ON CERTAIN ELEMENTS OF C^*-ALGEBRAS

BY

J. A. ERDOS[1]

1. Introduction

Let A be a C^*-algebra; that is, a Banach algebra with involution satisfying the C^*-condition $\| aa^* \| = \| a \|^2$. It is well known that A can be faithfully represented as an algebra of operators on some Hilbert space. However it is clear that the rank of the image of a given element may vary in different representations. The main purpose of this paper is to give a necessary and sufficient condition for an element to have an operator of rank one as its image under some faithful representation of A.

We shall call an element s of a C^*-algebra A a *single* element if, whenever $asb = 0$ for some a, b in A, we have that at least one of as, sb is zero.

It is easy to see that an operator of rank one is a single element of any algebra of operators that contains it. The condition of being single has been used by Ringrose [8], [9] as a property of rank one operators in triangular and nest algebras that is invariant under algebraic isomorphisms. Munn [6] has also used the condition, imposing it on every element of an algebraic semigroup, but there does not seem to be any obvious connection between [6] and what follows here.

It will be shown that there exists a faithful representation of A such that the image of every non-zero single element of A is an operator of rank one. This is done in Sections 2 and 3. In Section 4 the theory developed is applied to give the standard representation of a dual C^*-algebra. This section also contains characterizations of dual C^*-algebras and certain W^*-algebras, including a characterisation of the algebra of all bounded linear operators on a Hilbert space. The final section contains an example showing that the main result cannot be generalised to all Banach algebras.

In general, the terminology used will be as in Rickart [7] and Dixmier [1], [2]. At certain points, noted in the text, representations that are not adjoint preserving will be considered. Otherwise representations will be, by definition adjoint preserving. All the algebras considered will be over the complex field.

2. Algebraic properties

The set of single elements of the C^*-algebra A will be denoted by σ. Note that the zero element is a member of σ.

LEMMA 2.1. *If $s \in \sigma$ then $s^* \in \sigma$. For any element x of A, $xs \in \sigma$ and $sx \in \sigma$.*

Received May 2, 1969.

[1] The results of this paper were obtained while the author was a Research Associate at the University of Illinois.

Proof. If $as^*b = 0$ then $b^*sa^* = 0$ and as s is single, either b^*s or sa^* is zero. So either s^*b or as^* is zero and thus s^* is single.

If $axsb = 0$ then, as s is single, either axs or sb is zero. So either axs or xsb is zero and thus xs is single. Similarly sx is single.

An alternative statement of the above lemma is that in the multiplicative semigroup of A the single elements form a selfadjoint semigroup ideal. If this ideal contains a non-zero element s then it contains a non-zero selfadjoint element s^*s. The next lemma shows that in this case it will contain a non-zero selfadjoint idempotent.

LEMMA 2.2. *If $s \in \sigma$ and s is normal then for some complex number λ,*
(i) $s^2 = \lambda s$,
(ii) $s = \lambda e$ *where e is a single selfadjoint idempotent.*

Proof. If $s = 0$ then we may take $\lambda = 0$ and $e = 0$. Suppose $s \neq 0$ and let C be the commutative C^*-algebra generated by s and s^*. By the Gelfand representation theorem [7, p. 190], C is isometrically *-isomorphic to the algebra $C_0(X)$ of all functions vanishing at infinity on a locally compact Hausdorff space X. If $c \in C$ let $\hat{c}$ be the image of c under the Gelfand representation. We show that the support of $\hat{s}$ (that is, $\{x : \hat{s}(x) \neq 0\}$) consists of exactly one point. As $s \neq 0$ we have that $\hat{s} \neq 0$ and so the support of $\hat{s}$ must contain at least one point. However, if the support of $\hat{s}$ contains two distinct points x_1 and x_2, since a locally compact Hausdorff space is completely regular, there exist functions f and g in $C_0(X)$ with disjoint supports such that $f(x_1) \neq 0$ and $g(x_2) \neq 0$. We then have $f\hat{s}g = 0$ with $f\hat{s} \neq 0$ and $\hat{s}g \neq 0$. But as the Gelfand representation is onto $C_0(X)$, there exist elements a and b in C such that $\hat{a} = f$ and $\hat{b} = g$. Then $asb = 0$ but $as \neq 0$ and $sb \neq 0$. This contradicts the fact that s is single. Hence $\hat{s}(x) = 0$ except at one point x_0.

