Exercises (matrices)

Exercise 1 Recall that $\mathbb{C}^n$ is equipped with the inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$

$$\langle x, y \rangle := y^* x = \sum_{i=1}^{n} x_i \overline{y_i}$$

Show that: $|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \quad \forall\, x, y \in \mathbb{C}^n$

Solution: For every $\lambda \in \mathbb{R}$ we have:

$$0 \leq \langle x + \lambda y, x + \lambda y \rangle = \lambda^2 \|y\|^2 + \lambda\left(2\operatorname{Re}(y^* x)\right) + \|x\|^2$$

$$\leq \|y\|^2 \cdot \lambda^2 + 2|y^* x|\lambda + \|x\|^2.$$

Hence it has at most one solution, i.e., $D \leq 0$

$$\Rightarrow \left(2|\langle x, y\rangle|\right)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\Rightarrow \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

Exercise 2. Recall that every $A \in M_n(\mathbb{C})$ defines a linear operator on $\mathbb{C}^n$. It follows that every A is automatically continuous with respect to the Euclidean $\|\cdot\|_2$-norm on $\mathbb{C}^n$. If we write

$$\|A\| = \sup_{\|x\|_2 \leq 1} \|Ax\|_2$$

for the operator norm in $M_n(\mathbb{C}) = B(\mathbb{C}^n)$ then show that $\|A\| = \|A^*\|$ and $\|A\|^2 = \|A^*A\|$.

Solution: For $x \in \mathbb{C}^n$ with $\|x\|_2 \leq 1$ we have:

$$\|Ax\|_2^2 = \langle Ax, Ax \rangle = (Ax)^* Ax = x^* A^* A x$$

$$= \langle A^* A x, x \rangle \leq \|A^* A x\|_2 \|x\|_2 \leq$$

$$\leq \|A^* A\| \leq \|A^*\| \cdot \|A\|.$$

Taking supremum gives:

$$\|A\|^2 \leq \|A^*A\| \leq \|A^*\|\,\|A\|$$

from which it follows first that $\|A\| \leq \|A^*\|$ (and by symmetry that $\|A\| = \|A^*\|$) and then that $\|A\|^2 = \|A^*A\|$. $\boxed{}$

<u>Exercise 3</u> Let $D \in M_n(\mathbb{C})$ be diagonal. Show that $\|D\| = \max\{|d_{ii}| \,;\, i = 1, \dots, n\}$.

<u>Solution</u>: Let i_0 such that $\max\limits_{i=1..n}\{|d_{ii}|\} = |d_{i_0 i_0}|$.

We first see that:

$$\|D e_{i_0}\|_2 = \|d_{i_0 i_0} e_{i_0}\|_2 = |d_{i_0 i_0}|$$

and so $|d_{i_0 i_0}| \leq \|D\|$.

Moreover for every $x \in \mathbb{C}^n$ with $\|x\|_2 \leq 1$ we have:

$$\|Dx\|_2^2 = \sum_{i=1}^{n} |d_{ii}|^2 |x_i|^2 \leq |d_{i_0 i_0}|^2 \sum_{i=1}^{n} |x_i|^2 = |d_{i_0 i_0}|^2$$

$\Rightarrow$ $\|D\| \leq |d_{i_0 i_0}|$ and the solution is complete $\boxed{}$

<u>Exercise 4</u> Let $U \in M_n(\mathbb{C})$ be a unitary. Then

$$\|UA\| = \|A\| \quad \text{and} \quad \|AU\| = \|A\| \quad \forall A \in M_n(\mathbb{C}).$$

<u>Solution</u> $\|UA\|^2 = \|(UA)^*UA\| = \|A^*U^*UA\| = \|A^*A\| = \|A\|^2$.

Moreover: $\|AU\| = \|(AU)^*\| = \|U^*A^*\| = \|A^*\| = \|A\|$

Since U^* is also unitary. $\boxed{}$

Exercise 5. Let $A \in M_n(\mathbb{C})$ such that $A = A^*$. Show that $\|A\| = \max\{|\lambda| : \lambda \text{ eigenvalue of } A\}$.

Solution Suppose first that A is diagonal. Then by exercise 3 we have:

$$\|A\| = \max_{i=1..n}\{|a_{ii}|\} = \max\{|\lambda| : \lambda \text{ eigenvalue of } A\}$$

Now ~~let~~ $A = A^*$ in $M_n(\mathbb{C})$. Then there is a unitary $U \in M_n(\mathbb{C})$ and a diagonal $D \in M_n(\mathbb{C})$ with entries the eigenvalues of A such that: $A = UDU^*$.

