(One-point) Compactification

Exercise 1. Show the following:

(a) Every closed subspace of a compact space is compact.

(b) Every compact subspace of a Hausdorff space is closed.

(c) The continuous image of a compact space is compact.

(d) If $f: X \to Y$ is continuous for X compact and Y Hausdorff, then f is closed. If in addition f is a bijection then it is a homeomorphism.

(e) A space which is compact and Hausdorff is T_4.

Solution (a) Let X compact and $Y \subseteq X$ closed. Let $\{U_\lambda\}_\lambda$ an open cover for Y and consider $V_\lambda \subseteq X$ open such that $U_\lambda = V_\lambda \cap Y$. Consider $\{V_\lambda\}_\lambda \cup \{X \setminus Y\}$ which is an open cover for X, and thus has a finite subcover $\{V_0, V_{\lambda_1}, \ldots, V_{\lambda_n}\}$ with $V_0 = X \setminus Y$ (wlog). Then $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$ is a finite open cover for Y.

(b) Let X be Hausdorff and $Y \subseteq X$ compact. Let $x \notin Y$ and for every $y \in Y$ consider disjoint open sets $x \in V_y$ and $y \in U_y$. Then $\{U_y \cap Y\}_{y \in Y}$ is an open cover for Y and thus has a finite subcover $\{U_{y_1} \cap Y, \dots, U_{y_n} \cap Y\}$. Then the set $V := V_{y_1} \cap \dots \cap V_{y_n}$ is an open n'hood of x and by construction $V \subseteq X - Y$. So $X - Y$ is open.

(c) Let $f : X \to Y$ continuous for X compact and Y Hausdorff. Let $\{V_\lambda\}_\lambda$ an open cover of $f(X)$ and consider $U_\lambda := f^{-1}(V_\lambda)$ which are open in X. Then $\{U_\lambda\}_\lambda$ is an open cover for X and thus there exists $\{U_1, \dots, U_n\}$ finite subcover. By definition $f(X) \subseteq f\left(\bigcup_{i=1}^n U_i\right) \subseteq \bigcup_{i=1}^n f(U_i)$ thus $f(X)$ is compact. $= \bigcup_{i=1}^n V_i$

(d) Now suppose C is closed in X. Hence C is compact;$^{(a)}$ thus $f(C)$ is compact$^{(c)}$ and so $f(C)$ is closed in Y. (b).

If f is in addition and bijection and f^{-1} is well-defined. As $f = (f^{-1})^{-1}$ is closed it follows that f^{-1} is continuous

(e) As X is Hausdorff, it is then T_1.
For normality: let $A, B \subseteq X$ closed and disjoint and choose $x \in B$. As in (b) take U_x, V_x open s.t. $A \subseteq U_x$ and $x \in V_x$. Then $\{V_x \cap B\}_{x \in B}$ is an open cover of B. As B is closed and thus compact there is a finite subcover $\{V_{x_1} \cap B, .. V_{x_n} \cap B\}$
Now take $U = \bigcap_{k=1}^{n} U_{x_k}$ and $V = \bigcup_{k=1}^{n} V_{x_k}$ which are disjoint and separate A and B. $\boxtimes$

Exercise 2. Let X be a locally compact, but not compact, Hausdorff space and define the space

$$X' = X \sqcup \{p\}$$

by adjoining a point. In X' let the family of basic n'hoods of points in X coming from the topology on X, and the basic n'hoods of p as $\{p\} \cup (X \setminus C)$ for $C \subseteq X$ compact.

(a) Show that this family is a basis of topology on X'

(b) X is open and dense in X'.

(c) X' is compact and Hausdorff

(d) If $X = \mathbb{R}$ then $X' \cong \mathbb{T}$.

(e) There is $f \in C_b(\mathbb{R})$ that does not extend to an $C(\mathbb{R}')$

(a) Omitted.

(b) X is open in X and thus open in X^{+}.

Let U be open n'hood of p. Then we write $U = (X \setminus C) \cup \{p\}$ for some compact $C \subseteq X$.

As X is not compact itself we get $C \subsetneq X$ and so $U \cap X \neq \emptyset$. Hence X is dense in X^{+}.

(c) Hausdorff: Let $x, y \in X^{+}$ s.t. $x \neq y$.