Let $\lambda = \hat{s}(x_0)$. Clearly $(\hat{s})^2(x_0) = \lambda\hat{s}(x_0)$. Also if $\hat{e} = \lambda^{-1}\hat{s}$, $\hat{e}(x_0) = 1$ and $\hat{e}(x) = 0$ when $x \neq x_0$. Hence $\hat{e}$ is a real idempotent function. As the Gelfand map is a *-isomorphism we have that $s^2 = \lambda s$ and that $e = \lambda^{-1}s$ is a selfadjoint single idempotent.

LEMMA 2.3. *For any single element s there exist selfadjoint single idempotents e and f such that $s = fse$.*

Proof. The case $s = 0$ is trivial. Suppose $s \neq 0$. From Lemma 2.2 there exist single selfadjoint idempotents e and f such that for some non-zero complex numbers λ and μ, $s^*s = \lambda e$ and $ss^* = \mu f$. We prove that $f = fse$. As e is idempotent,

$$\lambda(se - s)s^*s = (se - s)e = 0$$

and so, as s^* is single and $s^*s \neq 0$, we have

$$(se - s)s^* = 0.$$

Hence

$$(se - s)(se - s)^* = (se - s)es^* - (se - s)s^* = 0,$$

and the C^*-condition implies that $se - s = 0$. Similarly it can be shown that $fs = s$. Therefore as e and f are idempotent,

$$s = se = fs = fse.$$

COROLLARY 2.4. *The principal left ideal As of a non-zero single elements is equal to the principal left ideal of a single selfadjoint idempotent. Also $s \in As$. Similar statements hold for principal right ideals.*

Proof. If e is as in the lemma,

$$As = Ase \subseteq Ae.$$

But

$$Ae = As^*s \subseteq As.$$

Hence $As = Ae$ and as $s = se$, $s \in Ae = As$.

We are now in a position to prove the theorem below which contains the main algebraic fact connecting single elements with operators of rank one.

THEOREM 2.5. *If s and t are single elements then the set*

$$sAt = \{sat : a \in A\}$$

is a zero or a one-dimensional linear subspace of A. Also $s \in sAs$.

Proof. It is clear that sAt is a linear subspace of A. If $sAt = (0)$ there is nothing to prove and so we may suppose that $sAt \neq (0)$. Let e and f be the single selfadjoint idempotents such that $ss^* = \mu f$ and $t^*t = \lambda e$. Then $s = fs$ and $t = te$. Therefore

$$sAt = fsAte \subseteq fAe \quad \text{and} \quad fAe = ss^*At^*t \subseteq sAt.$$

Hence $sAt = fAe$ where e and f are single selfadjoint idempotents. If $s = t$ then $s = fse$ and so $s \in sAs$.

Now let x and y be non-zero elements of sAt. Then $x = fxe$ and $y = fye$. Using Corollary 2.4, $x \in Ae = Ay$. So for some $z \in A$, $x = zy$. It follows easily that

$$(*) \qquad x = fzfy.$$

To complete the proof we show that fAf consists of scalar multiples of f. If $a = a^*$ then faf is a selfadjoint single element. By Lemma 2.2,

$$faf = \lambda g$$

where g is a single selfadjoint idempotent. But clearly

$$g = fg = gf$$

and so

$$(g - f)fg = gfg - fg = 0.$$

As f is single and $fg = g \neq 0$, it follows that $(g - f)f = 0$ which shows that

$$f = gf = g.$$

Hence for any selfadjoint element a of A, $faf = \lambda f$. But any element z of A can be written as $z = a + ib$ where a and b are selfadjoint. Therefore

$$fzf = faf + ifbf = \lambda f + i\mu f.$$

Thus from $(*)$, if $k = \lambda + i\mu$, as $y = fy$,

$$x = kfy = ky.$$

This completes the proof.