Then:

$$\|A\| = \|UDU^*\| = \|D\| = \max_{i=1..n}\{|d_{ii}|\} =$$
$$= \max\{|\lambda| : \lambda \text{ eigenvalue of } A\}. \quad \boxed{}$$

Exercise 6. Let $A \in M_n(\mathbb{C})$. Show that:

$$\|A\| = \max\{|\lambda|^{1/2} : \lambda \text{ eigenvalue of } A^*A\}.$$

Find the norm of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in $M_2(\mathbb{C})$.

Solution We do just the first. We have:

$$\|A\|^2 = \underbrace{\|A^*A\|}_{\text{selfadjoint}} = \max\{|\lambda| : \lambda \text{ eigenvalue of } A^*A\}.$$

$\boxed{}$

Exercise 7 Show the following inequalities:

$$|A_{ij}| \leq \|A\| \leq \left(\sum_{i,j=1}^{n} |A_{ij}|^2\right)^{1/2} \leq \sum_{i,j=1}^{n} |A_{ij}|$$

Solution: we have that:

$$|A_{ij}| = \left\| \begin{bmatrix} 0 & \cdots & A_{ij} & \cdots & 0 \\ & & \vdots & & \\ & & 0 & & \end{bmatrix} \right\| = \| E_{ii} A E_{jj} \| \leq \|E_{ii}\| \, \|A\| \, \|E_{jj}\|$$

$$= \|A\| \quad \forall i,j.$$

Moreover for $x \in \mathbb{C}^n$ with $\|x\|_2 \leq 1$ we have:

$$\|Ax\|_2^2 = \left\| \begin{pmatrix} \sum_{j=1}^{n} A_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj} x_j \end{pmatrix} \right\|_2^2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} A_{ij} x_j \right|^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |A_{ij} x_j|^2 \right)^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |A_{ij}|^2 |x_j|^2 \right)^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |A_{ij}|^2 \right) \cdot \underbrace{\left(\sum_{j=1}^{n} |x_j|^2 \right)}_{\|x\|_2^2}$$

$$\leq \sum_{i,j=1}^{n} |A_{ij}|^2.$$

Taking supremum we have $\|A\|^2 \leq \sum_{i,j=1}^{n} |A_{ij}|^2$.

Finally it is immediate that:

$$\sum_{i,j=1}^{n} |A_{ij}|^2 \leq \left(\sum_{i,j=1}^{n} |A_{ij}| \right)^2$$

and the solution is complete. $\boxtimes$

<u>Remark</u> With that in hand we have that a sequence $(A_k)_k \subseteq M_n(\mathbb{C})$ converges (in operator norm) to $A \in M_n(\mathbb{C})$ if and only if $A_{ij} = \lim_k (A_k)_{ij}$.

This allows to make sense of the functional calculus for normal matrices:

Let $A \in M_n(\mathbb{C})$ normal and set $\sigma(A)$ for the set of eigenvalues of A. For every $f \in C(\sigma(A))$, by Stone-Weierstrass, we can find polynomials p_k such that $p_k \xrightarrow{k} f$ uniformly.

If $U^* A U = D$ is the diagonalisation of A we see that $p(D) = U^* p(A) U$ for every polynomial p.

Moreover we have that:

$$\|\cdot\|-\lim_k p_k(D) = \|-\lim_k \begin{pmatrix} p_k(d_{11}) & & \\ & \ddots & \\ & & p_k(d_{nn}) \end{pmatrix} = \begin{pmatrix} f(d_{11}) & & \\ & \ddots & \\ & & f(d_{nn}) \end{pmatrix}$$

Therefore the limit $\lim_k U p_k(D) U^*$ exists in $M_n(\mathbb{C})$ and we call it $f(A)$.

<u>Remark</u> Recall the function $Tr: M_n(\mathbb{C}) \to \mathbb{C}$ such that $Tr(A) = \sum_{i=1}^n A_{ii}$. Then we have:

$$\left.\begin{aligned} Tr(AB) &= \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki} \\ Tr(BA) &= \sum_{k=1}^n (BA)_{kk} = \sum_{k=1}^n \sum_{i=1}^n B_{ki} A_{ki} \end{aligned}\right\} \Rightarrow Tr(AB) = Tr(BA).$$

<u>Exercise 8</u>. Let $A = A^*$ in $M_n(\mathbb{C})$. Show that the following are equivalent:

(1) $\{\lambda \mid \lambda \text{ eigenvalue of } A\} \subseteq \mathbb{R}_+$.

(2) $x^* A x \geq 0 \quad \forall\, x \in \mathbb{C}^n$.

(3) there exists $C \in M_n(\mathbb{C})$ s.t. $A = C^* C$.

