1 C*-algebras

1.1 C*-algebras

Definition 1.1 *A* **C*-algebra** $\mathcal{A}$ *is a complex algebra equiped with an involution*[1] $a \to a^*$ *and a complete submultiplicative norm (i.e.* $\|ab\| \leq \|a\| \, \|b\|$*) satisfying the* **C*-condition**

$$\|a^*a\| = \|a\|^2 \qquad \text{for all } \ a \in \mathcal{A}.$$

If $\mathcal{A}$ has a unit $\mathbf{1}$ then necessarily $\mathbf{1}^* = \mathbf{1}$ and $\|\mathbf{1}\| = 1$.

Definition 1.2 *A* **morphism** $\phi : \mathcal{A} \to \mathcal{B}$ *between C*-algebras is a linear map that preserves products and the involution.*

We will see later that morphisms are automatically contractive, and 1-1 morphisms are isometric (algebra forces topology).

Basic Examples:

- $\mathbb{C}$
- $C(K)$: K compact Hausdorff, $f^*(t) = \overline{f(t)}$: abelian, unital.
- $C_0(X)$: X locally compact Hausdorff, $f^*(t) = \overline{f(t)}$: abelian, nonunital (iff X non-compact).
- $M_n(\mathbb{C})$: A^* = conjugate transpose,
  $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$: non-abelian, unital.
- $\mathcal{B}(\mathcal{H})$: involution defined by $\langle T^*x, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in \mathcal{H}$;
  $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$: non-abelian, unital.

Nonexamples:

- $A(\mathbb{D}) = \{f \in C(\mathbb{T}) : f \text{ extends to } \tilde{f} : \overline{\mathbb{D}} \to \mathbb{C} \text{ s.t. } \tilde{f}|_{\mathbb{D}} \text{ holomorphic}\}$
  ($\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$)

  A closed subalgebra of the C*-algebra $C(\mathbb{T})$ but not a *-subalgebra, because if $f \in A(\mathbb{D})$ then $\tilde{f}^*$ is not holomorphic unless it is constant: $A(\mathbb{D}) \cap A(\mathbb{D})^* = \mathbb{C}\mathbf{1}$: antisymmetric algebra.

[1]that is, a map on $\mathcal{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^*a^*$, $a^{**} = a$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$

- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > j\}$.

  A closed subalgebra of the C*-algebra $M_n(\mathbb{C})$ but not a *-subalgebra. Here $T_n \cap T_n^* = D_n$, the diagonal matrices: a maximal abelian selfadjoint algebra (masa) in M_n.

- $M_{oo}(\mathbb{C})$: infinite matrices with finite support.

  To define norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra, but not complete.

  Its completion is $\mathcal{K}$, the set of compact operators on ℓ^2: a non-unital, non-abelian C*-algebra.

1.2 Von Neumann algebras

$\mathcal{B}(\mathcal{H})$ has other natural topologies:
Say $T_i \xrightarrow{SOT} T$ iff $\|T_i x - Tx\| \to 0 \ \ \forall x \in \mathcal{H}$. A von Neumann algebra $\mathcal{M}$ is a selfadjoint unital subalgebra of $\mathcal{B}(\mathcal{H})$ which is SOT-closed.

C*-algebras : "Non-commutative topology"

von Neumann algebras: "Non-commutative measure theory".

1.3 Units

Every nonunital C*-algebra $\mathcal{A}$ embeds as a C*-algebra (i.e. isometrically and *-homomorphically) in the unital C*-algebra $\mathcal{A}^\sim = \{a + \lambda\mathbf{1} : a \in \mathcal{A}, \lambda \in \mathbb{C}\}$ (norm: later) so that $\mathcal{A}$ is an ideal of codimension 1 in $\mathcal{A}^\sim$.

