Spectrum of an operator

We will see some exercises concerning the spectrum of a $T \in B(X)$ for X: Banach space let us review here some elements of the theory.

Def. Let X be a Banach space and $T \in B(X)$.
We define:

(1) The point spectrum: $\sigma_p(T) := \{ \lambda \in \mathbb{C} \mid \ker(\lambda I - T) \neq \{0\} \}$

(2) The approximate point spectrum:

$$\sigma_a(T) := \{ \lambda \in \mathbb{C} \mid \forall \varepsilon > 0 \ \exists x_\varepsilon \in X : \|(\lambda I - T) x_\varepsilon\| < \varepsilon \|x_\varepsilon\| \}$$

(3) The compression spectrum:

$$\sigma_c(T) := \{ \lambda \in \mathbb{C} \mid \overline{(\lambda I - T) X} \neq X \}.$$

Remarks: (1) $\lambda \in \sigma_a(T) \iff \exists (x_n)_n \subseteq X$ with $\|x_n\|_X = 1 \ \forall n$ s.t. $(\lambda I - T) x_n \xrightarrow{n} 0$.

(2) It may not be that $\sigma_a(T)$ and $\sigma_c(T)$ are disjoint, for example when $X = \mathbb{C}^n$ then $\sigma_p(T) = \sigma_a(T) = \sigma_c(T)$.

(3) We have: $\sigma(T) = \sigma_a(T) \cup \sigma_c(T)$.

Prop. Let X be a Banach space and $T \in B(X)$. Then:

$$\partial \sigma(T) \subseteq \sigma_a(T).$$

Proof. let $(\lambda_n)_n \subseteq \mathbb{C} \setminus \sigma(T)$ such that $\lambda_n \xrightarrow{n} \lambda \in \partial\sigma(T)$. Since $\sigma(T)$ is closed we get that $\partial\sigma(T) \subseteq \sigma(T)$ and so $\lambda \in \sigma(T)$. Set $B := \lambda I - T$ and $B_n := \lambda_n I - T$. Then $B_n \in Inv(B(X))$ and $B_n \xrightarrow{n} B$.

Suppose that $\lambda \notin \sigma_a(T)$, and so there exists a $\delta > 0$ s.t. $\|Bx\| \geq 2\delta\|x\|$. For $\|x\|=1$ we have: $B_n x \xrightarrow{n} Bx$. Hence there exists $N > 0$ s.t.

$$\|B_n(x)\| \geq \|Bx\| - \|B_n(x) - Bx\| \geq \delta \quad \forall n \geq N$$

Thus $\|B_n(x)\| \geq \delta\|x\|$ for all $x \in X$, $n \geq N$.

let $y \in X$ and for $n \in \mathbb{N}$ take $x_n \in X$ s.t. $B_n x_n = y$.

Then:
$$\|Bx_n - y\| \leq \|B_n x_n - y\| + \|(B - B_n)x_n\|$$
$$= |\lambda - \lambda_n|\|x_n\| < |\lambda_n - \lambda| \frac{\|y\|}{\delta}$$

Hence $\overline{BX} = X$ showing that $\lambda \notin \sigma_c(T)$. But then $\lambda \notin \sigma_a(T) \cup \sigma_c(T) = \sigma(T)$ which is a contradiction.

Remark. Suppose $T \in B(H)$ for H Hilbert space. Then:
$$\sigma(T^*) = \{\bar{\lambda} \mid \lambda \in \sigma(T)\}$$
$$\sigma_p(T^*) = \{\bar{\lambda} \mid \lambda \in \sigma_c(T)\}$$
$$\sigma_c(T^*) = \{\bar{\lambda} \mid \lambda \in \sigma_p(T)\}.$$

In what follows, recall that if H is a Hilbert space then:

- $|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \quad \forall x, y \in H$
- $\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x, y \in H \quad T \in B(H)$
- $\|T\|^2 = \sup\limits_{\|x\|\leq 1} \|Tx\|^2 = \sup\limits_{\|x\|\leq 1} \langle T^*Tx, x\rangle$

$$\leq \|T^*T\| \leq \|T^*\| \cdot \|T\| \text{ and so}$$

$$\|T\| = \|T^*\| \text{ and } \|T\|^2 = \|T^*T\|.$$

<u>Prop</u>. Let H be a Hilbert space and $T \in B(H)$ is a normal operator ($T^*T = TT^*$). Then:

$$\sigma(T) = \sigma_a(T).$$

<u>Proof</u>. Let $\lambda \notin \sigma_a(u)$ and $x \perp (\lambda I - T)H$. Then $x \in \ker(\bar{\lambda} I - T^*)$ and $\|(\lambda I - T)x\| \geq \delta\|x\|$ for $\delta > 0$

Since T is normal we have:

$$\|(\bar{\lambda} I - T^*)x\|^2 = \langle(\bar{\lambda} I - T^*)^*(\bar{\lambda} I - T^*)x, x\rangle \overset{\text{normality}}{=\!=\!=}$$

$$= \langle(\bar{\lambda} I - T^*)(\lambda I - T)x, x\rangle = \|(\lambda I - T)x\|^2$$

$$\geq \delta\|x\|^2.$$

This shows that $\bar{\lambda} \notin \sigma_p(T^*) \implies \lambda \notin \sigma_c(T)$ and so $\lambda \notin \sigma(T)$. Hence $\sigma(T) = \sigma_a(T)$. $\blacksquare$

Prop. Let H be Hilbert space and let $T \in B(H)$ with $T = T^*$ (selfadjoint). We set:

$$m := \inf \{ \langle Tx, x \rangle \mid \|x\|_H = 1 \}$$

$$M := \sup \{ \langle Tx, x \rangle \mid \|x\|_H = 1 \}.$$

Then we have: (1) $\sigma(T) \subseteq [m, M]$,

(2) $m, M \in \sigma(T)$.

(3) $\|T\| = \max\{|m|, |M|\}$.

In particular we have that a selfadjoint $T \in B(H)$ is positive if and only if: $\langle Tx, x \rangle \geq 0$ for all $x \in H$.

Proof. • Let $\lambda > M$ and we will show that $\lambda \notin \sigma(T)$.

We have: $\langle Tx, x \rangle \leq M \|x\|^2 \ \forall x \in H \Rightarrow$

$\Rightarrow \langle \lambda x - Tx, x \rangle \geq (\lambda - M) \|x\|^2 \ \forall x \in H$

$\Rightarrow \lambda I - T$ is injective.

as $\sigma(T) = \sigma_a(T)$

This suffices as then $\lambda I - T$ is also onto and so $\lambda \notin \sigma(T)$.

• Likewise: if $\lambda < m$ then $\lambda \notin \sigma(T)$.

• Next we show that $M \in \sigma(T)$. We set

$$f : H \times H \longrightarrow \mathbb{C} \text{ with}$$

$$f(x, y) = \langle (MI - T)x, y \rangle.$$

Then f is a sesquilinear form and so it satisfies the Cauchy-Schwartz inequality:

$$|\langle (MI-T)x,y\rangle| = |f(x,y)| \leq f(x,x)^{1/2} f(y,y)^{1/2}$$
$$= \langle (MI-T)x,x\rangle^{1/2} \langle (MI-T)y,y\rangle^{1/2}$$

Hence: $\|(MI-T)x\| \leq c \cdot \langle (MI-T)x,x\rangle \quad \forall x \in H$.

By definition of M there is a sequence $(x_n)_n \subseteq H$ with $\|x_n\| = 1$ and $\langle Tx_n, x_n\rangle \xrightarrow{n} M$.

Hence: $\langle (MI-T)x_n, x_n\rangle = M\|x_n\|^2 - \langle Tx_n, x_n\rangle \xrightarrow{n} 0$ and so $M \in \sigma(T)$, otherwise we would have

$$x_n = (MI-T)^{-1}(MI-T)x_n \xrightarrow{n} 0 \quad (\text{contradiction}).$$

Now since T is selfadjoint in the C^*-algebra $B(H)$) we get: $\|T\| = r(T) = \max\{|m|, |M|\}$.

