Shifts, Contractions, Dilations, . . .

Notes by A.K., January 2012

Contents

1 Preliminaries

1.1 Reminder

If[1] $f : \mathbb{T} \to \mathbb{C}$ is a Borel function and $1 \le p < \infty$, we say $f \in L^p(\mathbb{T})$ if

$$\|f\|_p^p \equiv \int_0^{2\pi} |f(e^{ix})|^p \frac{dx}{2\pi} \equiv \int |f|^p dm < +\infty$$

and we say $f \in L^\infty(\mathbb{T})$ if f is essentially bounded, which means that there is an $M > 0$ s.t. the set $X_M = \{e^{ix} : |f(e^{ix})| > M\}$ has measure zero[2]; the least such M is denoted $\|f\|_\infty$.

We identify functions when they are almost everywhere (a.e.) equal, that is, when they differ on a set of measure zero. Thus

$$C(\mathbb{T}) \subsetneqq L^\infty(\mathbb{T}) \subsetneqq L^2(\mathbb{T}) \subsetneqq L^1(\mathbb{T}).$$

For $f \in L^1(\mathbb{T})$ define

$$\hat{f}(n) = \int_0^{2\pi} f(e^{ix}) e^{-inx} dm(x), \qquad n \in \mathbb{Z}.$$

The map

$$\mathcal{F} : f \to (\hat{f}(n))_{n\in\mathbb{Z}}$$

is the Fourier transform.

Proposition 1.1 *If $f \in L^1(\mathbb{T})$ satisfies $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$ (a.e.).*

Note that $L^2(\mathbb{T})$ is a Hilbert space for the scalar product

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{ix}) \overline{g(e^{ix})} dm(x)$$

and the family

$$\{\zeta_n : n \in \mathbb{Z}\} \quad \text{where} \quad \zeta_n(e^{ix}) = e^{inx}$$

is orthonormal: $\langle \zeta_n, \zeta_m \rangle = \delta_{nm}$.

Proposition 1.1 shows that no nonzero element of L^2 can be orthogonal to the family $\{\zeta_n : n \in \mathbb{Z}\}$: hence it must be *an orthonormal basis of L^2*.

Therefore for each $f \in L^2$ we have

$$f = \sum_{n=-\infty}^{+\infty} \hat{f}(n)\zeta_n \qquad (L^2 \text{ convergence})$$

$$\text{and} \ \ \|f\|_2^2 = \sum_{n=-\infty}^{+\infty} |\hat{f}(n)|^2 \qquad \text{(Parseval)}.$$

[1] notes11, 15 Jan. 2012

[2] that is, given any $\varepsilon > 0$, the set X_M can be covered by a countable number of intervals of total length at most ε

1.2 The space H^2

Definition 1 *For $1 \leq p \leq \infty$,*

$$H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \hat{f}(-k) = 0 \text{ for all } k = 1, 2, \ldots\}.$$

Given $f \in H^2(\mathbb{T})$, consider the power series

$$\tilde{f}(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

Since $\sum_{n=0}^{+\infty} |\hat{f}(n)|^2 = \|f\|_2^2 < \infty$, and so $\limsup |\hat{f}(n)| \leq 1$, the power series has radius of convergence *at least 1*, hence converges in the open unit disc $\mathbb{D}$ and defines an analytic function $\tilde{f} : \mathbb{D} \to \mathbb{C}$. Conversely, if an analytic function $g : \mathbb{D} \to \mathbb{C}$ has a power series $g(z) = \sum a_n z^n$ such that the coefficients satisfy $\sum |a_n|^2 < \infty$, then (by completeness of L^2) we may define $g^* \in L^2(\mathbb{T})$ by $g^* = \sum a_n \zeta_n$ and we find that $\widehat{g^*}(n) = \langle g^*, \zeta_n \rangle = a_n$ when $n \geq 0$ while $\widehat{g^*}(-k) = 0$ for $k = 1, 2, \ldots$. Thus $g^* \in H^2(\mathbb{T})$ and $\widetilde{g^*} = g$.

Using the linear map $f \to \tilde{f}$ and its inverse, $g \to g^*$ we identify $H^2(\mathbb{T})$ with the space $H^2(\mathbb{D})$ of all analytic functions on the disc with square-summable power series.

It can be shown that the "boundary function" may be obtained directly from g as follows:

Theorem 1.2 (Fatou) *If $g \in H^2(\mathbb{D})$, then for* almost all $e^{ix} \in \mathbb{T}$ *the limit* $\lim_{r \nearrow 1} g(re^{ix})$ *exists and equals $g^*(e^{ix})$.*

2 Invariant subspaces

Definition 2 *If H is a Hilbert space and $T : H \to H$ is bounded linear. i.e. $T \in B(H)$, a closed linear subspace $E \subseteq H$ is called T-**invariant** if $T(E) \subseteq E$.*

Let $T : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ be defined by

$$Tf = \zeta_1 f \qquad (f \in H^2(\mathbb{T})),$$

where $\zeta_1(z) = z^1$ $(z \in \mathbb{T})$. Note that T is an isometry[3] (so $T^*T = I$) but is not onto, since $\zeta_0 \perp T(H^2)$. Since $T(\zeta_n) = \zeta_{n+1}$ we have in fact

$$\bigcap_{n \geq 0} T^n(H^2) = \{0\}.$$

Indeed, $\zeta_k \perp T^n(H^2)$ for all $k < n$. Hence if $f \in \bigcap_{n \geq 0} T^n(H^2)$ then $f \perp \zeta_k$ for all $k \in \mathbb{Z}_+$ and so, since $\{\zeta_n : n \in \mathbb{Z}_+\}$ is an orthonormal basis of H^2, it follows that $f = 0$.

Now let $\phi \in H^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$. Note that since $|\phi| = 1$ a.e., ϕ defines a bounded, in fact an isometric operator T_ϕ on H^2 by the formula [4]

$$T_\phi f = \phi f, \qquad f \in H^2.$$

Therefore the set

$$\phi H^2 = \{\phi f : f \in H^2\}$$

is a closed subspace of H^2 because T_ϕ is isometric.

[3] *Exercise*: Note that T^* is *not* "multiplication by $\bar{\zeta}_1$" (which does not preserve H^2); what is it?
[4] *Exercise*: Why does $f \to \phi f$ map H^2 into H^2?