For example if $\mathcal{A} = C_0(X)$, then $\mathcal{A}^\sim \simeq C(X \cup \{\infty\})$ (where $X \cup \{\infty\}$ is the one-point compactification of X). The map

$$C_0(X)^\sim \to C(X \cup \{\infty\}) : (f, \lambda) \to f + \lambda\mathbf{1}$$

is an isomorphism.

2 Spectral Theory

2.1 The spectrum

Definition 2.1 *If $\mathcal{A}$ is a unital C*-algebra and $\mathcal{A}^{-1}$ denotes the group of invertible elements of $\mathcal{A}$, the* **spectrum** *of an element $a \in \mathcal{A}$ is*

$$\sigma(a) = \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin \mathcal{A}^{-1}\}.$$

If $\mathcal{A}$ is non-unital, the spectrum of $a \in \mathcal{A}$ is defined by

$$\sigma(a) = \sigma_{\mathcal{A}^{\sim}}(a).$$

In this case, necessarily $0 \in \sigma(a)$.

In a unital C*-algebra , if $\|\mathbf{1} - x\| < 1$ then $\sum_{n\geq 0}(\mathbf{1} - x)^n$ converges to an element y such that $xy = yx = \mathbf{1}$. The proof is the same as the case $\mathcal{A} = \mathbb{C}$. Hence if $|\lambda| > \|a\|$ then $\left\|1 - (1 - \frac{a}{\lambda})\right\| < 1$ so $\lambda \notin \sigma(a)$: the spectrum is bounded. Also the spectrum is closed (to prove this, prove that $\mathcal{A}^{-1}$ is open): hence the spectrum is compact.

Lemma 2.1 *The set $\mathcal{A}^{-1}$ is open in $\mathcal{A}$ and the map $x \to x^{-1}$ is continuous (hence a homeomorphism) on $\mathcal{A}^{-1}$.*

Proof We have seen that if $\|\mathbf{1} - x\| < 1$ then $x \in \mathcal{A}^{-1}$. Thus $\mathbf{1}$ is an interior point of $\mathcal{A}^{-1}$. To show that every $y \in \mathcal{A}^{-1}$ is interior, just notice that the map $x \to yx$ is a homeomorphism of $\mathcal{A}^{-1}$ (with inverse $z \to y^{-1}z$) and it maps $\mathbf{1}$ to y.[2]

To show that inversion is continuous, let $a, b \in \mathcal{A}^{-1}$. Then

$$\begin{aligned}\left\|a^{-1} - b^{-1}\right\| &= \left\|b^{-1}(b-a)a^{-1}\right\| = \left\|(b^{-1} - a^{-1})(b-a)a^{-1} + a^{-1}(b-a)a^{-1}\right\| \\ &\leq \left\|b^{-1} - a^{-1}\right\| \|b-a\| \left\|a^{-1}\right\| + \left\|a^{-1}\right\|^2 \|b-a\|\end{aligned}$$

hence

$$\left\|a^{-1} - b^{-1}\right\| (1 - \|b-a\| \left\|a^{-1}\right\|) \leq \left\|a^{-1}\right\|^2 \|b-a\| .$$

It follows that

$$\lim_{b\to a} \left\|b^{-1} - a^{-1}\right\| = 0. \qquad \square$$

[2]In fact the ball $\{x \in \mathcal{A} : \|x - y\| < \frac{1}{\|y^{-1}\|}\}$ is in $\mathcal{A}^{-1}$.

The fact that the spectrum is nonempty is proved by contradiction: the function $\lambda \to (\lambda\mathbf{1} - a)^{-1}$ is 'analytic' on its domain $\mathbb{C} \setminus \sigma(a)$ and is bounded; so if $\sigma(a) = \emptyset$, it would be 'entire' and bounded, hence constant ('Liouville')

The **spectral radius**

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

satisfies $\rho(a) \leq \|a\|$. The **Gelfand-Beurling** formula is

$$\rho(a) = \lim_n \|a^n\|^{1/n} \leq \|a\| .$$

Exercise 2.2 *Any morphism $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras extends uniquely to a unital morphism $\tilde{\phi} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ by $\tilde{\phi}(a, \lambda) = (\phi(a), \lambda)$.*

If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital morphism between unital algebras, then $\sigma(\phi(a)) \subseteq \sigma(a)$ for all $a \in \mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi(\mathbf{1}) \neq \mathbf{1}$, then $\sigma(\phi(a)) \subseteq \sigma(a) \cup \{0\}$.

