Exercises

Exercise 1 Let S be a set and set

$$\ell^\infty(S) = \{ f : S \to \mathbb{C} \mid \sup_{s \in S} |f(s)| < \infty \}.$$

Equip $\ell^\infty(S)$ with the pointwise operations and with: $\|\cdot\|_\infty : \ell^\infty(S) \longrightarrow \mathbb{R}_+$ such that

$$\|f\|_\infty = \sup_{s \in S} |f(s)|.$$

Show that the pair $(\ell^\infty(S), \|\cdot\|_\infty)$ is a Banach algebra.

Solution. First we note that $\|\cdot\|_\infty$ is well defined.
For $f, g \in \ell^\infty(S)$ and $\lambda \in \mathbb{C}$ we have:

(1) $\|f\|_\infty \geq 0$ by definition.
Moreover $\|f\|_\infty = 0 \iff \sup_{s \in S} |f(s)| = 0 \iff$

$$\iff f(s) = 0 \ \forall s \in S \iff f = 0.$$

(2) $|(\lambda f)(s)| = |\lambda|\,|f(s)| \ \forall s \in S \overset{\sup_{s\in S}}{\Longrightarrow}$

$$\Rightarrow \|\lambda f\|_\infty = |\lambda|\,\|f\|_\infty \Rightarrow \lambda f \in \ell^\infty(S).$$

(3) $|(f+g)(s)| = |f(s) + g(s)| \leq |f(s)| + |g(s)|, \ \forall s \in S$

$$\overset{\sup_{s\in S}}{\Longrightarrow} \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty < \infty \Rightarrow f+g \in \ell^\infty(S).$$

The above show that $\ell^\infty(S)$ is a linear space, and that $\|\cdot\|_\infty$ defines a norm on $\ell^\infty(S)$.

Moreover we have:

$$|(f\cdot g)(s)| = |f(s)\,g(s)| = |f(s)|\,|g(s)| \quad \forall s\in S$$

$$\overset{\sup_{s\in S}}{\Longrightarrow} \quad \|f\cdot g\|_\infty = \sup_{s\in S} |f(s)|\,|g(s)| \leq$$

$$\leq \sup_{s\in S} |f(s)| \cdot \sup_{s\in S} |g(s)| \leq \|f\|_\infty \cdot \|g\|_\infty$$

$\Longrightarrow f\cdot g \in \ell^\infty(S)$.

So $(\ell^\infty(S), \|\cdot\|_\infty)$ is a normed algebra.

Completeness: let $(f_n)_n \subseteq \ell^\infty(S)$ be a $\|\cdot\|_\infty$-Cauchy sequence. Hence $\forall \varepsilon>0$ $\exists n_0$ such that:

$$\|f_n - f_m\|_\infty < \varepsilon \quad \forall n,m \geq n_0 \quad \left(\Longrightarrow |f_n(s) - f_m(s)| < \varepsilon \ . \ \forall n,m\geq n_0 \text{ and } s\in S\right)$$

Thus for every $s\in S$ the sequence $((f_n(s)) \subseteq \mathbb{C}$ is $|\cdot|$-Cauchy, and as $\mathbb{C}$ is complete we have that it has a limit. Therefore the function:

$$f(s) := \lim_n f_n(s) \quad \forall s\in S$$

is well defined.

(a) $f\in \ell^\infty(S)$: For $\varepsilon = 1$ there exists n_0 s.t.

$$\|f_n - f_m\|_\infty < 1 \quad \forall n,m \geq n_0 .$$

So for $n \geq n_0$ and n_0 we have:

$$\|f_n\|_\infty \leq \|f_n - f_{n_0}\|_\infty + \|f_{n_0}\|_\infty < 1 + \|f_{n_0}\|_\infty .$$

Hence $(f_n)_n$ is bounded as:

$$\|f_n\|_\infty \leq \max\{1 + \|f_{n_0}\|_\infty, \|f_1\|_\infty, \dots, \|f_{n_0}\|_\infty\} =: M$$

Now, back to f, we have:

for $s \in S$ and $\varepsilon = 1$ there exists m_0 s.t. $|f(s) - f_{m_0}(s)| < 1$

Then: $|f(s)| \leq |f(s) - f_{m_0}(s)| + |f_{m_0}(s)| \leq 1 + \underbrace{M}_{\text{independent of } s} < \infty$

Thus $\sup_{s \in S} |f(s)| \leq 1 + M \Rightarrow f \in \ell^\infty(S)$.