- Likewise we get that $m \in \sigma(T)$.
- Finally if $T = T^*$ and $\sigma(T) \subseteq \mathbb{R}^+$ then we get that $\langle Tx,x\rangle \geq m \geq 0 \quad \forall \|x\|=1 \Rightarrow \langle Tx,x\rangle \geq 0 \ \forall x \in H$

Conversely if $T = T^*$ and $\langle Tx,x\rangle \geq 0 \ \forall x \in H$ then $m = \inf\{\langle Tx,x\rangle \mid \|x\|_H = 1\} \geq 0$ and so

$$\sigma(T) \subseteq [m, M] \subseteq \mathbb{R}^+.$$

Exercise 1 Consider $C[0,1] = \{f \mid f:[0,1]\to\mathbb{R} \text{ cont's}\}$ and for $h \in C[0,1]$ define $T_h(f) = hf$.

(1) Find $T_h \circ T_g$ (verify it is equal to T_{hg})

(2) Prove that $\ker T_h = (0)$ when $h(x)$ is a polynomial.

(3) Find an $h \in C[0,1]$ such that $\ker T_h \neq (0)$

(4) find an $h \in C[0,1]$ such that T_h has at least two eigenvalues.

(5) Find an $h \in C[0,1]$ such that T_h has infinitely many eigenvalues.

Soln. (1) omitted

(2) we see that the set $\{x \in [0,1] \mid h(x) = 0\}$ is discrete and as it does not contain an interval we have that $h(x) f(x) = 0 \ \forall x \in [0,1] \Longrightarrow f = 0$.

(3) for $h = 0$ trivially we have $T_h = 0$ and so $\ker T_h = C[0,1]$. For a non-trivial example let
$$h(x) = \begin{cases} x - 1/2 & x \in (1/2, 1] \\ 0 & x \in [0, 1/2] \end{cases}$$

Then we get:
$$\ker T_h = \{f \in C[0,1] \mid f(x) = 0 \ \forall x \in [1/2, 1]\}.$$

(4) This happens if we choose h to be "different" constant on disjoint intervals. For example for

$$h(x) = \begin{cases} 0 & x \in [0, 1/3] \\ x - 1/3 & x \in (1/3, 2/3] \\ 1/3 & x \in (2/3, 1] \end{cases}$$

(5) Likewise by cutting $[0,1]$ in countably many pieces and extending linearly for example take $I_k = \left[\frac{1}{2k}, \frac{1}{2k-1}\right]$ and define h such that $h(0) = 0$ and $h|_{I_k} = \lambda_k = \frac{1}{k}$. Then T_h has the eigenvalues $\{1/k : k \in \mathbb{N}\}$.

(However we cannot do the same and find $h \in C[0,1]$ so that T_h has uncountably many eigenvalues.)

$\square$

<u>Exercise 2</u>. Find the spectral radius for

$$T : \ell^1 \to \ell^1 \text{ such that } T(x_1, x_2, \dots) = (2x_1, x_1, x_2, x_3, \dots)$$

<u>Solution</u>: Recall that $\ell^1 = \{(x_n)_n \subseteq \mathbb{C} \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$ with norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$.

By induction we have:

$$T^k(x_1, x_2, \dots) = (2^k x_1, 2^{k-1} x_1, \dots, x_1, x_2, x_3, \dots)$$

and $\|T^k(x)\|_1 = (2^{k+1}-1)|x_1| + |x_2| + \dots$

Hence $\|T^k(x)\|_1 \leq (2^{k+1}-1)\|x\|_1 \quad \forall x \in \ell^1$.