Also, ϕH^2 is T-invariant:

$$T(\phi H^2) = \zeta_1 \phi H^2 = \phi(\zeta_1 H^2) \subseteq \phi H^2$$

because $\zeta_1 H^2 \subseteq H^2$.
In fact,

$$\bigcap_{n\geq 0} T^n(\phi H^2) \subseteq \bigcap_{n\geq 0} T^n(H^2) = \{0\}.$$

A function $\phi \in H^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ is called an *inner function* Examples are: ζ^n ($n \in \mathbb{N}$) and $f(z) = \exp \frac{z-1}{z+1}$.

Theorem 2.1 (Beurling) *A closed nonzero subspace $E \subseteq H^2(\mathbb{T})$ is T-invariant if and only if there exists $\phi \in H^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2$. Moreover, ϕ is essentially unique in the sense that if $E = \psi H^2$ where $|\psi| = 1$ a.e. then $\frac{\phi}{\psi}$ is (a.e. equal to) a constant (of modulus 1).*

Proof. Suppose that $E \subseteq H^2$ is a closed nonzero T-invariant subspace. The space $T(E)$ is a closed subspace of E because T is isometric. Moreover, $T(E) \neq E$ because

$$\bigcap_{n\geq 0} T^n(E) \subseteq \bigcap_{n\geq 0} T^n(H^2) = \{0\}.$$

Thus there exists $\phi \in E$ of norm 1, such that $\phi \perp T(E)$.

Claim 1. The sequence $\{\phi, T(\phi), T^2(\phi), \dots\}$ is an orthonormal sequence in E.
Proof. Since $\phi \in E$ which is T-invariant we have $T^n(\phi) \in E$ for all $n \in \mathbb{N}$. Moreover $\|T^n(\phi)\|_2 = \|\phi\|_2 = 1$. Also, if $m, n \in \mathbb{N}$ with $m > n$ we have

$$T^m(\phi) \in T^m(E) \subseteq T^{n+1}(E) = T^n(T(E)).$$

Thus $T^m(\phi) \in T^n(T(E))$. But $T^n(\phi) \perp T^n(T(E))$ since $\phi \perp T(E)$ by construction and T^n is isometric. Thus

$$T^n(\phi) \perp T^m(\phi). \qquad \square$$

Claim 2. For all nonzero $k \in \mathbb{Z}$ we have $\displaystyle\int \zeta_k |\phi|^2 dm = 0.$
Proof. For $k > 0$,

$$\int \zeta_k |\phi|^2 dm = \int (\zeta_k \phi)\bar{\phi} dm = \langle \zeta_k \phi, \phi \rangle = \left\langle T^k(\phi), \phi \right\rangle = 0$$

by the previous claim. For $k = -n < 0$,

$$\int \zeta_k |\phi|^2 dm = \int \phi(\overline{\zeta_n \phi}) dm = \langle \phi, \zeta_n \phi \rangle = \langle \phi, T^n(\phi) \rangle = 0. \qquad \square$$

It follows from this claim that the function $\psi = |\phi|^2$, which is in L^1, satisfies $\hat{\psi}(k) = 0$ for all $k \in \mathbb{Z}$ except $k = 0$. By Proposition 1.1, ψ must be a multiple of $\zeta_0 = \mathbf{1}$ and hence a.e. equal to a constant. Hence so is $|\phi|$. Since $\int |\phi|^2 dm = 1$, the constant must be 1.

This shows that $|\phi(z)| = 1$ a.e.

Claim 3. $E = \phi H^2$.

Proof. First, $\phi H^2 = T_\phi(H^2)$ and T_ϕ is an isometry since $|\phi| = 1$ a.e.
Since $\{\zeta_n : n \in \mathbb{Z}_+\}$ is an orthonormal basis of H^2, the set

$$\{T_\phi\zeta_0, T_\phi\zeta_1, T_\phi\zeta_2, \dots\} = \{\phi, \zeta_1\phi, \zeta_2\phi, \dots\} = \{\phi, T(\phi), T^2(\phi), \dots\}$$

is an orthonormal basis of ϕH^2, and is contained in E since $\phi \in E$ which is T-invariant. We conclude that $\phi H^2 \subseteq E$.

To prove that equality in fact holds, suppose $f \in E$ is orthogonal to ϕH^2; we show that $f = 0$. Indeed, for all $n = 0, 1, 2, \dots$ we have

$$f \perp \phi\zeta_n \quad \Rightarrow \quad \int f\overline{\phi\zeta_n}dm = 0 \quad \Rightarrow \quad \int f\bar{\phi}\zeta_{-n}dm = 0.$$

On the other hand if $k = 1, 2, \dots$ then $\langle \zeta_k f, \phi \rangle = 0$ since $\zeta_k f = T^k(f) \in T^k(E) \subseteq T(E)$ while $\phi \perp T(E)$ by definition; thus

$$0 = \langle \zeta_k f, \phi \rangle = \int \zeta_k f\bar{\phi}dm.$$

This shows that the L^2 function $f\bar{\phi}$ has all its Fourier coefficients equal to 0 and hence must vanish (a.e.). Since $|\phi| = 1$ a.e. this shows that $f = 0$.

Uniqueness

If $\phi H^2 = \psi H^2$ where $|\phi| = |\psi| = 1$ a.e. then $\overline{\psi}\phi H^2 = H^2$, so that $\overline{\psi}\phi = \overline{\psi}\phi\mathbf{1} \in H^2$. Similarly $\overline{\phi}\psi H^2 = H^2$, so that $\overline{\phi}\psi \in H^2$. Thus the function $h = \overline{\psi}\phi$ and its complex conjugate are both analytic, which can only happen if h is a constant (alternatively, $h \in H^2$ means $\hat{h}(-n) = 0$ for $n = 1, 2, \dots$ while $\bar{h} \in H^2$ means $\hat{h}(+n) = 0$ for $n = 1, 2, \dots$; hence h is constant).