Proposition 2.3
(i) $a = a^* \implies \sigma(a) \subseteq \mathbb{R}$
(ii) $a = b^*b \implies \sigma(a) \subseteq \mathbb{R}^+$
(iii) $u^*u = 1 = uu^* \implies \sigma(u) \subseteq \mathbb{T}$

Proof of (iii) We have $\rho(u) \leq \|u\| = 1$ so $\sigma(u) \subseteq \mathbb{D}$. Also u^{-1} is unitary so $\sigma(u^{-1}) \subseteq \mathbb{D}$. Thus if $|\lambda| < 1$ the element $x = (\lambda^{-1} - u^{-1})$ is invertible. But then $(\lambda - u)u^{-1} = \lambda(u^{-1} - \lambda^{-1})$ is invertible and hence so is $\lambda - u$. Therefore $\lambda \notin \sigma(u)$ so $\sigma(u) \subseteq \{\lambda : |\lambda| = 1\}$.

Proof of (i) Let $u(t) = \exp(ita)$ ($t \in \mathbb{R}$) (power series). Note that $u(t)^* = \exp(-ita)$ because $a = a^*$. Show that $u'(t) = au(t) = u(t)a$ as in Calculus. It follows that if $f(t) = u(t)u(-t)$ then $f'(t) = 0$ for all $t \in \mathbb{R}$ so $f(t) = f(0) = 1$ hence $u(t)u(t)^* = u(t)^*u(t) = 1$. Thus by (iii) we have $\sigma(\exp ita) \subseteq \mathbb{T}$.

Let $\lambda \in \sigma(a)$. Then

$$\begin{aligned}\exp(ia) - \exp(i\lambda) &= e^{i\lambda}(\exp i(a - \lambda) - 1) = e^{i\lambda} \sum_{n=1}^{\infty} \frac{i^n}{n!}(a - \lambda)^n \\ &= e^{il}(a - \lambda)b\end{aligned}$$

where $b \in \mathcal{A}$ commutes with $a - \lambda$. Thus $\exp(ia) - \exp(i\lambda)$ cannot be invertible. Therefore $e^{i\lambda} \in \sigma(\exp(ia)) \subseteq \mathbb{T}$ and so $\lambda \in \mathbb{R}$.

Second proof Let $a = a^*$. If $\lambda + i\mu \in \sigma(a)$ for some $\lambda, \mu \in \mathbb{R}$ with $\mu \neq 0$, then the element $a - (\lambda + i\mu)\mathbf{1} = -\mu(\frac{\lambda\mathbf{1}-a}{\mu} + i\mathbf{1})$ is not invertible. Thus, replacing a by the selfadjoint element $b = \frac{\lambda\mathbf{1}-a}{\mu}$, it suffices to show that $-i \notin \sigma(b)$. So suppose that $-i \in \sigma(b)$ and let $n \in \mathbb{N}$. Then $n+1 \in \sigma(n\mathbf{1}+ib)$ because $(n+1)\mathbf{1}-(n\mathbf{1}+ib) = i(-i\mathbf{1}+b)$ is not invertible. Therefore $|n+1| \leq \|n\mathbf{1} + ib\|$ hence

$$(n+1)^2 \leq \|n\mathbf{1} + ib\|^2 \stackrel{(c^*)}{=} \|(n\mathbf{1} + ib)^*(n\mathbf{1} + ib)\| \stackrel{(b=b^*)}{=} \left\|n^2\mathbf{1} + b^2\right\| \leq n^2+\left\|b^2\right\|.$$

Thus $2n + 1 \leq \|b^2\|$ for all n, a contradiction.