For $x_1 = 1,\ x_2 = x_3 = \dots = 0$ we have equality and so

$$\|T^k\| = 2^{k+1} - 1 < 2^{k+1}.$$

$$\Rightarrow \quad 2 < \|T^k\|^{1/k} < 2 \cdot 2^{1/k} \quad \left(\xrightarrow{k\to\infty}\right)$$

$$\Rightarrow \quad r(T) = 2.$$

Note that if $x = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ then $Tx = 2x$ and so this x is an eigenvector for T. $\boxtimes$

Exercise 3 Find the spectral radius of the operator $T: \ell^\infty \to \ell^\infty$ such that

$$T(x_1, x_2, \dots) = (x_1 + x_2, x_1, x_2, \dots).$$

Solution. If F_n is the Fibonacci sequence, we can show by induction that:

$$T^k(x_1, x_2, \dots) = (F_{k+1}x_1 + F_k x_2, F_k x_1 + F_{k-1}x_2, \dots$$
$$\dots, F_2 x_1 + F_1 x_2, x_1, x_2, \dots)$$

Hence $\|T^k x\|_\infty \leq (F_{k+1} + F_k)\|x\|_\infty = F_{k+2}\|x\|_\infty$

For $x = (1, 1, 0, \dots)$ we have equality and so

$$\|T^k\| = F_{k+2} \Rightarrow r(T) = \lim_k \|T^k\|^{1/k} = \frac{1+\sqrt{5}}{2} =: \varphi$$

(the golden mean). Note that $(\varphi, 1, \varphi^{-1}, \varphi^{-2}, \dots)$ is an eigenvector for T. $\boxtimes$

Exercise 1. Let X be a Banach space and $T \in B(X)$

(1) Show that if $T \in Inv(B(X))$ then $\|T^{-1}\| \geq \frac{1}{\|T\|}$

Find a T s.t. $\|T^{-1}\| > \frac{1}{\|T\|}$ and a T s.t. $\|T^{-1}\| = \frac{1}{\|T\|}$

(2) Suppose λ is an eigenvalue of T. Show that $|\lambda| \leq \|T\|$ and that $|\lambda| \leq r(T)$. Find a T such that $\|T\| = |\lambda| = r(T)$.

Solution (1) We have $1 = \|I_X\| = \|T^{-1} \circ T\| \leq \|T^{-1}\| \, \|T\|$ and since $T \neq 0$ we get $\|T^{-1}\| \geq \frac{1}{\|T\|}$

For $T = \begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}$ we have $T^{-1} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ and so

$$\|T^{-1}\| = 2 > 1 = \frac{1}{\|T\|}.$$

For $T = I_X$ we have $\|T^{-1}\| = \|I_X\| = 1 = \frac{1}{\|T\|}$

For H Hilbert space and T unitary we have that $\|T^{-1}\| = 1 = \|T\|$ and so $\|T^{-1}\| = \frac{1}{\|T\|}$.

(2) If λ is an eigenvalue of T then $\exists\, x \in X$ with $\|x\| = 1$ such that $Tx = \lambda x \Rightarrow \|Tx\| = |\lambda| \Rightarrow$
$\Rightarrow \|T\| \geq |\lambda|$.

Moreover by induction λ^k is an eigenvalue for T^k for all $k \in \mathbb{N}$ and so $|\lambda| = \lim_k |\lambda^k|^{1/k} \leq \lim_k \|T^k\|^{1/k} = r(T)$.

The latter holds for any selfadjoint matrix with λ_0 eigenvalue so that $|\lambda_0| = \max\{|\lambda| \mid \lambda \text{ eigenvalues of } T\}$

$\boxtimes$

Exercise 5. Let $X = \ell^2$ and $S \in B(\ell^2)$ defined by

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

(1) Show that S is an isometry and find S^*.

(2) Show that $\sigma_p(S) = \phi$. $\sigma_a(S) = \partial \mathbb{D}$ and $\sigma_c(S) = \mathbb{D}$.

(3) Conclude that $\sigma(S) = \overline{\mathbb{D}}$.

Solution. (1) We have that:

$$\| S(x_1, x_2, \dots)\|_2 = \|(0, x_1, x_2, \dots)\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} = \|(x_1, x_2, \dots)\|$$

This shows that S is an isometry and so $\|S\| = 1$.

