Introduction

In Functional Analysis we study spaces X with linear structure and norm.

(All l.sp. over $\mathbb{C}$). X is a Banach space if it is complete wrt norm.

F.A. goal: We know X if we know its dual $X^* = \{ f : X \to \mathbb{C} \mid f \text{ lin + bdd} \}$.

Because by HB $X \hookrightarrow X^{**}$ by $x \longmapsto \hat{x}$ with $\hat{x}(x^*) = x^*(x)$.

But X^{**} is a function space (lin.)

When $\|\cdot\|$ on X comes from inner product then X becomes a Hilbert space.

We write H for Hilbert spaces.

Now H is well-understood so shift goes to $B(H) = \{ T : H \to H \mid T \text{ lin + bdd} \}$.

Example $M_n(\mathbb{C})$. for $H = \mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$

For every $T \in B(H)$ there exists T^* s.t.

$$\langle Tx, y \rangle_H = \langle x, T^* y \rangle$$

__Example__: $T \in M_n$ then $T^* = (\overline{T}_{ji})_{ij}$.

It follows that $B(H)$ satisfies C^*-identity

$$\| T^* T \| = \| T \|^2 .$$

__C^*-algebra__: $(C, \|\cdot\|)$ Banach *algebra s.t

$$\| c^* c \| = \| c \|^2 \quad \forall c .$$

__Goal of class__: Describe $(C, \|\cdot\|)$.

__Answer__: Every C^*-algebra is a closed $*$-subalgebra of some $B(H)$.

Concrete picture	abstract picture
$\subseteq B(H)$	C^*-identity.

__Moreover__: C abelian C^*-algebra iff $C \simeq C_0(X)$.

To answer the first (last) we will need to look at:

- set of "positive" elements in C (answer: of the form c^*c.)
- positive functionals on C, i.e. $f(cc^*) \in \mathbb{R}_+$.
- construct H s.t. $C \hookrightarrow B(H)$.

But for this we need to make sense of positivity, and first show the second.

This uses a more general theory of commutative Banach algebras:

- Spectrum $\sigma(a) = \{\lambda : \lambda I - a \notin Inv(A)$

<u>Example</u>: $a \in M_n$ then $\sigma(a) =$ eigenvalues.

- pivotal in this is the one-dim representations (non-zero).

$\tau : A \longrightarrow \mathbb{C}$

$\rightsquigarrow$ connections with spectral theory.

Further goals:

- Import spectral theorem for normal operators from M_n to $B(H)$
- Import notion of compact operators

ex. $c_0 \subseteq \ell^\infty$.

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Why? and where do next?

- Trend of studying objects by repn's on H. rather than in themselves. (groups, graphs etc)
- Relaxing conditions on morphisms (completely positive maps systems, operator algebras)
- Abstract definition of operator subspaces of some $B(H)$.