COROLLARY 2.6. *If s is a non-zero single element of A then As is a minimal left ideal and sA is a minimal right ideal.*

Proof. If L is a left ideal contained in As and l is a non-zero element of L then $l = as$ for some $a \in A$ and

$$L \supseteq Al = Aas \supseteq As^*a^*as$$

From Corollary 2.4, $s^*s \in s^*As$ and as s^*As is one-dimensional,

$$s^*a^*as = ks^*s = \lambda e.$$

But $Ae = As$ and so $L \subseteq As^*a^*as = As$. Therefore As is a minimal left ideal Similarly sA is a minimal right ideal.

4. Representations

The main purpose of this section is to construct an isometric representation of the C^*-algebra A as an algebra of operators on a Hilbert space such that the image of each non-zero single element is an operator of rank one. The initial part of this construction is based on Chapter IV, § 10 of [7].

We first prove a result which may be of independent interest showing the stability of single elements under algebraic homomorphisms.

LEMMA 3.1. *If s is single and φ is any homomorphism (not necessarily adjoint preserving or continuous) of A into any linear algebra then $\varphi(s)$ is single in $\varphi(A)$.*

Proof. Let K be the kernel of φ. From Corollary 2.6, As is a minimal left ideal. Hence either $As \cap K = As$ or $As \cap K = (0)$. If $\varphi(s) = 0$ then it is single. If $\varphi(s) \neq 0$ then as $s \in As$, $As \cap K = (0)$.

Now if $\varphi(asb) = 0$ then $asb \in K$ and by the above, since sb is single, if $asb \neq 0$ then $Asb \subseteq K$. But $sb \in Asb$ and so $\varphi(sb) = 0$. If $asb = 0$ then either as or sb is zero. So in either case we have $\varphi(as) = 0$ or $\varphi(sb) = 0$ and hence $\varphi(s)$ is single in $\varphi(A)$.

Suppose that the C^*-algebra A contains a non-zero single element. Then by Lemma 2.2, A contains a non-zero selfadjoint single idempotent e. We show that an inner product can be defined on the elements of the ideal Ae. If x and

y are in Ae, $x = xe$ and $y = ye$. Therefore by Theorem 2.5 there exists a complex number $\lambda(x, y)$ such that

$$y^*x = ey^*xe = \lambda(x, y)e.$$

Define

$$\langle x, y\rangle = \lambda(x, y).$$

THEOREM 3.2. *The ideal Ae, with the C^*-algebra norm and the inner product defined above, is a Hilbert space.*

Proof. A routine verification shows that $\langle x, y\rangle$ is conjugate linear. If $x \in Ae$,

$$\langle x, x\rangle e = x^*x \neq 0$$

and so $\langle x, y\rangle$ is definite. To show positivity, note first that $\langle e, e\rangle = 1$ and that $\langle x, x\rangle = \overline{\langle x, x\rangle}$ so $\langle x, x\rangle$ is real. If $\langle x, x\rangle$ takes negative values then there exists $y \in Ae$ such that $\langle y, y\rangle = -1$. By multiplying y by a complex number of modulus one, we may arrange that the real part of $\langle y, e\rangle$ is zero. Then

$$\langle y + e, y + e\rangle = -1 + 1 + 2 \operatorname{Re} \langle y, e\rangle = 0$$

contradicting that $\langle x, y\rangle$ is definite.

To identify the inner product norm with the algebra norm we use the C^*-condition. First

$$\| e \| = \| e^2 \| = \| ee^* \| = \| e \|^2$$

and so $\| e \| = 1$. Therefore, since $\langle x, x\rangle \geq 0$,

$$\langle x, x\rangle = \langle x, x\rangle \| e \| = \| \langle x, x\rangle e \| = \| x^*xe \|,$$

and since $x = xe$,

$$\| x^*xe \| = \| ex^*xe \| = \| xe \|^2 = \| x \|^2.$$

Finally, to prove completeness it is now sufficient to prove that Ae is closed in A. If (x_i) is a sequence of elements of Ae converging to x,

$$x - xe - \lim_{i\to\infty} (x_i - x_i e) = 0.$$

Therefore $x \in Ae$ and so the proof is complete.