This concludes the proof of the Theorem. $\square$

Remark 2.2 *Note the dual role played by ϕ (and also by ζ_1) in the above proof: On the one hand ϕ is a vector in H^2 and is moved around by the operator T (we say ϕ is a* wandering vector*), and on the other it "is" an operator T_ϕ acting on the space H^2.*

3 Shifts

Definition 3 *A closed subspace L of a Hilbert space H is said to be* **wandering** *for an isometry $A \in \mathcal{B}(H)$ if the subspaces $L, A(L), A^2(L), \dots$ are pairwise orthogonal.*

Notation If $\{M_n\}$ is a family of pairwise orthogonal closed subspaces of a Hilbert space H, the orthogonal direct sum

$$\bigoplus_{n=0}^{\infty} M_n = M_0 \oplus M_1 \oplus M_2 \oplus \dots$$

is the smallest closed subspace $\bigvee M_n$ of H containing each M_n. This consists of all ξ of the form

$$\xi = \sum_{n=0}^{\infty} \xi_n \qquad \text{with } \xi_n \in M_n \text{ and } \quad \sum_{n=0}^{\infty} \|\xi_n\|^2 < \infty.$$

Thus if L is an A-wandering subspace we may form the orthogonal direct sum

$$\bigoplus_{n=0}^{\infty} A^n(L) := M_+(L).$$

Remark 3.1 *Note that we may recover the wandering subspace from $M_+(L)$:*

$$L = M_+(L) \ominus A(M_+(L)) := M_+(L) \cap A(M_+(L))^{\perp}.$$

Indeed, L is contained in $M_+(L)$ and is orthogonal to each $A^{n+1}(L)$, $(n \geq 0)$, hence to their orthogonal direct sum, which is $A(M_+(L))$; and conversely, if a vector $\xi = \sum_{k\geq 0} A^k x_k$ is in $M_+(L)$ (i.e. each x_k is in L) and is orthogonal to $A(M_+(L))$ hence to all $A^{n+1}(L)$, then for all $\eta \in L$ and $n \geq 0$ we have

$$0 = \left\langle \xi, A^{n+1}\eta \right\rangle = \sum_{k\geq 0} \left\langle A^k x_k, A^{n+1}\eta \right\rangle = \left\langle A^{n+1}x_{n+1}, A^{n+1}\eta \right\rangle = \langle x_{n+1}, \eta \rangle$$

and so $x_{n+1} = 0$; hence $\xi = x_0 \in L$.

Definition 4 *A* ***(unilateral) shift*** *on a Hilbert space H is a map $S \in \mathcal{B}(H)$ such that*
(a) $\|Sx\| = \|x\|$ for all $x \in H$ (S is an isometry) and
(b) There is an S-wandering subspace L such that $M_+(L) = H$.
The number $\dim L$ *is called the* **multiplicity** *of the shift.*

Note that, by Remark 3.1, the wandering subspace L is uniquely determined by S, and in fact, since $M_+(L) = H$,

$$L = H \ominus S(H) = S(H)^{\perp} = \ker(S^*).$$

Thus the multiplicity of a shift is uniquely defined.
Conversely,

Remark 3.2 *Two shifts $S \in B(H)$ and $S_1 \in B(H_1)$ are unitarily equivalent if and only if their wandering subspaces L and L_1 are of the same dimension.*
Thus the number $\dim L$ *uniquely determines S up to unitary equivalence.*

Indeed, if L and L_1 have the same dimension, choose any unitary $U : L \to L_1$ and define

$$V : H \to H_1 : \sum S^n(x_n) \to \sum S_1^n(Ux_n).$$

It is clear that V is invertible:

$$V^{-1}\left(\sum S_1^n(y_n)\right) = \sum S^n(U^{-1}y_n)$$

and it is isometric because

$$\left\|\sum S_1^n(Ux_n)\right\|^2 = \sum \|S_1^n(Ux_n)\|^2 = \sum \|x_n\|^2 = \left\|\sum S^n(x_n)\right\|^2.$$

For example the operator $Tf(z) = zf(z), \quad f \in H^2$ is a shift.[5] The vector ζ is a *wandering vector* for T, i.e. the family $\{T^n\zeta : n \in \mathbb{Z}_+\}$ is orthogonal.

We will need an easy observation

Remark 3.3 *If $T \in B(H)$ is an isometry and P a projection, then the projection onto $TP(H)$ is TPT^*.*

Proof. If $\xi = T(\eta) \in TP(H)$ then $TPT^*\xi = TPT^*T\eta = TP\eta = T\eta = \xi$ (since $T^*T = I$ and $\eta \in P(H)$), and if $\zeta \perp TP(H)$ then $TPT^*\zeta = 0$ since for all $\xi \in H$ we have $\langle TPT^*\zeta, \xi\rangle = \langle \zeta, TPT^*\xi\rangle = 0$ because $P(T^*\xi) \in P(H)$ so $TPT^*\xi \in TP(H)$. □

Theorem 3.4 (Wold Decomposition) *If $A \in \mathcal{B}(H)$ is an isometry, there is a unique decomposition $H = H_s \oplus H_u$ into A-reducing subspaces such that the restriction A_s of A to H_s is a shift (if nonzero) and the restriction A_u of A to H_u is unitary (if nonzero).*

*Moreover, if $L = H \ominus A(H) = A(H)^\perp = \ker A^*$, then L is an A-*wandering *subspace, i.e. the family $\{A^n(L) : n \in \mathbb{Z}_+\}$ is a family of closed mutually orthogonal subspaces.*

We have

$$H_s = M_+(L) = \bigoplus_{n\ge 0} A^n(L) = \{x \in H : A^{*n}x \to 0\} \qquad \textit{and} \qquad H_u = \bigcap_{n\ge 0} A^n(H).$$

Proof. (i) If $A^n x \in A^n(L)$ and $A^m y \in A^m(L)$ with $k = n - m > 0$ then, since A^m is isometric,

$$\langle A^n x, A^m y\rangle = \left\langle A^m A^k x, A^m y\right\rangle = \left\langle A^k x, y\right\rangle = 0$$

because $A^k x \in A(H)$ (since $k \ge 1$) while y is in L which is orthogonal to $A(H)$. Thus $A^n(L) \perp A^m(L)$.

(ii) Define $\quad H_s = \bigoplus_{n\ge 0} A^n(L)$. We show that

$$H_s = \{x \in H : A^{*n}x \to 0\}. \tag{**}$$

If $P(L)$ is the projection onto $L = (A(H))^\perp$, then $P(L) = I - AA^*$; by Remark 3.3 the projection $P(A^n(L))$ onto $A^n(L)$ is $A^n P(L) A^{*n} = A^n(I - AA^*)A^{*n}$. Now $x \in H_s$ if and only if

$$x = \sum_{n=0}^{\infty} P(A^n(L))x = \lim_{N\to\infty} \sum_{n=0}^{N-1} A^n(I - AA^*)A^{*n}x = x - \lim_{N\to\infty} A^N A^{*N} x \tag{*}$$

[5] Observe that under the unitary $\mathcal{F} : H^2 \to \ell^2 : \zeta_n \to e_n$ (of course $\mathcal{F}$ is the restriction to H^2 of the Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$) the operator T is transformed into the (multiplicity one) shift $S : \ell^2 \to \ell^2$ given by $Se_n = e_{n+1}$.

i.e. if and only if $\lim_{N\to\infty} A^N A^{*N} x = 0$, equivalently if and only if $\lim_N \|A^{*N}x\| = 0$ (A^N is an isometry).