For S^* verify that the operator

$$(x_1, x_2, \dots) \longmapsto (x_2, x_3, \dots)$$

does the trick.

(2) Let $\lambda \in \mathbb{C}$ s.t. $Sx = \lambda x$ for some $0 \neq x \in \ell^2$. Then:

$$Sx = \lambda x \implies (0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$$

So that either $\lambda = 0$ or $x_1 = 0$. But S is an isometry and so injective; hence $\lambda \neq 0$ which forces $x_1 = 0$. Then $\lambda x_2 = x_1 = 0$ implies $x_2 = 0$ and inductively $x = 0$ which is a contradiction.

Next we show that $\sigma_p(S^*) = \mathbb{D}$. ~~As~~ is $\mathbb{D}$ is closed when taking conjugates we will conclude that $\sigma_c(S) = \mathbb{D}$.

On one hand: for $\lambda \in \mathbb{C}$ with $S^*x = \lambda x$ we get:

$$(x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \dots)$$

Then $x_{n+1} = \lambda^n x_1$. As $x \in l^2$ we obtain:

$$\infty > \|x\|_2^2 = |x_1|^2 \sum_{n=1}^{\infty} \lambda^n \Rightarrow |\lambda| < 1.$$

Conversely if $|\lambda| < 1$ then for $x = (1, \lambda, \lambda^2, \dots)$ we get $S^* x = \lambda x$. λ_0 $\|x\|_2 < \infty$.

Next we show that if $\lambda \in \mathbb{D}$ then $\lambda \notin \sigma_a(S)$. This is equivalent to showing that $\lambda I - S$ is bounded below. Indeed for $x \in l^2$ we get:

$$\|(S - \lambda I)x\| \geq \big|\|Sx\| - \|\lambda x\|\big| = (1 - |\lambda|)\|x\|.$$

(3) We have $\sigma(S) = \sigma_a(S) \cup \sigma_c(S) = \partial\mathbb{D} \cup \mathbb{D} = \overline{\mathbb{D}}$.

Alternatively, since $\|S\| = 1$ we have $\sigma(S) \subseteq \overline{\mathbb{D}}$.

But also $\mathbb{D} \subseteq \sigma(S)$ and as $\sigma(S)$ is closed we get

$$\overline{\mathbb{D}} \subseteq \sigma(S) \subseteq \overline{\mathbb{D}} \Rightarrow \sigma(S) = \overline{\mathbb{D}}.$$ ☒

<u>Exercise 6.</u> Let $T \in B(l^2)$ given by:

$$T(x_1, x_2, \dots) = \left(0, x_1, \tfrac{1}{2}x_2, \dots, \tfrac{1}{n}x_n, \dots\right)$$

Then $\sigma(T) = \{0\}$. In particular $\sigma_p(T) = \varphi$ and $\sigma_a(T) = \sigma_c(T) = \{0\}$.

(Note: $T = S \circ D$ for S the shift and $De_n = \frac{1}{n}e_n$. We have $\sigma(D) = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\sigma(S) = \overline{\mathbb{D}}$, so the spectrum does not compile with composition.)

Solution First we observe that $\|T^k\| = \frac{1}{k!}$ (by induction). Hence

$$r(T) = \lim_k \|T^k\|^{1/k} = \lim_k \left(\frac{1}{k!}\right)^{1/k} = 0$$

(note: $k! > \left(\frac{k}{2}\right)^{k/2} \Rightarrow \left(\frac{1}{k!}\right)^{1/k} < \left(\frac{2}{k}\right)^{1/2} \xrightarrow{k} 0$)

Hence $\sigma(T) \subseteq B(0, r(T)) = \{0\}$. As the spectrum is non-empty we get $\sigma(T) = \{0\}$.

Now note that $\|Te_n\|_2 = \frac{1}{n} \xrightarrow{n} 0 \Rightarrow 0 \in \sigma_a(T)$.

However $T = S \circ D$ is injective (as so are S and D) and so $0 \notin \sigma_p(T) \subseteq \sigma(T) = \{0\} \Rightarrow \sigma_p(T) = \phi$.