We now introduce some notation. From Lemma 2.1 it is clear that the set of finite sums of single elements of a C^*-algebra forms a *-ideal. We shall denote this ideal by S and its closure by Σ. For any non-zero single idempotent e we write H_e for the ideal Ae considered as a Hilbert space. Let σ_e be the set $AeA = \{aeb : a, b \in A\}$, denote the set of finite sums of elements of σ_e by S_e and the closure of S_e by Σ_e. Clearly S_e and Σ_e are two-sided *-ideals of A. We define the representation ρ_e of A into the algebra of bounded linear operators on He by

$$\rho_e(a)x = ax$$

where $x \in H_e$. Since $H_e = Ae$ is a left ideal, it is easily verified that ρ_e is a representation. That ρ_e is adjoint preserving follows from $y^*ax = (a^*y)^*x$. We now prove some further properties of ρ_e. The operator $x \to \langle x, p \rangle q$ will be denoted by $p \otimes q$.

Theorem 3.3. *If s is a single element of A then $\rho_e(s)$ has rank one or zero. Every rank one operator on H_e is the image under ρ_e of some element of ρ_e. Also if $s \in \sigma_e$ then*

$$\| \rho_e(s) \| = \| s \|.$$

Proof. The range of $\rho_e(s)$ is sAe. If s is single then by Theorem 2.5, sAe is zero or one dimensional. Hence $\rho_e(s)$ has rank one or zero. Now let p and q be non-zero vectors of H_e. If $x \in H_e$, by the definition of the inner product,

$$p^*x = \langle x, p \rangle e$$

and since $q = qe$, $qp^* \in \sigma_e$ and we have

$$\rho_e(qp^*)x = qp^*x = \langle x, p \rangle q.$$

Hence $\rho_e(qp^*) = p \otimes q$ and every rank one operator on H_e is of this form.

To prove the remaining statement note first that if aeb is a non-zero element of σ_e then the fact that e is single implies that $aebb^*e \neq 0$ and hence that $\rho_e(aeb) \neq 0$. Now for any non-zero element s of σ_e,

$$\begin{aligned} \| \rho_e(s) \|^2 &= \| \rho_e(s)^*\rho_e(s) \| \\ &= \| \rho_e(s^*s) \| \\ &= \sup\{ \| \rho_e(s^*s)x \| : x \in H_e, \| x \| \leq 1\} \\ &= \sup\{ \| s^*sx \| : x \in Ae, \| x \| \leq 1\}. \end{aligned}$$

Lemma 2.2 shows that

$$s^*s = kf$$

where, as $s \neq 0$, f is a non-zero single selfadjoint idempotent and k is a non-zero complex number. From the proof of Theorem 3.2, $\| f \| = 1$ and so, taking norms,

$$\| s^*s \| = \| s \|^2 = | k |.$$

Hence

$$\| \rho_e(s) \|^2 = \sup\{ | k| \cdot \| fx \| : x \in Ae, \| x \| \leq 1\}.$$

Since $f \in Ae$, there exists $y \in Ae$ such that $fy \neq 0$. Taking $x = \| fy \|^{-1}fy$ we have that

$$\| \rho_e(s) \|^2 \geq | k | = \| s \|^2.$$

But clearly $\| \rho_e(s) \| \leq \| s \|$ and so for all $s \in \sigma_e$, $\| \rho_e(s) \| = \| s \|$.

A non-zero representation ρ of A on a Hilbert space H is defined to be (topologically) *irreducible* if for all non-zero h in H, $\{\rho(a)h : a \in A\}$ is dense in H. Since $\rho_e(A)$ contains every operator of rank one on H_e we have:

COROLLARY 3.4. *ρ_e is irreducible.*

THEOREM 3.5. *If ρ is a non-zero continuous irreducible representation (not necessarily adjoint preserving,) of A on a Hilbert space H and for some single element s, $\rho(s) \neq 0$ then ρ is similar to ρ_e where e is the single selfadjoint idempotent such that $s = se$. If ρ is an adjoint preserving representation then ρ is unitarily equivalent to ρ_e.*