The proof of **(ii)** is non-trivial: see later.

Lemma 2.4 *If $aa^* = a^*a$ (we say a is* **normal***) then $\rho(a) = \|a\|$. This is not true in general: consider any $a \neq 0$ with $a^2 = 0$.*

Proof

$$\|a\|^4 = \|a^*a\|^2 = \|(a^*a)^*(a^*a)\| = \left\|(a^2)^*a^2\right\| = \|a^2\|^2$$

hence $\|a\|^2 = \|a^2\|$ and inductively $\|a\|^{2^n} = \|a^{2^n}\|$ for all n. Thus $\rho(a) = \lim \left\|a^{2^n}\right\|^{2^{-n}} = \|a\|$. □

Proposition 2.5 *There is at most one norm on a *-algebra making it a C*-algebra.*

Proof The norm is defined algebraically:

$$\|a\|^2 = \|a^*a\| = \rho(a^*a).$$

Dependence of spectrum on the algebra If $\mathcal{A}$ is a unital C*-algebra and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ containing the identity, then every $b \in \mathcal{B}$ satisfies

$$\sigma_{\mathcal{A}}(b) \subseteq \sigma_{\mathcal{B}}(b).$$

Indeed if $\lambda \notin \sigma_{\mathcal{B}}(b)$ then $\lambda\mathbf{1} - b$ has an inverse in $\mathcal{B}$ hence also in $\mathcal{A}$. But equality need not hold:

For example suppose $\mathcal{A} = C(\mathbb{T})$, the continuous functions on the unit circle. Let $\mathcal{B}$ be the disk algebra and $b \in \mathcal{B}$ be the function $b(z) = z$. The function b^{-1} given by $b^{-1}(z) = \frac{1}{z}$ is continuous on $\mathbb{T}$, but does not have an extension to $\overline{\mathbb{D}}$ which is holomorphic in $\mathbb{D}$. It is remarkable that if $\mathcal{B}$ is a C*-subalgebra this cannot happen:

Proposition 2.6 (Permanence of spectrum) *If $\mathcal{A}$ is a unital C*-algebra and $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$ containing the identity, then every $b \in \mathcal{B}$ satisfies*

$$\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b).$$

Proof It is enough to show that if b has an inverse in $\mathcal{A}$, then this inverse is contained in $\mathcal{B}$.

Suppose first that $b = b^* \in \mathcal{A}^{-1}$. Since $\sigma_{\mathcal{B}}(b) \subseteq \mathbb{R}$, for each $n \in \mathbb{N}$ we have $\frac{i}{n} \notin \sigma_{\mathcal{B}}(b)$. Thus if $x_n = b - \frac{i}{n}\mathbf{1}$, all x_n^{-1} belong to $\mathcal{B}$. But since $x_n \to b$ and inversion is continuous *on the space* $\mathcal{A}^{-1}$, $x_n^{-1} \to b^{-1}$. Since $x_n^{-1} \in \mathcal{B}$ and $\mathcal{B}$ is closed, it follows that $b^{-1} \in \mathcal{B}$ as required.

For the general case, if $b \in \mathcal{B}$ is invertible in $\mathcal{A}$, so is b^* (verify) and hence so is $x = b^*b$. But x is selfadjoint, so by the previous paragraph $x \in \mathcal{B}^{-1}$. If $y = x^{-1}$, we have $yb^*b = yx = \mathbf{1}$ and so

$$b^{-1} = (yb^*b)b^{-1} = (yb^*)(bb^{-1}) = yb^*$$

hence $b^{-1} \in \mathcal{B}^{-1}$, which completes the proof.