Moreover $\langle Te_n, e_1 \rangle = 0 \Rightarrow e_1 \perp \overline{(T-0)\ell^2} \Rightarrow 0 \in \sigma_c(T)$

$\blacksquare$

Exercise 7. Let $T : \ell^\infty \to \ell^\infty$ given by:

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

Find the spectrum of T.

Solution. We can see that $\|Tx\|_\infty \leq \|x\|_\infty$ and $\|Te_2\|_\infty = \|e_2\|_\infty$ and so $\|T\| = 1$. Hence $\sigma(T) \subseteq \overline{\mathbb{D}}$. Moreover for $\lambda \in \mathbb{D}$ we have that $x = (1, \lambda, \lambda^2, \ldots) \in \ell^\infty$ for which

$$Tx = (\lambda, \lambda^2, \ldots) = \lambda(1, \lambda, \lambda^2, \ldots) = \lambda x.$$

Hence $\sigma_p(T) = \overline{\mathbb{D}}$. Thus:

$$\overline{\mathbb{D}} = \overline{\sigma_p(T)} \subseteq \sigma(T) \subseteq \overline{\mathbb{D}} \Rightarrow \sigma(T) = \sigma_p(T) = \overline{\mathbb{D}}.$$

$\blacksquare$

Exercise 8 Let $T: \ell^1 \to \ell^1$ given by:

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

(1) Find $\sigma(T)$.

(2) Show that $\mathrm{Ran}(T-I)$ is dense in ℓ^1 by showing that it contains all sequences with at most finite non-zero entries.

(3) Show that $\mathrm{Ran}(T-I) \neq \ell^1$. Can you give an explicit vector in the difference?

Solution (1) We start by computing $\|T\|$. We have:

$$\|T(x_1, x_2, \dots)\|_1 = \sum_{n=2}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} |x_n| = \|x\|_1 \Rightarrow \|T\| \leq 1$$

Moreover $\|Te_2\| = \|e_1\|_1 = 1$ and so $\|T\| = 1$.

Hence $\sigma(T) \subseteq \overline{\mathbb{D}}$.

A similar computation as before gives that $\sigma_p(T) = \mathbb{D}$. Thus we conclude:

$$\overline{\mathbb{D}} = \overline{\sigma_p(T)} \subseteq \overline{\sigma(T)} = \sigma(T) \subseteq \overline{\mathbb{D}} \Rightarrow \sigma(T) = \overline{\mathbb{D}}.$$

(2) We have that $b = (b_1, \dots, b_n, 0, \dots) \in \mathrm{Ran}(T-I)$ if and only if:

$$(b_1, \dots, b_n, 0, \dots) = (T-I)x = (x_2 - x_1, x_3 - x_2, \dots)$$

which gives $x_1 = b_1 + \dots + b_n$, $x_2 = b_2 + \dots + b_n$; $x_n = b_n$ and $x_m = 0 \ \forall m > n$.

Hence $c_c \subseteq \mathrm{Ran}(T-I)$ and as c_c is dense in ℓ^1, so is $\mathrm{Ran}(T-I)$.

(3) Since $1 \in \sigma(T)$ then $T-I$ is not invertible. Since $1 \notin \sigma_p(T)$ then $T-I$ is injective. So $T-I$ cannot be surjective. For an explicit vector take:

$$x = \left(\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \dots\right) \notin \operatorname{ran}(T-I)$$

Indeed if there is $y \in \ell^1$ with $(T-I)x = y$ then $y_{n+1} = y_1 - 1 + \frac{1}{n+1}$ since

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \left(1-\frac{1}{2}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

we need $y \in \ell^1$ and so $y_n \xrightarrow{n} 0$. Hence $y_1 = 1$ and so $y_n = \frac{1}{n}$. But $(1, \frac{1}{2}, \dots) \notin \ell^1$ $\boxtimes$

Exercise 9. Let X be a Banach space.