Proof. Since $\rho(s) = \rho(s)\rho(e) \neq 0$, $\rho(e)$ is a non-zero idempotent on H. Hence there exists a vector h of H such that $\| h \| = 1$ and $\rho(e)h = h$. Define the operator T from H_e to H by

$$Tx = \rho(x)h$$

where $x \in H_e$. Since ρ is continuous, there exists a positive constant k such that $\| \rho(a) \| \leq k \| a \|$. Therefore

$$\| Tx \| \leq k \| x \|$$

Also as $x = xe$, $x^*x = \lambda e$ for some constant λ and $\| x^*x \| = | \lambda |$. Then

$$\| \rho(x^*) \| \|\rho(x)h \| \geq \| \rho(x^*x)h \| = \| \lambda h \| = | \lambda | = \| x \|^2.$$

Hence as $\| \rho(x^*) \| \leq k \| x^* \| = k \| x \|$,

$$\| Tx \| = \| \rho(x)h \| \geq k^{-1} \| x \|$$

for $x \in H_e$. Now as for all $a \in A$, $\rho(a)h = \rho(a)\rho(e)h = \rho(ae)h$, and so as ρ is irreducible the set $\{\rho(x)h : x \in H_e\}$ is a dense subset of H. From above, the operator mapping $\rho(x)h$ onto x is injective and bounded on this dense subset of H. Hence it may be extended by continuity to a bounded operator from H to H_e which is the inverse of T. Then for all $x \in H_e$

$$T^{-1}\rho(a)Tx = T^{-1}\rho(ax)h = ax = \rho_e(a)x$$

and so ρ is similar to ρ_e.

If ρ is adjoint preserving it is automatically continuous and $\| \rho(a) \| \leq \| a \|$ for all a in A, (see [2, 1.3.7]). Putting $k = 1$ in the above inequalities shows that in this case T is isometric and invertible and consequently T is unitary. This proves the theorem.

COROLLARY 3.6. *For any non-zero single element s of A there is one and only one unitary equivalence class $[\rho]$ of irreducible (adjoint preserving) representations such that $\rho(s) \neq$ for $\rho \in [\rho]$. Also if $\rho \in [\rho]$ then $\rho(s)$ has rank one and $\| \rho(s) \| = \| s \|$.*

Proof. Immediate from Theorem 3.3 and Theorem 3.5.

THEOREM 3.7. *There exists an isometric representation of the C^*-algebra A such that the image of each non-zero single element has rank one.*

Proof. Let $\{[\rho_\gamma] : \gamma \in \Gamma\}$ (Γ an index set) be the set of all unitary equivalence classes of irreducible representations of A. Let $\{\rho_\gamma : \gamma \in \Gamma\}$ be a set consisting of one representative from each equivalence class. Let H_γ be the Hilbert space of ρ_γ. Define a Hilbert space H and a representation π of A on H by

$$H = \oplus \{H_\gamma : \gamma \in \Gamma\},$$
$$\pi(a) = \oplus \{\rho_\gamma(a) : \gamma \in \Gamma\}.$$

Theorem 2.7.3 of [2] states that there exists a set $\{\rho_i : i \in I\}$ of irreducible representations of A such that

$$\| a \| = \sup\{\| \rho_i(a) \| : i \in I\}.$$

Since $\| \rho(a) \|$ depends only on the equivalence class of ρ, we have

$$\| \pi(a) \| = \sup\{\| \rho_\gamma(a) \| : \gamma \in \Gamma\} \geq \| a \|.$$

Since the opposite inequality holds for all representations, (1.3.7 of [2]), π is isometric. If s is any non-zero single element of A, Corollary 3.6 shows that $\rho_\gamma(s) \neq 0$ for exactly one γ and for this γ, $\rho_\gamma(s)$ has rank one. Hence $\pi(s)$ has rank one.

4. Applications

Let A be a C^*-algebra, σ the set of single elements of A, S the set of finite sums of members of σ, and Σ the closure of S. We first identify S as the socle of A (see [7, p, 46]). In view of Corollary 2.6 it is sufficient to prove the following result.