This shows (∗∗).

On the other hand, $y \perp H_s$ if and only if $y \perp A^n(L)$ for all $n \geq 0$, equivalently if

$$0 = P(A^n(L))y = A^n(I - AA^*)A^{*n}y \iff A^nA^{*n}y = A^{n+1}A^{*n+1}y$$

for all n, and so $y = A^nA^{*n}y$ for all n. But A^nA^{*n} is the projection onto $A^n(H)$. Therefore $y \perp H_s$ if and only if $y \in A^n(H)$ for all $n \geq 0$. In other words, $y \perp H_s$ iff $y \in \bigcap_{n\geq 0} A^n(H)$. We have shown that

$$(H_s)^{\perp} = \bigcap_{n\geq 0} A^n(H) := H_u.$$

(iii) If $P = P(H_u)$ then, for all $x \in H$, $\quad Px = \lim_n A^nA^{*n}x$. Thus

$$P(AH_u)x = APA^*x = A\lim_n A^nA^{*n}A^*x = \lim_n A^{n+1}A^{*(n+1)}x = Px$$

$$\text{and so} \quad P(AH_u) = APA^* = P \quad \text{hence} \quad PA = APA^*A = AP.$$

The second relation shows that A reduces H_u and the first relation shows that A maps H_u onto H_u. Hence $A|_{H_u}$ is a unitary operator on H_u.

Finally, $H_s = H_u^{\perp}$ also reduces A.

Uniqueness It remains to prove that if $H = K_s \oplus K_u$, is an arbitrary decomposition so that $A|_{K_s}$ is a shift and $A|_{K_u}$ is unitary, then $K_u = H_u$ and $K_s = H_s$. But if $A|_{K_s}$ is a shift then $L' := K_s \ominus A(K_s)$ is A-wandering; and it will be enough to prove that $L = L'$, for then $K_s = M_+(L') = M_+(L) = K_s$ and their orthogonal complements will also be equal. Now [6]

$$L = H \ominus AH = (K_u \oplus K_s) \ominus (AK_u \oplus AK_s) = (K_u \oplus K_s) \ominus (K_u \oplus AK_s) = K_s \ominus AK_s = L'.$$

Remark 3.5 *It follows that an isometry $A \in \mathcal{B}(H)$ is a shift if and only if it satisfies $\|A^{*n}x\| \to 0$ for all $x \in H$. Equivalently if and only if $\bigcap_{n\geq 0} A^n(H) = 0$.*

[6]In more detail: Given $x \in L$ write $x = x_s + x_u$ with $x_s \in K_s$ and $x_u \in K_u$. But $x \perp A(K_u \oplus K_s)$ and $A(K_u \oplus K_s) = AK_u \oplus AK_s = K_u \oplus AK_s$ (note that $A(K_u) = K_u$ since $A|_{K_u}$ is unitary). Thus $x \perp K_u$ so $x = x_s \in K_s$. But also $x \perp A(K_s)$, so $x \in K_s \ominus A(K_s) \subseteq L'$. This shows that $L \subseteq L'$; the same argument using the decomposition $H = H_s \oplus H_u$ yields $L' \subseteq L$.

4 The Beurling - Lax - Halmos Theorem

We wish to generalise Beurling's Theorem (Theorem 2.1) to characterise invariant subspaces of a shift of arbitrary multiplicity.

In the multiplicity one case, invariant subspaces M were shown to be of the form $M = T_\phi(H^2)$ where $\phi \in H^2$ was a suitable function. Note that T_ϕ is an isometry which commutes with the operator T. This is the form of Beurling's Theorem that generalises. Indeed, it can be shown that, conversely, any isometry $A \in B(H^2)$ which commutes with T is necessarily of the form $A = T_\phi$ (see [2, Problem 242]).

Let E be a Hilbert space and define

$$H = E \otimes \ell^2 = \{\xi = (x_n) : x_n \in E,\ \sum_{n\geq 0} \|x_n\|_E^2 < \infty\}.$$

This is a Hilbert space with scalar product

$$\langle (x_n), (y_n) \rangle = \sum_n \langle x_n, y_n \rangle_E$$

(the sum converges absolutely). Completeness is proved just like the case of ℓ^2.

We denote the sequence $(0, \dots, 0, x, 0, \dots)$ (with x at the n-th place) by the symbol $x \otimes e_n$; the linear span of $\{x \otimes e_n : x \in E, n \in \mathbb{Z}_+\}$ is dense in H [7] and

$$(x_n) = \sum_{n=0}^{\infty} x_n \otimes e_n.$$

Let $S \in B(H)$ be given by

$$S((x_0, x_1, x_2, \dots)) = (0, x_0, x_1, x_2, \dots)$$

i.e.

$$S(x \otimes e_n) = x \otimes e_{n+1} \quad (x \in E, n \in \mathbb{Z}_+).$$

This is an isometry, called **the unilateral shift of multiplicity** $\dim E$.

Suppose $V \in B(H)$ is a partial isometry. Let $M = V(H)$. This is a closed subspace since V is isometric on the orthogonal complement of its kernel.

Remark 4.1 *If $VS = SV$ then M is S-invariant.*

Proof. If $\xi \in M$ there exists $\eta \in H$ such that $\xi = V\eta$. Then

$$S(\xi) = S(V\eta) = V(S\eta) \in V(H) = M. \qquad \square$$

Conversely,

Theorem 4.2 *Let $M \subseteq H$ be a closed S-invariant subspace. Then there exists a partial isometry $V \in B(H)$ which commutes with S such that*

$$M = V(H).$$

We will need the following

Lemma 4.3 *If H is a separable Hilbert space and $P \in B(H)$ is a projection, then for any orthonormal basis $\{f_i : i \in I\}$ of H we have*

$$\dim P(H) = \sum_{i \in I} \|Pf_i\|^2.$$

[7] H is the orthogonal direct sum of its subspaces $E_n := \{x \otimes e_n : x \in E\}$ which are all isomorphic to E.

