_{ij} = \overline{\varphi(E_{ij})}$ and note that:

$$(A^* E_{ij})_{kk} = \sum_{l=1}^{n} (A^*)_{kl} (E_{ij})_{lk} = \delta_{jk} (A^*)_{ki}$$

Hence:

$$\mathrm{Tr}(A^* E_{ij}) = \sum_{k=1}^{n} \delta_{jk} (A^*)_{ki} = (A^*)_{ji} = \overline{A_{ij}} = \varphi(E_{ij})$$

Since $\{E_{ij} : i,j = 1\dots n\}$ form a basis for $M_n(\mathbb{C})$ we get:

- $\varphi(x) = \mathrm{Tr}(A^* x) \quad \forall x \in M_n(\mathbb{C})$.
- If $\varphi(x) = \mathrm{Tr}(B^* x)$ then:

$$B_{ij} = \overline{\varphi(E_{ij})} = A_{ij} \quad \forall i,j \Rightarrow A = B.$$

For the norm of φ we have: let $x \in M_n(\mathbb{C})$

Then: consider $A = W|A|$ polar decomposition

$$|\mathrm{Tr}(A^* x)| = |\mathrm{Tr}(|A| W^* x)| =$$

$$\overset{|A| = U^* D U}{=\!=} |\mathrm{Tr}(U^* D U W^* x)| =$$

$$= |\mathrm{Tr}(D(U W^* x U^*))| =$$

λ_i eigv of $|A|$

$$= \left|\sum_{i=1}^{n} \lambda_i \langle U W^* x U^* e_i, e_i \rangle\right|$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \, |\langle W^* x U^* e_i, U^* e_i \rangle|$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \, \|W^* x U^* e_i\| \cdot \|U^* e_i\| \leq$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \, \|W^* x U^*\| \cdot \|U^*\| \leq$$

$$\leq \sum_{i=1}^{n} |\lambda_i| \, \underbrace{\|W^*\|}_{1} \cdot \|x\| \cdot \|U\|^{2}$$

$$\leq \mathrm{Tr}(|A|) \cdot \|x\|$$

$$\Rightarrow \|\varphi(x)\| \leq \mathrm{Tr}(|A|) \cdot \|x\| \quad \forall x \in M_n(\mathbb{C})$$

$$\Rightarrow \|\varphi\| \leq \mathrm{Tr}(|A|).$$

Now for $A = W|A|$ we have that W is a partial isometry and so W^*W is a projection.

- If $\varphi = 0$ then $A = W = 0$ and $\|\varphi\| = 0 = Tr(|A|)$.
- If $\varphi \neq 0$ then $\|W\|^2 = \|W^*W\| = 1$ $(W^*W \neq 0)$.

Hence: $Tr(A^*W) = Tr(|A|)$ which shows that $\|\varphi\| \geq Tr(|A|)$. $\boxed{\checkmark}$

Claim 3. The association $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})^d$ which sends $A \longmapsto Tr(A^* \cdot)$ is a bijection. Hence the map: $\|\cdot\|_1 : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ given by: $\|A\|_1 := Tr(|A|)$ is a norm on $M_n(\mathbb{C})$.

proof. Follows from Claim 2 and its proof. $\boxed{\checkmark}$

Question. Can you provide a matrix algebra proof for claim 3? That is show directly that $\|\cdot\|_1$ is a norm on $M_n(\mathbb{C})$ without passing through $M_n(\mathbb{C})^d$?

Question Show that $\|\cdot\|_2 : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ given by $\|X\|_2 = Tr(X^*X)^{1/2}$ defines a norm induced by an inner product on $M_n(\mathbb{C})$.

Claim 4. Let $\varphi \in M_n(\mathbb{C})^d$ and A_φ the associated matrix such that $\varphi(x) = Tr(A_\varphi^{\circ} X)$ $\forall x \in M_n(\mathbb{C})$.

Show that:

(i) φ is selfadjoint iff A_φ is selfadjoint

(ii) φ is positive iff A_φ is positive.

proof. (i) Sps $\varphi(x^*) = \overline{\varphi(x)}$ $\forall x \in M_n(\mathbb{C})$.

Then: $(A_\varphi^*)_{ij} = \overline{A_{ji}} = \overline{\overline{\varphi(E_{ji})}} = \varphi(E_{ji}) =$

$$= \varphi(E_{ij}^*) = \overline{\varphi(E_{ij})} = (A_\varphi)_{ij} \quad \forall i,j.$$

$\Rightarrow A^* = A$.

Conversely, suppose $A_\varphi^* = A_\varphi$. Then:

$$\overline{\varphi(x)} = \overline{Tr(A_\varphi^* x)} \overset{!}{=} Tr((A_\varphi^* x)^*) =$$

$$= Tr(x^* A_\varphi) = Tr(A_\varphi x^*) =$$

$$= Tr(A_\varphi^* x^*) = \varphi(x^*) \quad \forall x \in M_n(\mathbb{C}).$$

(ii) Sps $\varphi(x^*x) \geq 0$ $\forall x \in M_n(\mathbb{C})$. Since φ is selfadjoint then A_φ is selfadjoint and we write $A_\varphi = U^* D U$ for $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ with the λ_i's being the eigenvalues of A_φ.

It suffices to show that $\lambda_i \geq 0$ $\forall i = 1, \dots, n$.

We have that:

$$U^* E_{ii} U = (E_{ji} U)^* (E_{ij} U) \geq 0.$$

Hence:

$$0 \leq \varphi(U^* E_{ii} U) = \mathrm{Tr}(A_\varphi^* U^* E_{ii} U) =$$

$$= \mathrm{Tr}(U A_\varphi U^* E_{ii}) = \mathrm{Tr}(D E_{ii}) = \lambda_i.$$

Conversely suppose that $A_\varphi \geq 0$ and so we can write $A_\varphi = B^* B$ for $B \in M_n(\mathbb{C})$. Then for every $X \in M_n(\mathbb{C})$ we have:

$$\varphi(X^* X) = \mathrm{Tr}(A_\varphi^* X^* X) = \mathrm{Tr}(B^* B X^* X)$$

$$= \mathrm{Tr}(B X^* X B^*) = \mathrm{Tr}((XB^*)^* (XB^*)) \geq 0$$

$\Rightarrow \varphi \geq 0.$ $\boxed{\therefore}$

<u>Check</u>: If φ is positive then we know from general theory that $\|\varphi\| = \varphi(I_n)$.

On the other hand $\mathrm{Tr}(|A_\varphi|) \underset{A_\varphi \geq 0}{=} \mathrm{Tr}(A_\varphi) = \varphi(I_n)$.

Hence we verify that $\mathrm{Tr}(|A_\varphi|) = \|\varphi\|$ (or that $\|\varphi\| = \varphi(I_n)$).