LEMMA 4.1. *If M is a minimal left ideal of A then $M = As$ for some single element s of A. A similar statement holds for right ideals.*

Proof. If $0 \neq s \in M$, $As \subseteq M$. As $s^*s \neq 0$ and M is minimal, $As = M$. We prove that s is single. If $asb = 0$ and $as \neq 0$ then as before $Aas = M$. Hence $Mb = Aasb = (0)$ and so $sb = 0$. A similar proof holds for right ideals.

For the definition of a dual algebra we refer to [7, Chapter II § 8].

THEOREM 4.2. *The C^*-algebra A is dual if and only if $\Sigma = A$.*

Proof. This is an immediate consequence of Lemma 4.1 and Theorem 2.1 of [5] which states that a C^*-algebra is dual if and only if its socle is dense.

If $as = 0$ for all $s \in \sigma$ implies that $a = 0$, we shall say that A is *separated by its single elements*. Since σ is a selfadjoint set an equivalent property is: $sa = 0$ for all $s \in \sigma$ implies that $a = 0$. We now show that for any C^*-algebra the quotient by some ideal results in an algebra with this property. However if

$\sigma = (0)$ this quotient is trivial. Let B be the left annihilator of Σ, that is, $B = \{b : bx = 0 \text{ for all } x \in \Sigma\}$. Since Σ is a selfadjoint ideal, it follows easily that B is also a selfadjoint ideal. We now consider the C^*-algebra A/B. The equivalence class $a + B$ of an element will be denoted by $[a]$.

LEMMA 4.3. *If s is a single element in A then $[s]$ is a single element in A/B and A/B is separated by its single elements.*

Proof. That $[s]$ is single follows from Lemma 3.1. If $[a][t] = 0$ for all single elements $[t]$ of A/B then in particular $[a][s] = 0$ for all single elements s of A. Then $as = 0$ for all $s \in \sigma$ and so $a\Sigma = (0)$. That is, $a \in B$ and $[a] = 0$.

For an algebra that is separated by its single elements a faithful representation can be found without appealing to general representation theory. We first prove the following result for all C^*-algebras. (Recall that $\sigma_e = AeA$.)

LEMMA 4.4. *If e and f are single selfadjoint idempotents, either $\sigma_e = \sigma_f$ or $\sigma_e \cap \sigma_f = (0)$. Hence ρ_e and ρ_f are unitarily equivalent if and only if $\sigma_e = \sigma_f$.*

Proof. Suppose $\sigma_e \cap \sigma_f \neq (0)$. Then for some a, b in A, $afb \in \sigma_e$. Since $f^2 = f$ this implies that $(fa^*af)f(fbb^*f)$ is in σ_e, and the fact that f is single shows that this element is not zero. Hence by Theorem 2.5, $f \in \sigma_e$ and so $\sigma_f \subseteq \sigma_e$. The opposite inclusion is proved in the same way. Therefore $\sigma_e = \sigma_f$.

If $\sigma_e = \sigma_f$ then Theorem 3.3 and Corollary 3.6 show that ρ_e and ρ_f are unitarily equivalent. Conversely if ρ_e and ρ_f are unitarily equivalent then as $\rho_e(e) \neq 0$ we must have $\rho_e(f) \neq 0$. Hence there exists $x \in Ae$ such that $\rho_e(f)x = fx \neq 0$. Then $(xf)^*(fx) = x^*fx \neq 0$ and since $x = xe$, by Theorem 2.5, $x^*fx = \lambda e$. Therefore $e \in \sigma_e \cap \sigma_f$ and thus from above, $\sigma_e = \sigma_f$.

Let A be a C^*-algebra that is separated by its single elements. By Zorn's lemma there exists a set E of non-zero single selfadjoint idempotents that is maximal subject to the condition that $\sigma_e \cap \sigma_f = (0)$ when $e, f \in E$ and $e \neq f$. It follows easily that

$$\sigma = \bigcup\{\sigma_e : e \in E\}.$$

Let

$$\rho = \oplus\{\rho_e : e \in E\}.$$

Then, as in the proof of Theorem 3.6, $\rho(s)$ has rank one for all non-zero single elements of A. Hence if $a \in A$ and $as \neq 0$ for some single element s of A, then $\rho(a)\rho(s) \neq 0$ and so $\rho(a) \neq 0$. Therefore the fact that A is separated by its single elements implies that ρ is faithful. By 1.8.1 of [2], ρ is consequently isometric. If this same construction is carried out with an arbitrary C^*-algebra the resulting representation will have as kernel the annihilator of Σ.