Proof. Let $\{y_k : k \in K\}$ be an orthonormal basis of $P(H)$. Since each Pf_i is in $P(H)$, by Parseval we have

$$\|Pf_i\|^2 = \sum_{k\in K} |\langle Pf_i, y_k\rangle|^2$$

and so
$$\sum_{i\in I}\|Pf_i\|^2 = \sum_{i\in I}\sum_{k\in K} |\langle Pf_i, y_k\rangle|^2 = \sum_{k\in K}\sum_{i\in I} |\langle f_i, Py_k\rangle|^2 = \sum_{k\in K}\sum_{i\in I} |\langle f_i, y_k\rangle|^2 = \sum_{k\in K}\|y_k\|^2$$

by Parseval again, since $\{f_i : i \in I\}$ is an orthonormal basis of H. But the last sum equals the cardinality of K, i.e. the dimension of $P(H)$. □

Proof of the Theorem. Define

$$L = M \ominus S(M) = M \cap (SM)^{\perp}.$$

This is a nonzero subspace because $S(M) \neq M$. Indeed, if $m \in \mathbb{Z}_+$ is the smallest integer for which there exists $\xi = (x_n) \in M$ with $x_m \neq 0$, then all $S(\eta) \in S(M)$ have their first m coordinates equal to 0 and so $\xi \notin S(M)$.

Let P be the projection onto M and let Q be the projection onto L. Then SPS^* is the projection onto $S(M)$ (Remark 3.3) and so

$$Q = P - SPS^*.$$

Claim 1. L is a wandering subspace, i.e. the subspaces $S^n(L)$, $n \geq 0$ are pairwise orthogonal.

Proof. [8] If $S^m(\xi) \in S^m(L)$ and $S^n(\eta) \in S^n(L)$ with $k = m - n > 0$ then

$$\langle S^m(\xi), S^n(\eta)\rangle = \left\langle S^n S^k(\xi), S^n(\eta)\right\rangle = \left\langle S^k(\xi), \eta\right\rangle \qquad (S^n \text{ is isometric}).$$

But $S^k(\xi) \in S(M)$ (note $k \geq 1$) because $S(\xi) \in S(M)$ which is S-invariant; on the other hand $\eta \in L$ and $L \perp S(M)$, so $\langle S^k(\xi), \eta\rangle = 0$. □

Thus we may form the sum

$$N = \bigoplus_{n=0}^{\infty} S^n(L) = L \oplus S(L) \oplus S^2(L) \oplus \dots$$

This consists of all ξ of the form

$$\xi = \sum_{n=0}^{\infty} S^n(\xi_n) \qquad \text{with } \xi_n \in L \text{ and } \quad \sum_{n=0}^{\infty} \|S^n\xi_n\|^2 = \sum_{n=0}^{\infty}\|\xi_n\|^2 < \infty.$$

Claim 2. $N = M$.

Proof. Since $L \subseteq M$ so $S^n(L) \subseteq S^n(M) \subseteq M$, we see that M contains each $S^n(L)$, hence it must contain N.

Now take $\xi \in M \cap N^{\perp}$. Then $\xi \in M$ and $\xi \perp L$; thus $Q\xi = 0$, i.e. $(P - SPS^*)\xi = 0$ and so $\xi = P\xi = SPS^*\xi$. Thus $\xi \in S(M)$; but $\xi \in (S(L))^{\perp}$ so $SQS^*\xi = 0$, i.e. $(SPS^* - S^2PS^{*2})\xi = 0$ and so $\xi = SPS^*\xi = S^2PS^{*2}\xi$, i.e. $\xi \in S^2(M)$. Continuing

[8] This generalises the argument in part (i) of the proof of Theorem 3.4.

inductively, we conclude that $\xi \in S^n(M)$ i.e. $\xi = S^n P S^{*n}\xi$ for all $n \geq 0$. But then $\xi = 0$, because if $\xi = \sum_{k\geq 0} x_k \otimes e_k$ then $S^{*n}\xi = \sum_{k\geq n} x_k \otimes e_{k-n}$ so

$$\|\xi\|^2 = \|S^n P S^{*n}\xi\|^2 \leq \|S^{*n}\xi\|^2 = \sum_{k\geq n} \|x_k\|^2 \to 0\,. \qquad \square$$

Claim 3. There exists a partial isometry $U : H \to E$ with initial space L.

Proof. It is enough to prove that there exists an isometry $W : L \to E$; then $U : H \to E$ will be the extension of W to H, defined by setting $U(\xi) = 0$ for $\xi \in L^\perp$.

Now the existence of an isometry $W : L \to E$ will follow if we prove that $\dim L \leq \dim E$.

Let $\{u_k : k \in K\}$ be an orthonormal basis of E. Since H is generated by $\{x \otimes e_n : x \in E, n \in \mathbb{Z}_+\}$, the set $\{u_k \otimes e_n : k \in K, n \in \mathbb{Z}_+\}$ is an orthonormal basis of H. Thus by Lemma 4.3

$$\dim L = \sum_{k\in K}\sum_{n\in\mathbb{Z}_+} \|Q(u_k \otimes e_n)\|^2 = \sum_{k\in K}\sum_{n\in\mathbb{Z}_+} \langle Q(u_k \otimes e_n), u_k \otimes e_n\rangle\,.$$

Now $u_k \otimes e_n = S^n(u_k \otimes e_0)$ and $Q = P - SPS^*$, so

$$\begin{aligned}\langle Q(u_k \otimes e_0), u_k \otimes e_0\rangle &= \langle P(u_k \otimes e_0), (u_k \otimes e_0)\rangle - \langle SPS^*(u_k \otimes e_0), u_k \otimes e_0\rangle \\ &= \langle P(u_k \otimes e_0), (u_k \otimes e_0)\rangle\end{aligned}$$

(because $S^*(x \otimes e_0) = 0$) and for $n > 0$,

$$\begin{aligned}\langle Q(u_k \otimes e_n), u_k \otimes e_n\rangle &= \langle PS^n(u_k \otimes e_0), S^n(u_k \otimes e_0)\rangle - \langle SPS^*S^n(u_k \otimes e_0), S^n(u_k \otimes e_0)\rangle \\ &= \langle PS^n(u_k \otimes e_0), S^n(u_k \otimes e_0)\rangle - \left\langle PS^{n-1}(u_k \otimes e_0), S^{n-1}(u_k \otimes e_0)\right\rangle.\end{aligned}$$