(1) For $T \in B(X)$ show that: $\operatorname{Ran}(T)$ is not dense in X if and only if there exists $f \in X^*$ s.t. $f \circ T = 0$.

(2) Let $S : \ell^1 \to \ell^1$ s.t. $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Find $\sigma(S)$.

Solution (1) If such an $f \in X^*$ exists then $\overline{\operatorname{Ran} T} \subseteq \ker f \neq X$ and so $\overline{\operatorname{Ran} T} \subsetneq X$. Conversely if $\overline{\operatorname{Ran} T} \subsetneq X$ then $\exists\, x \notin \overline{\operatorname{Ran} T}$ and by Hahn-Banach there exists $f \in X^*$ such $f(x) = 1$ and $f|_{\overline{\operatorname{Ran} T}} = 0$.

(2) First we note that S is an isometry as:

$$\|Sx\|_1 = \sum_{n=1}^{\infty} |x_n| = \|x\|_1 .$$

Hence $\|S\| = 1$ and S is injective. As with ℓ^2 we can show that $\sigma_p(S) = \phi$.

Now for $\lambda \in \overline{\mathbb{D}}$ consider the vector $y = (1, \lambda, \lambda^2, \dots) \in \ell^\infty$ which defines $f_y \in (\ell^1)^*$ by:

$$f_y(x_1, \dots, x_n, \dots) = \sum_{n=1}^{\infty} x_n y_n .$$

Moreover for $x \in \ell^1$ we have:

$$f_y(S-\lambda)x = f_y(-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \dots) =$$

$$= -(\lambda x_1 + \lambda(\lambda x_2 - x_1) + \lambda^2(x_3 - \lambda x_2) + \dots) = \dots = 0$$

By (1) it follows that $\mathrm{Ran}(S - \lambda I)$ is not dense in ℓ^1, and so $\lambda \in \sigma(S)$. Consequently

$$\overline{\mathbb{D}} \subseteq \sigma(S) \subseteq \overline{\mathbb{D}} \Rightarrow \sigma(S) = \overline{\mathbb{D}} .$$

∎

<u>Exercise 10</u>. Let $S : \ell^\infty \to \ell^\infty$ such that:

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

(1) Find an open ball in ℓ^∞ that is disjoint from $\mathrm{Ran}(S - I)$.

(2) Determine $\sigma(S)$.

<u>Solution</u>: (1) Let $x_0 = (1, 1, \dots) \in \ell^\infty$ and take the ball $B(x_0, 1/2)$.

If $y \in B(x_0, \frac{1}{2})$ then $\mathrm{Re}(y_n) > \frac{1}{2}$ as $\|x_0 - y\|_\infty < \frac{1}{2}$.
Suppose there is $x \in \ell^\infty$ s.t. $(I-S)x = y$ for such a y. Then we would have:

$$x_n = y_1 + y_2 + \dots + y_n \quad \forall n$$

$$\Rightarrow \quad \mathrm{Re}(x_n) = \mathrm{Re}(y_1) + \dots + \mathrm{Re}(y_n) \geq \frac{n}{2}$$

$$\Rightarrow \quad \|x\|_\infty \geq \frac{n}{2} \quad \forall n \quad \text{contradiction.}$$

(2) We have $\|S\| = 1$ and so $\sigma(S) \subseteq \overline{\mathbb{D}}$. Moreover $\sigma_p(S) = \phi$. Let $\lambda \in \mathbb{D}$ and take $y = (1, \lambda, \lambda^2, \dots) \in \ell^1$.
Then $f_y \in (\ell^\infty)^*$ and it follows that

$$f_y(S - \lambda I) = 0.$$

Hence $\mathrm{Ran}(S - \lambda I)$ is not dense in ℓ^∞ which shows that $\lambda \in \sigma(S)$. We conclude:

$$\mathbb{D} \subseteq \sigma(S) \subseteq \overline{\mathbb{D}} \Rightarrow \sigma(S) = \overline{\mathbb{D}}.$$ $\boxtimes$

<u>Exercise 11</u>. Consider $T: \ell^1 \to \ell^1$ such that:

$$T(x_1, x_2, \dots) = (x_1 + x_2 + x_3,\ x_2 + x_3 + x_4,\ \dots)$$

Find $\sigma(T)$. What is $\sigma(T) \cap \mathbb{R}$?