2.2 Commutative C*-algebras

Theorem 2.7 (Gelfand-Naimark 1) *Every commutative C*-algebra $\mathcal{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is the set of nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$ with the topology of pointwise convergence. The map is the Gelfand transform: $a \to \hat{a}$ where $\hat{a}(\phi) = \phi(a)$ $(\phi \in \hat{\mathcal{A}})$. The algebra $\mathcal{A}$ is unital iff $\hat{\mathcal{A}}$ is compact.*

In more detail: $\hat{\mathcal{A}}$ is the set of all *nonzero* multiplicative linear forms (*characters*) $\phi : \mathcal{A} \to \mathbb{C}$, (necessarily $\|\phi\| \leq 1$ and, when $\mathcal{A}$ is unital, $\|\phi\| = \phi(\mathbf{1}) = 1$) equipped with the w*-topology: $\phi_i \to \phi$ iff $\phi_i(a) \to \phi(a)$ for all $a \in \mathcal{A}$.

When $\mathcal{A}$ is non-abelian there may be no characters (consider $M_2(\mathbb{C})$ or $\mathcal{B}(\mathcal{H})$, for example).

When $\mathcal{A}$ is abelian there are 'many' characters: for each $a \in \mathcal{A}$ there exists $\phi \in \hat{\mathcal{A}}$ such that $\|a\| = |\phi(a)|$.

When $\mathcal{A}$ is unital $\hat{\mathcal{A}}$ is compact and $\mathcal{A}$ is isometrically *-isomorphic to $C(\hat{\mathcal{A}})$.

When $\mathcal{A}$ is abelian but non-unital every $\phi \in \hat{\mathcal{A}}$ extends uniquely to a character $\tilde{\phi} \in \widehat{\mathcal{A}^{\sim}}$ by $\tilde{\phi}(\mathbf{1}) = 1$, and there is exactly one $\phi_\infty \in \widehat{\mathcal{A}^{\sim}}$ that vanishes on $\mathcal{A}$. Thus $\mathcal{A}$ is *-isomorphic to $C_0(\hat{\mathcal{A}})$, the algebra of those continuous functions on the 'one-point compactification' $\hat{\mathcal{A}} \cup \{\phi_\infty\}$ which vanish at ϕ_∞.

2.3 Positivity

Definition 2.2 *An element $a \in \mathcal{A}$ is* ***positive*** *if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$. We write $\mathcal{A}_+ = \{a \in \mathcal{A} : a \geq 0\}$. If a, b are selfadjoint, we define $a \leq b$ by $b - a \in \mathcal{A}_+$.*

Examples 2.8 *In $C(K)$: $f \geq 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in K$ because $\sigma(f) = f(K)$.*

In $\mathcal{B}(\mathcal{H})$: $T \geq 0$ iff $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.

Remark 2.9 *Any morphism $\pi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras preserves order:*

$$a \geq 0 \quad \Rightarrow \quad \pi(a) \geq 0.$$

Proof If $a = a^*$ and $\sigma(a) \subseteq [0, +\infty)$ then $\pi(a)^* = \pi(a^*)$ and

$$\sigma(\pi(a)) \subseteq \sigma(a) \cup \{0\} \subseteq [0, +\infty)$$

so $\pi(a) \geq 0$.

Remark 2.10 *If $a = a^*$ then* $\qquad -\|a\| \mathbf{1} \leq a \leq \|a\| \mathbf{1}$.

Proof For $\|a\| \mathbf{1} - a \geq 0$ observe that $\|a\| \mathbf{1}$ is selfadjoint and

$$\sigma(\|a\| \mathbf{1}) = \{\|a\| - \lambda : \lambda \in \sigma(a)\} \subseteq \mathbb{R}_+$$

because $\lambda \in \mathbb{R}$ and $\lambda \leq \|a\|$ for $\lambda \in \sigma(a)$.