Thus for each $k \in K$,

$$\begin{aligned}\sum_{n=0}^{m} \langle Q(u_k \otimes e_n), u_k \otimes e_n\rangle &= \langle P(u_k \otimes e_0), (u_k \otimes e_0)\rangle \\ &\quad + \langle PS(u_k \otimes e_0), S(u_k \otimes e_0)\rangle - \langle P(u_k \otimes e_0), (u_k \otimes e_0)\rangle \\ &\quad + \dots\ \dots \\ &\quad + \langle PS^m(u_k \otimes e_0), S^m(u_k \otimes e_0)\rangle - \left\langle PS^{m-1}(u_k \otimes e_0), S^{m-1}(u_k \otimes e_0)\right\rangle \\ &= \langle PS^m(u_k \otimes e_0), S^m(u_k \otimes e_0)\rangle\end{aligned}$$

and so

$$\dim L = \lim_{m\to\infty} \sum_{k\in K} \langle PS^m(u_k \otimes e_0), S^m(u_k \otimes e_0)\rangle \leq \sum_{k\in K} \|u_k\|^2 = \dim E. \qquad \square$$

Now define $V : H \to H$ as follows: Let $\xi = \sum_{n=0}^{\infty} x_n \otimes e_n \in H$, so that $\sum_n \|x_n\|^2 = \|\xi\|^2 < +\infty$. Observe that $x_n \in E$ so $U^*x_n \in L$ and the vectors $S^n(U^*x_n)$ are pairwise orthogonal; thus the series $\sum_n S^n(U^*x_n)$ converges in H and we may define

$$V\left(\sum_{n=0}^{\infty} x_n \otimes e_n\right) = \sum_{n=0}^{\infty} S^n(U^*x_n).$$

This is a contraction:

$$\|V\xi\|^2 = \left\|\sum_{n=0}^{\infty} S^n(U^*x_n)\right\|^2 = \sum_n \|S^n(U^*x_n)\|^2 = \sum_n \|U^*x_n\|^2 \leq \sum_n \|x_n\|^2 = \|\xi\|^2\,.$$

Claim 4. V is a partial isometry.

Proof. Let $F = U(L) \subseteq E$ be the final space of the partial isometry U. Note that U^* is a partial isometry with initial space F and final space L. Consider the subspace

$$X := \left\{ \xi = \sum_{n=0}^{\infty} x_n \otimes e_n : \ x_n \in F \right\} \subseteq H.$$

If $\xi \in X$, then each coordinate x_n is in F and so $\|U^* x_n\| = \|x_n\|$. Thus

$$\|V\xi\|^2 = \sum_n \|U^* x_n\|^2 = \sum_n \|x_n\|^2 = \|\xi\|^2 .$$

and so $V|_F$ is isometric.

If $\xi \perp X$, then each coordinate x_n is in $F^{\perp}$ [9] and so $U^* x_n = 0$. Thus $V\xi = 0$, showing that V vanishes on $X^{\perp}$. □.

Claim 5. $V(H) = M$.

Proof. Since the range of U^* is L, it is clear that that $V(H)$ lies in the direct sum of the subspaces $S^n(L)$, namely M. On the other hand, given $n \geq 0$ and $\xi \in L$, letting $x = U\xi \in F$ we have $V(x \otimes e_n) = S^n(U^* x) = S^n(\xi)$. Thus $V(H)$ contains all subspaces $S^n(L)$, $n \in \mathbb{Z}_+$, hence also their direct sum M. □

Claim 6. $VS = SV$.

Proof. This is obvious: For all $x \in E$ and $n \geq 0$,

$$\begin{aligned} VS(x \otimes e_n) &= V(x \otimes e_{n+1}) = S^{n+1}(U^* x) \\ SV(x \otimes e_n) &= S(S^n(U^* x)) = S^{n+1}(U^* x). \end{aligned} \qquad \square$$

[9]because for all $x \in F$ and all $n \geq 0$ we have $\xi \perp (x \otimes e_n)$, hence $x_n \perp x$.

5 Dilations of a contraction

Theorem 5.1 *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that*

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad (n \geq 1).$$

Remark 5.2 *The condition $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ forces the subspace $\mathcal{H} \subseteq K$ to be* semi-invariant *under U, i.e. of the form $\mathcal{H} = \mathcal{H}_2 \cap \mathcal{H}_1^{\perp}$, where $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and both spaces are U-invariant.*

Thus the matrix of U with respect to the (ordered) decomposition $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H} \oplus \mathcal{H}_2^{\perp}$ takes the form [10]

$$U = \begin{bmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{bmatrix}.$$

Indeed, define

$$\mathcal{H}_2 = \overline{[U^n y : y \in \mathcal{H}, n \geq 0]} \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H}_2 \cap \mathcal{H}^{\perp}.$$

Then we have two closed subspaces of $\mathcal{K}$ such that $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and $\mathcal{H} = \mathcal{H}_2 \cap \mathcal{H}_1^{\perp}$. Clearly $\mathcal{H}_2$ is U invariant (because $[U^n y : y \in \mathcal{H}]$ is) and we need to show that $\mathcal{H}_1$ is also U invariant.

Thus if P_i denotes the projection onto $\mathcal{H}_i$, we have to show that $UP_1 = P_1 U P_1$. But $P_1 = P_2 - P = P_2 P^{\perp}$ and $UP_2 = P_2 U P_2$ because $U(\mathcal{H}_2) \subseteq \mathcal{H}_2$, hence

$$P_1 U P_1 = P_2 U P_2 P^{\perp} - PUP_2 P^{\perp} = UP_2 P^{\perp} - PUP_2 P^{\perp} = UP_1 - PUP_2 P^{\perp}$$

and so it suffices to show that $PUP_2P^{\perp} = 0$, equivalently that $PUP_2 = PUPP_2$, or $PUx = PUPx$ for $x \in \mathcal{H}_2$. In fact it suffices to show the last equality when $x = U^n y$ for some $n \in \mathbb{Z}_+$ and $y \in \mathcal{H}$ (for then it will follow for arbitrary $x \in \mathcal{H}_2$ by linearity and continuity).

But for $x = U^n y$, since $PUPU^nP = (PUP)(PU^nP) = T^{n+1} = PU^{n+1}P$ we have

$$\begin{aligned} PUPx &= PUPU^n y = PUPU^n P y \qquad (\text{because } y \in \mathcal{H}) \\ &= PU^{n+1} P y = PU^{n+1} y = PU(U^n y) = PUx \end{aligned}$$

as required. This shows that $\mathcal{H}_1$ is U-invariant as well.