If A is a dual C^*-algebra, from Theorem 4.2, $A = \Sigma$ and so clearly A is separated by its single elements. From Theorem 3.3 it follows that the image of Σ under ρ_e is the closure of the set of all operators of finite rank on H_e.

It is a well known result that this is the set of all compact operators on H_e. Therefore ρ is easily seen to be the standard representation of a dual C^*-algebra as the $C(\infty)$ sum of C^*-algebras each of which is the set of all compact operators on some Hilbert space, (see [4, Theorem 8.3, p. 412]).

We now turn to a characterization of certain W^*-algebras. The set of seminorms on a C^*-algebra A defined by $a \to \| as \|$ for all single elements s of A determines a locally convex topology on A. We call this the *s-topology*. The s-topology is Hausdorff if and only if the single elements separate A.

THEOREM 4.5. *A C^*-algebra is isometrically isomorphic to a direct sum of type I factors if and only if*

(i) *the s-topology is Hausdorff,*

(ii) *the unit ball A_1 of A is complete in the uniform structure associated with the s-topology.*

Proof. Suppose (i) and (ii) hold. Then the representation ρ constructed above is isometric and it follows from Theorem 3.3 that the image under ρ of the s-topology coincides with the strong topology on $\rho(A)$. Then the unit ball of $\rho(A)$ is strongly complete and hence strongly closed. Therefore $\rho(A)$ is strongly closed and by Theorem 1 p. 40 of [1] it is weakly closed.

Let $\mathfrak{B}(H_e)$ be the set of all bounded linear operators on H_e. Clearly

$$\rho(A) \subseteq \oplus\{\mathfrak{B}(H_e) : e \in E\}$$

and $\rho(A)$ contains each operator of finite rank on H_e for all e in E. Hence by von Neumann's double commutant theorem (see [1, Theorem 2, Corollary 1, p. 43]), as $\rho(A)$ is weakly closed,

$$\rho(A)'' = \rho(A) = \oplus\{\mathfrak{B}(H_e) : e \in E\}$$

and $\mathfrak{B}(H_e)$ is a type I factor.

Conversely if A is isometrically isomorphic to a direct sum of type I factors then for a suitable set $\{H_\gamma : \gamma \in \Gamma\}$ of Hilbert spaces, A is isometrically isomorphic to the direct sum $\oplus\{\mathfrak{B}(H_\gamma) : \gamma \in \Gamma\}$, (see [1, p. 121]). As this algebra has the properties (i) and (ii), the theorem follows.

COROLLARY 4.6. *A C^*-algebra is isometrically isomorphic to the algebra of all bounded linear operators on some Hilbert space if and only if, in addition to* (i) *and* (ii) *of Theorem* 4.5,

(iii) $\sigma_e = \sigma_f$ *for all non-zero single selfadjoint idempotents e and f of A.*

Proof. Lemma 4.4 shows that $\rho = \rho_e$ for some non-zero single selfadjoint idempotent e. Hence the direct sum has only one member. The converse is obvious.

Suppose now that A is any C^*-algebra that can be faithfully represented as a W^*-algebra on some Hilbert space H. Different conditions on A for this to hold have been given by Kadison [3] and Sakai [10]. By abuse of notation we suppose that A is a W^*-algebra of operators on H. We may also suppose that $\{ax : a \in A, x \in H\}$ is dense in H. Let σ be the set of single elements of A and let $H_1 = \{sx : s \in \sigma, x \in H\}$. Using Lemma 1.1 it is easy to see that H_1 is invariant under both A and the commutant A' of A. Thus by the double commutant theorem the orthogonal projection f on H_1 is in the centre of A. Let $g = I - f$. Then A can be decomposed into the direct sum

$$A = A_f \oplus A_g .$$

It is easy to see that A_f corresponds to the part of the factor decomposition of A that consists of a direct sum of type I factors. Also A_g is the annihilator B of Σ introduced prior to Lemma 4.3. Thus in this case B is a direct summand. In the next section we show that this is not true in general.