- If $x, y \in X$ then they are separated in X and thus in X^{+}.
- So assume $x \in X$ and $y = p$. By local compactness there is a compact n'hood C of x. Then the open sets C° and $(X \setminus C) \cup \{p\}$ separate x from p.

Compactness: Let $\{U_\lambda\}_\lambda$ open cover of X^{+} and let U_{λ_0} containing p, so that $X^{+} \setminus U_{\lambda_0}$ is compact. Then $\{U_\lambda\}_{\lambda \neq \lambda_0}$ are an open cover of $X^{+} \setminus U_{\lambda_0}$ giving a finite subcover $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$. Then the family $\{U_{\lambda_1}, \dots, U_{\lambda_n}\} \cup \{U_{\lambda_0}\}$ is a finite subcover for X^{+}.

(compactness follows even when X is compact, but not Hausdorff-ness.) (e.g. $\mathbb{Q}$, discrete topology)

(d) The function $\cot : (0, 2\pi) \to \mathbb{R}$ gives a homeomorphism. Define:

$$h : [0, 2\pi) \to \mathbb{R} : \quad h(x) = \begin{cases} \cot(x) & x \in (0, 2\pi) \\ p & x = 0 \end{cases}$$

Recall that $[0, 2\pi) \simeq \mathbb{T}$ by $x \longmapsto (\sin x, \cos x)$.

Define an open n'hood of 0 by $(2\pi - a, 2\pi) \cup [0, \alpha)$

This function now gives the required homeomorphism

(e) Let $f = \sin$. Then $f \in C_b(\mathbb{R})$. Suppose f has an extension (denoted again by f) on $C(\mathbb{R}')$.

Then $\forall \varepsilon > 0 \ \exists U$ open n'hood of p such that

$$f(U) \subseteq (f(p) - \varepsilon, f(p) + \varepsilon) \quad \text{for}$$

Then $\mathbb{R}' \setminus U$ is compact in $\mathbb{R}$ and thus bounded and so $\mathbb{R}' \setminus U \subseteq [m, M]$.

For any $y \in [-1, 1]$ there is $x \in \mathbb{R} \setminus [m, M]$ such that $f(x) = y$. So for ε sufficiently small (and U arbitrary) $f(U)$ is not contained in any interval $(f(p) - \varepsilon, f(p) + \varepsilon)$. So f does not extend.

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Exercise 3. Every locally compact and Hausdorff is $T_{3\frac{1}{2}}$.

Solution. (i) If X is compact then it is T_4 and so by Urysohn it is $T_{3\frac{1}{2}}$.

(ii) If X is not compact, then $X^{\perp}$ is compact and Hausdorff and so $T_{3\frac{1}{2}}$. As X is a subspace of a $T_{3\frac{1}{2}}$ it is itself $T_{3\frac{1}{2}}$ ☒

Recall: $T_{3\frac{1}{2}}$ means ~~The~~ and separations of points from closed sets via a continuous bounded function.

Remark: Let X be a locally compact and Hausdorff space and recall:

$$C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X)$$

where $C_c(X) = \{ f : X \to \mathbb{C} \mid f \text{ continuous and has compact support} \}$

As $X \subseteq X^{\perp}$ we can indirectly apply Urysohn's lemma on $X^{\perp}$ and then pass to X

<u>Remark</u>. Every $f \in C_0(X)$ extends to a function $\tilde{f} \in C(X^+)$ with $\tilde{f}(x) = \begin{cases} f(x) & x \in X \\ 0 & x = p \end{cases}$.

Moreover $C(X^+)$ admits the constant function $\mathbb{1}$ as its unit. We can then show that the map:

$$\Phi : C_0(X)^1 \longrightarrow C(X^+)$$
$$(f, \lambda) \longmapsto \tilde{f} + \lambda \mathbb{1}$$

is a $*$-isomorphism. We can also see the norm $\|(f, \lambda)\|_1 = \|f\|_\infty + |\lambda|$ is equivalent to the supremum norm $\|\tilde{f} + \lambda \mathbb{1}\|_\infty$, but not the same (take any $f \in C_0(X)$ with $\|f\|_\infty = 1$ and then $\|(f, 1)\|_1 = 2$ while

$$\|\tilde{f} + \lambda \mathbb{1}\|_\infty = \max\{\|f\|_\infty, |\lambda|\} = 1 \,.).$$

We can make $C_0(X)^1$ a C^*-algebra in 2 ways:

- abstractly as $A^1 \subseteq B(A)$
- inducing the norm of $C(X^+)$ on $C_0(X)^1$.