Proposition 2.11 *Every positive element has a unique positive square root. In fact*

$$a \in \mathcal{A}_+ \quad \Longleftrightarrow \quad \text{there exists } b \in \mathcal{A}_+ \text{ such that } a = b^2.$$

Proof If $a \geq 0$, consider the C*-subalgebra $\mathcal{C} = C^*(a)$ of $\mathcal{A}$ generated by a; it is (*-isomorphic to) the algebra $C_o(X)$ for some X. Note that $a \in \mathcal{C}_+$ since $\sigma_{\mathcal{C}}(a) = \sigma_{\mathcal{A}}(a)$. The *-isomorphism and its preserve order. Since $a \geq 0$, we have $\hat{a} \geq 0$. Look at the function $\sqrt{\hat{a}} \in C_o(X)$. This is the image of some $b \in \mathcal{A}$, which must be positive because $\sqrt{\hat{a}} \geq 0$, and $(\hat{b})^2 = \hat{a}$, so $b^2 = a$.

Conversely if $a = b^2$, look at the C*-subalgebra $C^*(b)$ of $\mathcal{A}$ generated by b; it is (*-isomorphic to) the algebra $C_o(Y)$ for some Y. Since $b \geq 0$, we have $\hat{b} \geq 0$, so $\hat{a} = \widehat{b^2} \geq 0$ and so $a \geq 0$.

Uniqueness: If a, b are as above and $c \geq 0$ satisfies $c^2 = a$ then observe that $ac = ca$. Since b is in $C^*(a)$ it follows that $bc = cb$. Now consider the C*-algebra $C^*(b, c)$: it is abelian and contains a, so we may view b, c, a as continuous functions on *the same space* and then it is clear that $b = c$.

Proposition 2.12 *For any C*-algebra the set $\mathcal{A}_+$ is a* cone*:*

$$a, b \in \mathcal{A}_+, \ \lambda \geq 0 \quad \Rightarrow \quad \lambda a \in \mathcal{A}_+, a + b \in \mathcal{A}_+.$$

Proof The first assertion is immediate from the definition. Hence, for the second one, it is enough to assume that $\|a\| \leq 1$ and $\|b\| \leq 1$ and prove that $\frac{a+b}{2} \geq 0$. Also, there is no loss in assuming that $\mathcal{A}$ is unital.

But we have the following characterization:

Lemma 2.13 *In a unital C*-algebra if $x = x^*$ and $\|x\| \leq 1$, then*

$$x \geq 0 \quad \Longleftrightarrow \quad \|\mathbf{1} - x\| \leq 1.$$

Thus if a and b are positive contractions then $\frac{a+b}{2}$ is a selfadjoint contraction and

$$\left\| \mathbf{1} - \frac{a+b}{2} \right\| = \frac{1}{2} \|(\mathbf{1} - a) + (\mathbf{1} - b)\| < 1$$

so that $\frac{a+b}{2} \geq 0$.

Proof of the Lemma Considering the C*-algebra generated by x and $\mathbf{1}$, there is no loss in assuming that x is a continuous function on a compact set. Then the Lemma is just an application of the triangle inequality: The assumption is that $-1 \leq x(t) \leq 1$ for all t and we need to conclude that

$$x(t) \geq 0 \quad \Longleftrightarrow \quad |1 - x(t)| \leq 1.$$

But this is obvious!

Theorem 2.14 *In any C^*-algebra a^*a is positive.*

Proof Of course a^*a is selfadjoint. So it can be written

$$a^*a = b - c \quad \text{where } b, c \geq 0,\ bc = 0$$

(to see this, consider a^*a as a function and let b and c be its positive and negative parts).