<u>Solution</u>. Let $U^* A U = D$ be the diagonalisation of A. Then we see that:

(1) $\iff$ $\{\lambda \mid \lambda \text{ eigenvalue of } D\} \subseteq \mathbb{R}_+$, since A and D have the same eigenvalues

(2) $\iff$ $(x^* U) D (U^* x) \geq 0 \ \forall x \in \mathbb{C}^n \iff y^* D y \geq 0 \ \forall y \in \mathbb{C}^n$ since U is an isomorphism on $\mathbb{C}^n$

(3) $\iff$ $U D U^* = C^* C \iff D = (CU)^* CU$.

Therefore it suffices to show the equivalences for A real + diagonal in $M_n(\mathbb{C})$.

(1) $\Rightarrow$ (2): $x^* A x = x^* \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix} x = \sum_{i=1}^n a_{ii} |x_i|^2 \geq 0$.

(2) $\Rightarrow$ (1): $x^* A x \geq 0 \ \forall x$. So for $x = e_i$ we get:
$a_{ii} = (e_i)^* A e_i \geq 0 \Rightarrow a_{ii} \geq 0$.

(1) $\Rightarrow$ (3): Follows by taking $C = \begin{pmatrix} a_{11}^{1/2} & & \\ & \ddots & \\ & & a_{nn}^{1/2} \end{pmatrix}$

(3) $\Rightarrow$ (2): $x^* A x = x^* C^* C x = \|Cx\|_2^2 \geq 0$ ∎

Remark. The assumption that $A = A^*$ is important. For example take $A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ with eigenvalue(s) $\{1\}$. But we see:

for $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $x^* A x = (1\ 1)\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1\ -3)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2$

Exercise 9. Show that the following are equivalent for $A \in M_2(\mathbb{C})$: with $A = A^*$:

$$\{\lambda \mid \lambda \text{ eigenvalue of } A\} \subseteq \mathbb{R}_+ \iff \det(A) \geq 0 \text{ and } Tr(A) \geq 0.$$

Solution. Let $U^* A U = D$ be the diagonalisation of A. We see that:

- $\det(A) = \det(U D U^*) = \det(U)\det(D)\det(U^*) = \det(D)$.
- $Tr(A) = Tr(U D U^*) = Tr(D U^* U) = Tr(D)$.

Hence it suffices to show this for A real diagonal, i.e. $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

In this case it is clear that $\{\lambda_1, \lambda_2\} \subseteq \mathbb{R}_+$, if and only if $\lambda_1 + \lambda_2 \geq 0$ and $\lambda_1 \lambda_2 \geq 0$, if and only if $Tr(D) \geq 0$ and $\det(D) \geq 0$. $\square$

Remark. This does not work for $n \geq 3$. For example take $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Exercise 10. Show that $Tr: M_n(\mathbb{C}) \to \mathbb{C}$ is faithful, i.e if $Tr(A^*A) = 0$ then $A = 0$.

Solution. let $U^*(A^*A)U = D$ the diagonalization and note that:

$$Tr(A^*A) = Tr(A^*AUU^*) = Tr(U^*(A^*A)U) = Tr(D) =$$

$$= \sum_{\lambda \in \sigma(A^*A)} \lambda.$$

However $\sigma(A^*A) \subseteq \mathbb{R}_+$ and so $Tr(A^*A) = 0 \Rightarrow \lambda = 0$ only eigenvalue $\Rightarrow D = 0 \Rightarrow A^*A = 0 \Rightarrow$

$\|A\| = \|A^*A\| = 0 \Rightarrow A = 0.$ $\square$

Exercise 11. Show that, if $\tau: M_2(\mathbb{C}) \to \mathbb{C}$ is an $*$-homomorphism then $\tau = 0$.

Solution. We have: $\tau(I_2) = \tau(I_2^2) = \tau(I_2)^2 \Rightarrow$
$\Rightarrow \tau(I_2) = 0$ or 1.

- If $\tau(I_2) = 0$, then: $\tau(A) = \tau(I_2 A) = \tau(I_2)\tau(A) = 0$ $\forall A \in M_2(\mathbb{C})$
- If $\tau(I_2) = 1$, then: $\tau(E_{ii}) = \tau(E_{ii}^2) = \tau(E_{ii})^2 \Rightarrow$
$\Rightarrow \tau(E_{ii}) = 0$ or 1.

If $\tau(E_{11}) = \tau(E_{22}) = 1$ then $\tau(I_2) = \tau(E_{11}) + \tau(E_{22}) \Rightarrow 1 = 2$
Hence one of them is 0 and the other 1. Assume wlog $\tau(E_{11}) = 1$ and $\tau(E_{22}) = 0$. Then:

$$1 = \tau(E_{11}) = \tau(E_{12}E_{22}E_{21}) = \tau(E_{12})\tau(E_{22})\tau(E_{21}) = 0$$

which is a contradiction. $\square$

Note: Adopt the proof to show the same for $M_n(\mathbb{C})$ $n \geq 2$.