Exercise 4 Let X be locally compact and Hausdorff space. Show the following:

(a) If $K \subseteq U \subseteq X$ with K compact and U open, then there exists $V \subseteq X$ open with $\overline{V}$ compact such that $K \subseteq V \subseteq \overline{V} \subseteq U$.

(b) For each $x \in X$ and $U \subseteq X$ open with $x \in U$ there exists open $V \subseteq X$ with $x \in V$ and $\overline{V}$ compact such that $\overline{V} \subseteq U$.

(c) (Urysohn's Lemma) If $K \subseteq U \subseteq X$ with K compact and U open, then there exists $f \in C_c(X)$ such that $0 \leq f \leq 1$, $f|_K = 1$ and $\operatorname{supp} f \subseteq U$.

(d) (Tietze Extension Theorem) If $K \subseteq X$ compact and real $f \in C(K)$, then there exists real $g \in C_c(X)$ extending f on X.

(e) $C_0(X)$ is the uniform closure of $C_c(X)$ in $C_b(X)$.

Solution: (In) all below we see X inside its one-point compactification X^+, by assuming that X is locally compact but not compact

(a) Since $K \subseteq X$ compact then $K \subseteq X^+$ compact and since X^+ is T_2 we get that K is closed in X^+. Since $X \subseteq X^+$ open then $U \subseteq X^+$ open. By normality of X^+ there exist $V \subseteq X^+$ open s.t. $K \subseteq V \subseteq \overline{V} \subseteq U$ and note that actually $\overline{V} \subseteq X$. As $\overline{V}$ is closed in X^+ then it is compact and also V is open in X.

(b) Apply for $K = \{x\}$.

(c) By (a) there exists $V \subseteq X$ open with $\overline{V}$ compact such that $K \subseteq V \subseteq \overline{V} \subseteq U$. Since $K \cap (X^+ \setminus V) = \phi$ and K, $X^+ \setminus V$ closed, by Urysohn's Lemma for the normal X^+ we get $g \in C(X^+)$ such that $0 \leq g \leq 1$, $g|_K = 1$, $g|_{X^+ - V} = 0$. We now set $f = g|_X$ which has the desired properties.

- $f \in C(X)$, $0 \leq f \leq 1$, $f|_K \equiv 1$.
- $f|_{X \setminus V} = g|_{X - V} = 0$ and so $f|_{X \setminus \overline{V}} = 0$

$$\{x \in X : f(x) \neq 0\} \subseteq \overline{V} \subseteq U.$$

Since $\overline{V}$ is compact and $\operatorname{supp} f \subseteq \overline{V} \subseteq U$ and so $f \in C_c(X)$.

(d) Since $K \subseteq X$ is compact then $K \subseteq X^1$ is compact and closed in X^1. By Tietze extension Theorem on X^1 we get an real extension $g \in C(X^1)$. Now take $g|_X$ for the extension of f and follow (e).

(e) We have shown that $C_0(X)$ is closed in $C_0(X)$. So it remains to show that $C_c(X)$ is dense in $C_0(X)$.

Let $f \in C_0(X)$ and $\varepsilon > 0$. Take $K \subseteq X$ compact such that $|f(x)| < \varepsilon/2$ $\forall x \in X - K$. By (c) there exists $g \in C_c(X)$ with $0 \leq g \leq 1$ s.t. $g|_K = 1$. Then the function $h := fg$ is supported on supp g and is continuous, i.e., $h \in C_c(X)$.

- If $x \in K$ then $|f(x) - h(x)| = |f(x) - f(x)| = 0$.
- If $x \notin K$ then $|f(x) - h(x)| = |f(x)||1 - g(x)| \leq$
$$\leq |f(x)| < \varepsilon/2.$$

Hence $\|f - h\|_\infty < \varepsilon$ and the proof is complete

$\square$