Example 5.3 *Suppose $z \in \mathbb{C}$ with $|z| < 1$ and let $T = zI$ acting on $\mathcal{H} = \mathbb{C}$.*

Try to construct a unitary dilation U on the space $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$:

$$U = \begin{bmatrix} a & b & c \\ 0 & z & d \\ 0 & 0 & e \end{bmatrix}.$$

Now U is unitary if and only if its rows and columns form orthonormal sets. This forces $|b|^2 = |d|^2 = 1 - |z|^2$ and we may choose $b = d = \sqrt{1 - |z|^2}$. But then the orthogonality of the second and third column give $b\bar{c} + z\bar{d} = 0$, so $c = -\bar{z}$. Now if the first row and third column are to have unit length then necessarily $a = 0$ and $e = 0$. Thus

$$U = \begin{bmatrix} 0 & b & -\bar{z} \\ 0 & \boxed{z} & b \\ 0 & 0 & 0 \end{bmatrix} \qquad (b = \sqrt{1 - |z|^2}).$$

[10] It would be *lower* triangular if we wrote the decomposition as $\mathcal{K} = \mathcal{H}_2^{\perp} \oplus \mathcal{H} \oplus \mathcal{H}_1$.

But this is not unitary! The first column and last row are 0. To remedy this, we need to add rows and columns:

$$U = \begin{bmatrix} s & 1 & 0 & 0 & 0 \\ 0 & 0 & b & -\bar{z} & 0 \\ 0 & 0 & \boxed{z} & b & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & t \end{bmatrix}.$$

Now the length of the first row and the last column force $s = t = 0$, and again the first column and last row are 0. Thus we need to add more rows and columns, and so on "ad infinitum": So U turns out to be an operator acting on the infinite dimensional (!) space $\ell^2(\mathbb{Z})$ given by

$$U = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & 1 & 0 & & & & \\ & 0 & b & -\bar{z} & & & \\ & & \boxed{z} & b & 0 & & \\ & & & 0 & 1 & \ddots & \\ & & & & \ddots & \ddots & \end{bmatrix}$$

It is not hard to verify that this operator is unitary, and it will follow anyway from the general construction (second method) below.

Proof of the Theorem. (First method) We dilate T in two steps:
(a) we dilate $(T, \mathcal{H})$ to an *isometry* $(V, \mathcal{K}_1)$ and
(b) we dilate the isometry $(V, \mathcal{H}_1)$ to a unitary $(U, \mathcal{K})$.
Then $(U, \mathcal{K})$ will be a unitary dilation of $(T, \mathcal{H})$.

(a) Dilation of a contraction to an isometry

The space $\mathcal{K}_1$ is defined to be

$$\mathcal{K}_1 = \mathcal{H} \otimes \ell^2(\mathbb{Z}_+)$$

(see Section 4). Recall that $\mathcal{K}_1$ consists of all families $(x_n)_{n\geq 0}$ (with $x_n \in \mathcal{H}$ for all n) which are square summable in norm. We identify $\mathcal{H}$ with the subspace $\mathcal{H}_0 = \{(h, 0, 0, \dots) : h \in \mathcal{H}\}$ of $\mathcal{K}_1$.

Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. It follows that the operator $I - T^*T$ is positive; hence we may define

$$D_T = (I - T^*T)^{1/2}.$$

Consider the operator $V \in \mathcal{B}(\mathcal{K}_1)$ given by

$$V = \begin{bmatrix} T & 0 & 0 & 0 & \dots \\ D_T & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \vdots & \vdots & & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix}$$

(the last matrix is written with respect to the decomposition $\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{H}^\perp$.) Explicitly,

$$V(h_0, h_1, h_2, \dots) = (Th_0, D_T h_0, h_1, h_2, \dots).$$

Clearly V dilates T. To see that it is isometric, note that

$$\begin{aligned}\|Th_0\|^2 + \|D_T h_0\|^2 &= \langle Th_0, Th_0\rangle + \langle D_T h_0, D_T h_0\rangle \\ &= \langle T^*Th_0, h_0\rangle + \left\langle D_T^2 h_0, h_0\right\rangle = \|h_0\|^2 .\end{aligned}$$

and thus

$$\begin{aligned}\|Vh\|^2 &= \|Th_0\|^2 + \|D_T h_0\|^2 + \sum_{n=1}^{\infty} \|h_n\|^2 \\ &= \|h_0\|^2 + \sum_{n=1}^{\infty} \|h_n\|^2 = \|h\|^2\end{aligned}$$

(b) Dilation of an isometry to a unitary

Let $V \in \mathcal{B}(\mathcal{K}_1)$ be an isometry, $V^*V = I$.

Observe that $(VV^*)V = V(V^*V) = V$. Also $(VV^*)^2 = V(V^*V)V^* = VV^*$ so VV^* is a projection. Let $P = I - VV^*$. This is also a projection and $PV = V - VV^*V = 0$.

Thus if

$$U : \mathcal{K}_1 \oplus \mathcal{K}_1 \to \mathcal{K}_1 \oplus \mathcal{K}_1 : \quad \text{is given by} \quad U = \begin{bmatrix} V^* & 0 \\ P & V \end{bmatrix}$$

then U is unitary. Indeed,

$$UU^* = \begin{bmatrix} V^* & 0 \\ P & V \end{bmatrix}\begin{bmatrix} V & P \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} V^*V & V^*P \\ PV & P^2 + VV^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$U^*U = \begin{bmatrix} V & P \\ 0 & V^* \end{bmatrix}\begin{bmatrix} V^* & 0 \\ P & V \end{bmatrix} = \begin{bmatrix} VV^* + P^2 & PV \\ V^*P & V^*V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Combining the two steps, we obtain the dilation U in the form

$$U = \begin{bmatrix} T^* & A^* & 0 & 0 \\ 0 & B^* & 0 & 0 \\ P_{11} & P_{12} & T & 0 \\ P_{21} & P_{22} & A & B \end{bmatrix} = \begin{bmatrix} V^* & 0 & 0 \\ P_1 & T & 0 \\ P_2 & A & B \end{bmatrix}$$

where $P_1 = [\ P_{11} \quad P_{12}\ ]$ and $P_2 = [\ P_{21} \quad P_{22}\ ]$. □

Remark 5.4 *Observe that, since the space $\mathcal{K}_1$ is in fact U-invariant (not just semi-invariant), we have $U^n|_{\mathcal{K}_1} = V^n$: thus U^n is in fact an* extension *of V^n (not merely a dilation).*

Proof of the Theorem. (Second method) Notice that since $\|T\| \leq 1$, the operators $I - T^*T$ and $I - TT^*$ are positive; hence we may define

$$D_T = (I - T^*T)^{1/2}, \quad D_{T^*} = (I - TT^*)^{1/2}.$$

These are called the 'defect operators': $D_T = 0$ iff T is an isometry, and $D_{T^*} = 0$ iff T^* is an isometry (then T is called a *co-isometry*). Note that

$$TD_T^2 = T - TT^*T = D_{T^*}^2 T, \qquad T^*D_{T^*}^2 = T^* - T^*TT^* = D_T^2 T^*$$

from which we obtain

$$T^*D_{T^*} = D_T T^*, \qquad TD_T = D_{T^*}T \tag{1}$$

by approximating the function $f(t) = \sqrt{t}$ by polynomials (p_n) uniformly for $t \in [0,1]$ (so that $D_T = \lim_n p_n(D_T^2)$ and $D_{T^*} = \lim_n p_n(D_{T^*}^2)$).