Let $x = ca^*$. Observe that

$$xx^* = ca^*ac = c(b-c)c = -c^3$$

and so, since $c \geq 0$,

$$-xx^* \in \mathcal{A}_+.$$

On the other hand, if we write $x = u + iv$ with u, v selfadjoint, we find

$$xx^* + x^*x = 2u^2 + 2v^2 \in \mathcal{A}_+$$

since $\mathcal{A}_+$ is a cone. Adding the last two relations

$$x^*x = -xx^* + (xx^* + x^*x) \in \mathcal{A}_+$$

again since $\mathcal{A}_+$ is a cone. Thus we have

$$\sigma(x^*x) \subseteq \mathbb{R}_+ \quad \text{and} \quad \sigma(xx^*) \subseteq \mathbb{R}_-.$$

But in any unital algebra we have $\sigma(kh) \subseteq \sigma(hk) \cup \{0\}$. Indeed if $\lambda \notin \sigma(hk)$ is nonzero then the element

$$y = \lambda^{-1}\mathbf{1} + \lambda^{-1}k(\lambda\mathbf{1} - hk)^{-1}h$$

satisfies $y(\lambda\mathbf{1} - kh) = (\lambda\mathbf{1} - kh)y = \mathbf{1}$ and so $\lambda \notin \sigma(kh)$.

It follows that $\sigma(xx^*) = \{0\}$. Thus $\|xx^*\| = 0$ (xx^* is selfadjoint) showing that $-c^3 = xx^* = 0$ and so $c = 0$. $\square$

Remark 2.15 *If $a \leq b$ then for all $c \in \mathcal{A}$*

$$c^*ac \leq c^*bc \leq \|c\|^2 b$$
$$a^t \leq b^t \quad \textit{(when } a \geq 0 \textit{ and } t \in (0,1)\textit{)}.$$

But if $0 \leq a \leq b \Rightarrow a^2 \leq b^2$, then $\mathcal{A}$ is commutative.

Proof Omitted

2.4 Approximate units

Every C*-algebra has an **approximate unit**, namely a net $\{u_\lambda\}$ such that $\|u_\lambda x - x\| \to 0$ and $\|x u_\lambda - x\| \to 0$. In fact $\{u_\lambda\}$ can be chosen contractive ($\|u_\lambda\| \le 1$), positive ($u_\lambda \ge 0$) and increasing ($\lambda \le \mu \Rightarrow u_\mu - u_\lambda \ge 0$).

If the algebra is separable, then the approximate unit may be chosen to be a sequence.

The general result is the following:

Theorem 2.16 *For any C*-algebra $\mathcal{A}$, the set*

$$\Lambda = \{u \in \mathcal{A}_+ : \|u\| < 1\}$$

is upward directed[3] *and an approximate unit.*[4]

But in special cases we can choose the approximate unit to have additional properties:

Examples 2.17 *In $C_0(\mathbb{R})$, we may choose $\{u_n\}_{n\in\mathbb{N}}$ where u_n is any continuous function such that $0 \le u_n \le 1$, $u_n(t) = 1$ for $t \in [-n, n]$ and $u_n(t) = 0$ for $|t| > n+1$. In particular we may choose the approximate unit to be in the ideal of functions of compact support, we can choose it to consist of piecewise linear, or infinitely differentiable functions.*

In $c_0(\mathbb{N})$, we may choose u_n to be the characteristic function of $\{0, 1, \ldots, n\}$. In $c_0(\Gamma)$, we can choose u_λ to be the characteristic function of a finite subset $\lambda \subseteq \Gamma$ (the indexing is by inclusion).

In $\mathcal{K}(\ell^2)$ (the 'non-commutative analogue of c_0') we may choose u_n to be the projection onto the subspace spanned by $\{e_0, e_1, \ldots, e_n\}$. Here the approximate unit may be chosen in the ideal of finite rank oeprators.

[3] i.e. if $u_1, u_2 \in \Lambda$ there is $u_3 \in \Lambda$ with $u_3 \ge u_j$ for $j = 1, 2$

[4] i.e. given $a \in \mathcal{A}$ and $\epsilon > 0$ there is $u_o \in \Lambda$ such that for all $u \in \Lambda$ with $u \ge u_o$ we have $\|ua - a\| < \epsilon$ and $\|au - a\| < \epsilon$