(b) $f = \|\cdot\|_\infty - \lim_n f_n$: For every $\varepsilon > 0$ $\exists n_0$ s.t.

$$\|f_n - f_m\| < \varepsilon/2 \quad \forall n, m \geq n_0$$

$$\Leftrightarrow \quad |f_n(s) - f_m(s)| < \varepsilon/2 \quad \forall n, m \geq n_0 \quad \forall s \in S$$

for $s \in S$, $\exists m_0$ s.t. $|f(s) - f_{m_0}(s)| < \varepsilon/2 \quad \forall m \geq m_0$.

Choose $m' \geq \max\{n_0, m_0\}$. Then for $n \geq n_0$

we get:

$$|f(s) - f_n(s)| \leq |f(s) - f_{m'}(s)| + |f_{m'}(s) - f_n(s)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence taking $\sup_{s \in S}$ we obtain:

$$\|f - f_n\|_\infty < \varepsilon \quad \forall n \geq n_0.$$

and the solution is complete. ∎

Exercise 2. Let Ω be a topological space and set

$$C_b(X) = \{ f : X \to \mathbb{C} \mid f \text{ continuous}, \sup_{\omega \in \Omega} |f(\omega)| < \infty \}.$$

Equip $C_b(X)$ with the pointwise operations and with: $\|\cdot\|_\infty : C_b(X) \to \mathbb{R}_+$ such that

$$\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|$$

Show that the pair $(C_b(\Omega), \|\cdot\|_\infty)$ is a Banach algebra.

Solution. Since the pointwise operations on continuous (and uniformly bounded) functions give a continuous (and uniformly bounded) function we get that $(C_b(\Omega), \|\cdot\|_\infty)$ is a subalgebra of $(\ell^\infty(\Omega), \|\cdot\|_\infty)$. Hence we just need to show that it is closed. This follows by the fact that the $\|\cdot\|_\infty$-limit of $\|\cdot\|_\infty$-bounded continuous functions is continuous.

Indeed. Let $(f_n)_n \subset C_b(\Omega)$ and $f \in \ell^\infty(\Omega)$ such that

$$\lim_n \|f - f_n\|_\infty = 0.$$

Fix $\omega \in \Omega$ and $\varepsilon > 0$. For $\varepsilon/3$ there exists n_0 such that:

$$\|f - f_{n_0}\|_\infty < \varepsilon/3 . \quad (*)$$

As f_{n_0} is continuous, for $\omega \in \Omega$ and $\varepsilon/3$ there exists an open n'hood V of ω such that:

$$|f_{n_0}(\omega') - f_{n_0}(\omega)| < \varepsilon/3 \quad \forall \omega' \in V.$$

Note that (*) gives:

$$|f(\omega) - f_{n_0}(\omega)| < \varepsilon/3 \text{ and } |f(\omega') - f_{n_0}(\omega')| < \varepsilon/3 \quad \forall \omega' \in \Omega.$$

Therefore for $\omega' \in V$ we get:

$$|f(\omega) - f(\omega')| \le |f(\omega) - f_{n_0}(\omega)| + |f_{n_0}(\omega) - f_{n_0}(\omega')| + |f_{n_0}(\omega') - f(\omega')| < 3 \cdot \varepsilon/3 = \varepsilon.$$

Hence f is continuous (and bounded). $\boxtimes$

<u>Exercise 3.</u> Let Ω be compact + Hausdorff space and set: $C(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ continuous}\}$.

Equip $C(\Omega)$ with the pointwise operations and with $\|\cdot\|_\infty : C(\Omega) \to \mathbb{R}_+$ such that

$$\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|.$$

Show that the pair $(C(\Omega), \|\cdot\|_\infty)$ is a Banach algebra.

