The mean ergodic theorem of von Neumann: a very elementary proof

Let [1] $T : H \to H$ be any map on a Hilbert space and let

$$Fix(T) = \{x \in H : Tx = x\}$$

be the fixed point set of T. If $x \in Fix(T)$, then the iterates $T^k x$ are all equal to x, hence $T^k x \to x$. On the other hand, if T is continuous and $(T^n x)$ converges to some y, then $Ty = T(\lim_n T^n x) = \lim_n T(T^n x) = \lim_n T^{n+1} x = y$, so $y \in Fix(T)$.

But, even for a unitary operator T, it can happen that the sequence $(T^n x)$ converges only in the trivial case $x = 0$. Example: the bilateral shift.

The situation is much better if we take averages:

Theorem 1 *Let $T \in \mathcal{B}(H)$ be a contraction. If*

$$S_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k \quad (n = 0, 1, \dots)$$

are the averages of the iterates T^k of T, then

(i) (S_n) converges strongly (i.e. pointwise) and

(ii) its limit is the orthogonal projection F onto the fixed point set

$$Fix(T) = \ker(I - T) = \{x \in H : Tx = x\}.$$

Proof. *(a)* Suppose first that $x = (I - T)(H)$, hence there exists $y \in H$ with $x = (I - T)y$. Then for each $k \in \mathbb{Z}_+$ we have $T^k x = T^k y - T^{k+1} y$, therefore

$$S_n x = \frac{1}{n}(y - T^n y)$$

$$\text{hence} \qquad \|S_n x\| \le \frac{1}{n} \|y - T^n y\| \le \frac{2\|y\|}{n} \to 0.$$

Thus $S^n x \to 0$ for all $x = (I - T)(H)$.

(b) It follows that for all $x \in \overline{(I - T)(H)}$ we have $S^n x \to 0$. Indeed given $\varepsilon > 0$ choose $z = (I - T)y \in (I - T)(H)$ so that $\|x - z\| < \varepsilon$, and then choose $n_0 \in \mathbb{N}$ such that $\|S_n z\| < \varepsilon$ for all $n \ge n_0$.

If $n \ge n_0$ then, since each S_n is a contraction,

$$\|S_n x\| \le \|S_n(x - z)\| + \|S_n z\| \le \|x - z\| + \|S_n z\| < 2\varepsilon.$$

(c) It remains to consider the case $x \in \overline{(I - T)(H)}^{\perp} = \ker(I - T^*)$, i.e. $x = T^* x$. But then $x = Tx$: indeed

$$\begin{aligned}\|x - Tx\|^2 &= \|x\|^2 + \|Tx\|^2 - 2\,\mathrm{Re}\,\langle x, Tx\rangle \\ &= \|x\|^2 + \|Tx\|^2 - 2\,\mathrm{Re}\,\langle T^* x, x\rangle = \|x\|^2 + \|Tx\|^2 - 2\|x\|^2 \le 0\end{aligned}$$

because T is a contraction; hence $\|x - Tx\|^2 = 0$.

Thus $x \in Fix(T)$ and so, as noted above, $S_n x = x$ for all n, hence $\lim S_n x = x$.

Therefore for all $x \in H$,

$$\lim_n S_n x = \lim_n S_n F x + \lim_n S_n F^{\perp} x = Fx + 0. \qquad \square$$

See also the very interesting blog, Terry Tao: The mean Ergodic Theorem.

[1] meanergo, A. Katavolos, 31 Jan 2012