The parallelogram law

Proposition 1 *Suppose*[1] *E is a complex topological vector space space and $p : E \to \mathbb{R}_+$ is a continuous function satisfying*

$$p(\lambda x) = |\lambda| p(x) \qquad (\lambda \in \mathbb{C}, x \in E)$$

and the parallelogram law

$$p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2 \qquad (x, y \in E).$$

Then there is a (unique) continuous sesquilinear form $\phi : E \times E \to \mathbb{C}$ such that

$$p(x)^2 = \phi(x, x) \qquad (x \in E)$$

(and p is a seminorm).

Proof Recall that any sesquilinear form $\phi : E \times E \to \mathbb{C}$ satisfies the polarisation identity

$$4\phi(x,y) = \phi(x+y, x+y) - \phi(x-y, x-y) + i\phi(x+iy, x+iy) - i\phi(x-iy, x-iy).$$

Thus if ϕ and ψ are two sesquilinear forms satsifying $\phi(x,x) = \psi(x,x)$ for all $x \in E$ then $\phi = \psi$. This proves uniqueness.

For existence, suppose p satisfies the parallelogram law, and define ϕ by polarisation:

$$4\phi(x,y) = p(x+y)^2 - p(x-y)^2 + i(p(x+iy)^2 - p(x-iy)^2).$$

We will show that ϕ is a sesquilinear form. Observe that, since the linear operations and p are continuous, ϕ is continuous.

It follows immdediately from the definition that

$$\phi(0, y) = 0 = \phi(y, 0) \quad \text{for all} \quad y \in E.$$

Also

$$(i) \;\; 4\phi(x,x) = (2p(x))^2 + i(|1+i|p(x))^2 - i(|1-i|p(x))^2 = 4p(x)^2 \geq 0$$

since $|1+i| = |1-i|$.

$$\begin{aligned}(ii) \;\; 4\phi(y,x) &= p(x+y)^2 - p(y-x)^2 + ip(y+ix)^2 - ip(y-ix)^2 \\ &= p(x+y)^2 - p(x-y)^2 + ip(i(x-iy))^2 - ip(-i(x+iy))^2 \\ &= p(x+y)^2 - p(x-y)^2 + ip(x-iy))^2 - ip((x+iy)^2 = 4\overline{\phi(x,y)}.\end{aligned}$$

We show that

$$(iii) \;\; \phi(x+z, y) = \phi(x,y) + \phi(z,y).$$

[1] pythag, 30 May 2007

Indeed the parallelogram law gives

$$\begin{aligned} p((x+y)+z)^2 &= 2p(x+y)^2 + 2p(z)^2 - p((x+y)-z)^2 \\ -p((x-y)+z)^2 &= -2p(x-y)^2 - 2p(z)^2 + p((x-y)-z)^2 \\ p((z+y)+x)^2 &= 2p(z+y)^2 + 2p(x)^2 - p((z+y)-x)^2 \\ -p((z-y)+x)^2 &= -2p(z-y)^2 - 2p(x)^2 + p((z-y)-x)^2 \end{aligned}$$

but $p(x+y-z) = p(z-y-x)$ and $p(x-y-z) = p(z+y-x)$, so adding

$$\begin{aligned} &2p(x+z+y)^2 - 2p(x+z-y)^2 = \\ &p((x+y)+z)^2 - p((x-y)+z)^2 + p((z+y)+x)^2 - p((z-y)+x)^2 = \\ &2p(x+y)^2 + 2p(z)^2 - p(x+y-z)^2 - 2p(x-y)^2 - 2p(z)^2 + p(x-y-z)^2 + \\ &2p(z+y)^2 + 2p(x)^2 - p(z+y-x)^2 - 2p(z-y)^2 - 2p(x)^2 + p(z-y-x)^2 = \\ &2p(x+y)^2 - 2p(x-y)^2 + 2p(z+y)^2 - 2p(z-y)^2 \end{aligned}$$

in other words

$$\operatorname{Re}\phi(x+z,y) = \operatorname{Re}\phi(x,y) + \operatorname{Re}\phi(z,y).$$

Replacing y with iy in the same calculation we conclude that

$$\operatorname{Im}\phi(x+z,y) = \operatorname{Im}\phi(x,y) + \operatorname{Im}\phi(z,y)$$

and hence (iii) is proved. The relation

$$(iii)' \;\; \phi(x, y+z) = \phi(x,y) + \phi(y,z)$$

follows from (iii) and (ii). From (iii) and induction we see that

$$\phi(nx,y) = n\phi(x,y) \;\text{ for all }\; n \in \mathbb{N}.$$

Since also

$$\begin{aligned} &4\phi(-x,y) = p(-x+y)^2 - p(-x-y)^2 + i(p(-x+iy)^2 - p(-x-iy)^2) \\ &= -\,(p(-(x+y))^2 - p(-(x-y))^2) - i(p(-(x+iy))^2 - p(-(x-iy))^2) = -4\phi(x,y) \end{aligned}$$

we have

$$\phi(nx,y) = n\phi(x,y) \;\text{ for all }\; n \in \mathbb{Z}.$$

Now if $m \in \mathbb{N}$ we have

$$\phi(x,y) = \phi(m\frac{1}{m}x,y) = m\phi(\frac{1}{m}x,y) \quad \text{and so} \;\; \frac{1}{m}\phi(x,y) = \phi(\frac{1}{m}x,y).$$

Now the relation

$$\lambda\phi(x,y) = \phi(\lambda x,y)$$

which has just been proved for all *rational* λ follows for all *real* λ by continuity.

It remains to show that

$$\phi(ix,y) = i\phi(x,y).$$

Indeed,

$$\begin{aligned} 4\phi(ix,y) =& p(ix+y)^2 - p(ix-y)^2 + i(p(ix+iy)^2 - p(ix-iy)^2) = \\ & p(y+ix)^2 - p(y-ix)^2 + i(p(y+x)^2 - p(y-x)^2) = \\ & i[p(y+x)^2 - p(y-x)^2 - i(p(y+ix)^2 - p(y-ix)^2)] = \\ & = 4i\overline{\phi(y,x)} = 4i\phi(x,y) \qquad \text{by } (ii) \end{aligned}$$

and this concludes the proof that ϕ is sesquilinear. Now the relation $p(x) = \sqrt{\phi(x,x)}$ together with the Cauchy-Schwartz inequality shows that p must be a seminorm.