<u>Solution</u>. As continuous image of compact is compact (in $\mathbb{C}$), by Heine-Borel we get $C(\Omega) = C_0(\Omega)$. $\boxtimes$

<u>Exercise 4</u>. Let Ω be a locally compact and Hausdorff space. We say that a function $f:\Omega\to\mathbb{C}$ vanishes at infinity if for each $\varepsilon>0$ the set $\{w\in\Omega \mid |f(w)|\geq\varepsilon\}$ is compact. Set

$$C_0(\Omega)=\{f:\Omega\to\mathbb{C} \mid f \text{ continuous and vanishes at infinity}\}$$

Equip $C_0(\Omega)$ with the pointwise operations and with: $\|\cdot\|_\infty : C_0(\Omega)\to\mathbb{R}_+$ such that

$$\|f\|_\infty=\sup_{w\in\Omega}|f(w)|.$$

Show that the pair $(C_0(\Omega),\|\cdot\|_\infty)$ is a Banach algebra.

<u>Solution</u>. First we note that $C_0(\Omega)\subseteq C_b(\Omega)$.
Indeed for $f\in C_0(\Omega)$ and $\varepsilon=1$ we have that the set $K:=\{w\in\Omega \mid |f(w)|\geq 1\}$ is compact.
Since f is continuous we have that

$$\sup_{w\in K}|f(w)|\leq M<\infty \text{ for some } M.$$

On the other hand by definition of K we have

$$\sup_{w\notin K}|f(w)|<1.$$

Hence

$$\sup_{w\in\Omega}|f(w)|\leq\max\{1,M\}<\infty.$$

We next make the observation:

$$C_0(\Omega) = \{ f : \Omega \to \mathbb{C} \mid f \text{ continuous and for every } \varepsilon > 0 \ \exists K \subseteq \Omega \text{ compact such that } |f(\omega)| < \varepsilon \ \forall \omega \notin K \}.$$

This allows to show that $(C_0(\Omega), \|\cdot\|_\infty)$ is a normed subalgebra of $(C_b(\Omega), \|\cdot\|_\infty)$.

For example for $f, g \in C_0(\Omega)$ we have that $f \cdot g$ is continuous. Moreover for $\varepsilon > 0$ there are

$K_1 \subseteq \Omega$ compact s.t. $|f(\omega)| < \sqrt{\varepsilon} \quad \forall \omega \notin K_1$

$K_2 \subseteq \Omega$ compact s.t. $|g(\omega)| < \sqrt{\varepsilon} \quad \forall \omega \notin K_2$.

Hence $K_1 \cap K_2 \subseteq \Omega$ is compact and for $\omega \notin (K_1 \cap K_2)$ we get $|(f \cdot g)(\omega)| = |f(\omega)| \cdot |g(\omega)| < \varepsilon$.

It remains to show that $(C_0(\Omega), \|\cdot\|_\infty)$ is closed in $(C_b(\Omega), \|\ \|)$. Let $(f_n)_n \subseteq C_0(\Omega)$ and $f \in C_b(\Omega)$ s.t. $\lim_n \|f - f_n\|_\infty = 0$.

Fix $\varepsilon > 0$. Then there is n_0 s.t. $\|f - f_{n_0}\|_\infty < \varepsilon/2$.

For f_{n_0} and $\varepsilon/2 > 0$ there is $K \subseteq \Omega$ compact such that $|f(\omega)| < \varepsilon/2 \ \forall \omega \notin K$. Hence for $\omega \notin K$ we get:

$$|f(\omega)| \leq |f(\omega) - f_{n_0}(\omega)| + |f_{n_0}(\omega)| \leq$$
$$\leq \|f - f_{n_0}\| + |f_{n_0}(\omega)| < 2 \cdot \varepsilon/2 = \varepsilon. \quad \boxtimes$$

Exercise 5 Let $(X, \|\ \|_X)$ be a Banach space and set $\|T\| = \sup_{\|x\| \le 1} \|T(x)\|_X$ for $T : X \to X$ linear.

Recall that $\|\cdot\|$ is a norm on

$$B(X) = \{T : X \to X \mid T \text{ linear}, \|T\| < \infty\}$$

where $B(X)$ is equipped with the pointwise operations for linear structure and composition for the multiplication. Then $(\mathbb{B}(X), \|\ \|)$ is a Banach algebra.

Solution. Adapt the proof of exercise 1 for the set $X_1 := \{x \in X \mid \|x\|_X \le 1\}$.

Also note that uniqueness of limits gives that the pointwise limit of linear operators is a linear operator. $\square$