<u>Solution</u>. If $S: \ell^1 \to \ell^1$ is the <u>left</u> shift we see that $T = I + S + S^2$. Therefore:

$$\sigma(T) = \{1 + \lambda + \lambda^2 \mid \lambda \in \sigma(S)\} = \{1 + \lambda + \lambda^2 \mid \lambda \in \overline{\mathbb{D}}\}.$$

For $\sigma(T) \cap \mathbb{R}$ let $r \in \mathbb{R}$ s.t.

$$r = 1 + \lambda + \lambda^2 \quad \text{for } \lambda \in \overline{\mathbb{D}}.$$

Then $\lambda^2+\lambda+(1-r)=0 \Rightarrow \lambda=-\frac{1}{2}\pm\sqrt{r-3/4}$.

The exact condition of at least one root sitting in $\bar{D}$ is that $0\leq r<3$ and so:

$$\sigma(T)\cap\mathbb{R}=[0,3].$$ ☒

(This is different from $\{1+\lambda+\lambda^2 : \lambda\in\sigma(S)\cap[-1,1]\}=[3/4,3]$ so we need to be careful with the Spectral Mapping Theorem.)

<u>Exercise 12</u>. Let A be a unital Banach algebra and let $a\in A$.

(1) Suppose that $p(a)=0$ for $p(z)\in\mathbb{C}[z]$. Show that $p(\lambda)=0$ for every $\lambda\in\sigma(a)$.

(2) Suppose that $a^2=a$. Show that $\sigma(a)\subseteq\{0,1\}$.

(3) Suppose that $a^2=a$ and $a\neq 0, 1_A$. What is $r(a)$? What is $r(1-a)$? Is it possible to have $\sigma(a)=\{0\}$ or $\sigma(a)=\{1\}$?

(4) Suppose that $a^2=a$. Show that aAa is a closed subalgebra of A with a as its unit.

<u>Solution</u>. (1) From the spectral mapping theorem we get: $p(\sigma(a))=\sigma(p(a))=\sigma(0)=\{0\}$. So $p(\lambda)=0$ for every $\lambda\in\sigma(a)$.

(2) For $p(z)=z^2-z$ we have $p(a)=0$ and so $0=p(\lambda)=\lambda^2-\lambda=\lambda(\lambda-1)$ for every $\lambda\in\sigma(a)$. Hence $\lambda=0$ or 1.

(3) If $a^2 = a$, then: $(1-a)^2 = 1 - 2a + a^2 = 1 - a$.

Sps $\sigma(a) = \{1\}$. Then a is invertible and so:

$$1 = aa^{-1} = a^2 a^{-1} = a$$

which is a contradiction as $a \neq 1$

Sps $\sigma(a) = \{0\}$. Then $1-a$ is invertible and so

$$1 = (1-a)(1-a)^{-1} = (1-a)^2 (1-a)^{-1} = 1-a$$

which gives $a = 0$, a contradiction.

Hence $\sigma(a) = \{0, 1\} = \sigma(1-a)$ and so

$$r(a) = 1 = r(1-a).$$

Alternatively: by $a^2 = a$ we get $a^n = a$ $\forall n$

then $r(a) = \lim_n \|a^n\|^{1/n} = \lim_n \|a\|^{1/n} = 1$.

and likewise $r(1-a) = 1$.

(4) Obviously aAa is a subalgebra of A (as $a \in A$) with a as its unit. Sps now that $(ax_n a)_n \subseteq aAa$ with $\lim_n ax_n a = x$, for some $x \in A$. Then:

$$axa = \lim_n a^2 x_n a^2 = \lim_n a x_n a = x$$

Hence $x = axa \in aAa$, showing that aAa is closed in A. $\boxtimes$