The space $\mathcal{K}$ is defined to be

$$\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{Z}).$$

Thus $\mathcal{K}$ consists of all families $(x_n)_{n\in\mathbb{Z}}$ (with $x_n \in \mathcal{H}$ for all n) which are square summable in norm, i.e. $\sum_{n\in\mathbb{Z}} \|x_n\|^2 < \infty$.
We identify $\mathcal{H}$ with the subspace $\mathcal{H}_0 = \{(\dots, 0, \boxed{h}, 0, \dots) : h \in \mathcal{H}\}$ of $\mathcal{K}$.

Consider the operator $U \in \mathcal{B}(\mathcal{K})$ given by

$$U = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & I & 0 & & & & \\ & 0 & D_T & -T^* & & & \\ & & \boxed{T} & D_{T^*} & 0 & & \\ & & & 0 & I & \ddots & \\ & & & & \ddots & \ddots & \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{bmatrix}.$$

Explicitly, for $h = (h_n)$, we have $Uh = h' = (h'_n)$, where

$$h'_{-1} = D_T h_0 - T^* h_1, \quad h'_0 = T h_0 + D_{T^*} h_1, \quad h'_j = h_{j+1},\ (j \neq -1, 0).$$

Clearly U dilates T. To see that it is isometric, consider

$$\begin{aligned} \|h'_{-1}\|^2 + \|h'_0\|^2 &= (\langle D_T^2 h_0, h_0\rangle + \langle T^* h_1, T^* h_1\rangle - 2Re\,\langle D_T h_0, T^* h_1\rangle) \\ &\quad + (\langle T h_0, T h_0\rangle + \langle D_{T^*}^2 h_1, h_1\rangle + 2Re\langle D_{T^*} h_1, T h_0\rangle) \\ &= \|h_0\|^2 + \|h_1\|^2 \qquad (\text{using } T D_T = D_{T^*} T). \end{aligned}$$

To prove that U is onto, given $(h'_n) \in \ell^2(\mathbb{Z})$ set $h_j = h'_{j-1},\ (j \neq -1, 0)$ and determine h_0 and h_1 by solving the system

$$\left.\begin{array}{r} h'_{-1} = D_T h_0 - T^* h_1 \\ h'_0 = T h_0 + D_{T^*} h_1 \end{array}\right\} \Rightarrow \left.\begin{array}{c} D_T h'_{-1} = (I - T^*T) h_0 - D_T T^* h_1 \\ T^* h'_0 = T^* T h_0 + T^* D_{T^*} h_1 \end{array}\right\}$$

$$\Rightarrow D_T h'_{-1} + T^* h'_0 = h_0 - D_T T^* h_1 + T^* D_{T^*} h_1 = h_0$$

$$\left.\begin{array}{r} h'_{-1} = D_T h_0 - T^* h_1 \\ h'_0 = T h_0 + D_{T^*} h_1 \end{array}\right\} \Rightarrow \left.\begin{array}{c} T h'_{-1} = T D_T h_0 - T T^* h_1 \\ D_{T^*} h'_0 = D_{T^*} T h_0 + (I - TT^*) h_1 \end{array}\right\}$$

$$\Rightarrow -T h'_{-1} + D_{T^*} h'_0 = -T D_T h_0 + D_{T^*} T h_0 + h_1 = h_1$$

where we have used relations (1). $\square$

6 von Neumann's inequality

Theorem 6.1 (von Neumann's inequality) *If $T \in \mathcal{B}(H)$ is a contraction and p is a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, then*

$$\|p(T)\|_{\mathcal{B}(H)} \leq \sup\{|p(z)| : z \in \mathbb{T}\}.$$

Proof. Let $U \in \mathcal{B}(K)$ be any unitary dilation of $T \in \mathcal{B}(H)$. Observe that $p(T) = P_H p(U)|_H$ and hence $\|p(T)\|_{\mathcal{B}(H)} \leq \|p(U)\|_{\mathcal{B}(K)}$. But U is a unitary operator so $\sigma(U) \subseteq \mathbb{T}$; thus by the spectral mapping theorem we have

$$\|p(U)\|_{\mathcal{B}(K)} = \sup\{|p(z)| : z \in \sigma(U)\} \leq \sup\{|p(z)| : z \in \mathbb{T}\}.$$

Thus $$\|p(T)\|_{\mathcal{B}(H)} \leq \|p(U)\|_{\mathcal{B}(K)} \leq \sup\{|p(z)| : z \in \mathbb{T}\}.$$ □

Example 6.2 *In particular if $T = wI$ where $w \in \mathbb{D}$, then for any polynomial p we obtain $p(T) = p(w)I$ and so*

$$|p(w)| = \|p(T)\| \leq \sup\{|p(z)| : z \in \mathbb{T}\}.$$

More generally, let $A(\mathbb{D})$ be the algebra of all continuous complex-valued functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$. This is a closed subalgebra of $C(\overline{\mathbb{D}})$: it consists of all $f \in C(\overline{\mathbb{D}})$ that are limits of polynomials (in z) uniformly in $\overline{\mathbb{D}}$. It follows that the last inequality is true for all $f \in A(\mathbb{D})$:

$$\sup\{|f(w) : w \in \overline{\mathbb{D}}\} \leq \sup\{f(z)| : z \in \mathbb{T}\}.$$

We have obtained a particular case of the *maximum modulus principle* of complex analysis by Operator Theory methods.

Remark 6.3 *von Neumann's inequality shows that, given a contraction $T \in B(H)$ the functional calculus $p \to p(T)$ extends by continuity to a* contractive homomorphism *$f \to f(T)$ from the disc algebra $A(\mathbb{D})$ into $B(H)$: a* representation *of the Banach algebra $A(\mathbb{D})$. Conversely, given any contractive representation $\pi : A(\mathbb{D}) \to B(H)$ we obtain a contraction $T = \pi(\zeta) \in B(H)$ (recall $\zeta(z) = z$ for $z \in \overline{\mathbb{D}}$) such that $\pi(f) = f(T)$ for all $f \in A(\mathbb{D})$.*

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