5. Counterexamples

As the definition of a single element is applicable to any algebra, it is natural to ask whether the foregoing theory can be extended to Banach algebras in general. We now show that without further conditions such a generalisation is impossible.

Let A be the algebra of all complex-valued functions continuous on the closed unit disc and holomorphic inside the unit disc of the complex plane, (see [7, p. 304]). This is a Banach algebra with no divisors of zero and hence every element is trivially single. However this algebra has no one-dimensional character of any kind and it can be observed that all the important results of Section 2 are false in this case. Note that A even has an involution defined by

$$f^*(z) = f(\bar{z})$$

which satisfies the condition that $f^*f = 0$ implies that $f = 0$.

It may be significant that the single elements of a C^*-algebra automatically satisfy a norm condition that is not satisfied by the elements of the above algebra. This condition is: for all single elements s,

$$\| asb \| \, \| s \| = \| as \| \, \| sb \|.$$

The condition is a norm analogue of the definition of a single element. The proof of it in a C^*-algebra is as follows. If s is a non-zero selfadjoint single idempotent e, then $\| e \| = 1$ and by Theorem 2.5, $ebb^*e = \lambda e$ and hence $\| eb \|^2 = | \lambda |$. But then

$$\| aeb \|^2 = \| aebb^*ea^* \| = \| \lambda aeea^* \| = \| eb \|^2 \| ae \|^2.$$

For a general single element s, $s = se$ where e is a selfadjoint single idempotent. Then from above,

$$\| asb \| = \| aseb \| = \| as \| \, \| eb \|$$

and

$$\| sb \| = \| seb \| = \| s \| \| eb \|.$$

Combining these two lines gives the result.

Lemma 4.1 is one result which can be proved for any semisimple Banach algebra. On this basis it is possible to develop the theory of semisimple annihilator algebras, (which are rich in minimal ideals), (see [7, Chapter II, §8]). However this is very similar to the standard treatment.

Finally we show that for a general C^*-algebra the annihilator ideal B of Σ is not always a direct summand. Let K be the algebra of all compact operators on some Hilbert space. Let C be some C^*-algebra with no identity and no non-zero single elements, (for example $C_0(X)$ where X is a locally compact Hausdorff space with no isolated points). Consider the algebra $K \oplus C$ and adjoin an identity to it. Then $\Sigma = K \oplus 0$ and $B = 0 \oplus C$. However any complementary subspace to B must contain the identity and hence is not an ideal.

References

1. J. Dixmier, *Les algebres d'operateurs dans l'espace Hilbertien*, Gauthier-Villar Paris, 1957.
2. ———, *Les C^*-algebres et leurs representations*, Gauthier-Villars, Paris, 1964.
3. R. V. Kadison, *Operator algebras with a faithful weakly-closed representation*, Ann. of Math., vol. 64 (1956), pp. 175–181.
4. I. Kaplansky, *Normed algebras*, Duke Math. J., vol. 16 (1949), pp. 399–481.
5. ———, *The structure of certain operator algebras*, Trans. Amer. Math. Soc., vol. 70 (1951), pp. 219–255.
6. W. D. Munn, *Brandt congruences on inverse semigrous*, Proc. London Math. Soc. (3), vol. 14 (1964), pp. 154–164.
7. C. E. Rickart, *Banach algebras*, Van Nostrand, Princeton, N.J., 1960.
8. J. R. Ringrose, *Algebraic isomorphisms between ordered bases*, Amer. J. of Math., vol. 83 (1961), pp. 463–478.
9. ———, *On some algebras of operators II*, Proc. London Math. Soc. (2), vol. 16 (1966), pp. 385–402.
10. S. Sakai, *A characterization of W^*-algebras*, Pacific J. Math., vol. 6 (1956), pp. 763–773.

King's College
London, England