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Another 'short' proof of the Riesz representation theorem

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1. *Introduction.* In [**2**], a short synthetic proof of the Riesz representation theorem was given; this used the Hahn–Banach theorem, the Stone–Čech compactification of a discrete space and the Caratheodory extension procedure for measures. In this note, we show how the theorem can be proved using ultrapowers in place of the Stone–Čech compactification. We also describe how the proof can be expressed in a non-standard way (a rather different non-standard proof has been given by Loeb [**4**]).

2. *The Riesz representation theorem.* We prove this in the same form as in [**2**]:

THEOREM. *If F is a positive linear functional on the space $C(X)$ of continuous real-valued functions on a compact Hausdorff space X, then there exists a unique Baire measure π on X such that*

$$F(f) = \int_X f\,d\pi \quad \textit{for each } f \textit{ in } C(X).$$

Proof. By the Hahn–Banach theorem there is a linear functional G on the Banach space $\mathrm{Ba}_b(X)$ of bounded Baire-measurable functions on X with the supremum norm, which extends F, and with $\|G\| = \|F\| = F(1)$. Since $\|G\| = G(1)$, G is a positive linear functional on $\mathrm{Ba}_b(X)$. If $A \in \mathrm{Baire}\,(X)$, let $\nu(A) = G(\chi_A)$. ν is a finitely additive set function.

Now let $\mathscr{U}$ be a free ultrafilter on the integers, and let $X/\mathscr{U}$ be the set-theoretic ultrapower of X. If $x/\mathscr{U} \in X/\mathscr{U}$, let $j(x) = \lim_{\mathscr{U}} x(i)$, the limit of $x(i)$ along the ultrafilter $\mathscr{U}$.

If A is a subset of X, let $A/\mathscr{U} = \{x/\mathscr{U}\colon x(i) \in A \text{ almost certainly}\}$. Since

$$A/\mathscr{U} \cup B/\mathscr{U} = (A \cup B)/\mathscr{U} \quad \text{and} \quad C(A)/\mathscr{U} = C(A/\mathscr{U})$$

the set $R = \{A/\mathscr{U}\colon A \text{ a Baire set}\}$ is an algebra. If $A/\mathscr{U} \in R$, let $\lambda(A/\mathscr{U}) = \nu(A)$. λ is finitely additive on R and, since every decreasing sequence of non-empty sets in R has a non-empty intersection, λ is *trivially* σ-additive. By the usual Caratheodory extension procedure, λ extends to a measure μ on the σ-algebra $\sigma(R)$ generated by R.

Now if E is a closed G_δ subset of X, there exists a decreasing sequence (O_i) of open subsets of X such that $E = \bigcap_{i=1}^\infty O_i$ and $\overline{O_{i+1}} \subseteq O_i$ for each i. It follows from the definition of j that $j^{-1}(E) = \bigcap_{i=1}^\infty (O_i/\mathscr{U})$. Thus j is $\sigma(R)$-to-Baire measurable. Let π be the image of μ under j, restricted to the Baire sets: if A is a Baire set in X then $\pi(A) = \mu(j^{-1}(A))$.

Next suppose that $f \in C(X)$ and that $0 \leqslant f \leqslant 1$. Let $C_{j,n} = \{x\colon f(x) \geqslant j/n\}$, let $D_{j,n,m} = \{x\colon f(x) > j/n - 1/m\}$ and let $I_{j,n,m}$ be the characteristic function of $D_{j,n,m}$. As above, $\overline{D_{j,n,m+1}} \subseteq D_{j,n,m}$ and $C_{j,n} = \bigcap_{m-1}^\infty D_{j,n,m}$, so that

$$\pi(C_{j,n}) = \lim_{m\to\infty} G(I_{j,n,m}).$$

Thus
$$\int f\,d\pi = \lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{1}{n}(C_{j,n})$$
$$= \lim_{n\to\infty}\lim_{m\to\infty} G\left(\sum_{j=0}^{n-1}\frac{1}{n}I_{j,n,m}\right).$$

But $\|\sum_{j=0}^{n-1}(1/n)\,I_{j,n,m}-f\|_\infty < 1/n+1/m$ for $m > n$, and so it follows from the continuity of G that $\int f d\pi = G(f)$. Thus π represents F; the uniqueness of π follows from the regularity of π and Urysohn's lemma.

3. *Non-standard methods.* The key to this proof is of course the fact that λ is *trivially* σ-additive. A similar fact is the key to the proof given in [**2**] and is also the key to the construction of Loeb measure in non-standard measure theory. In fact, the proof given here can be put (with slight adjustments) in a non-standard form, as we now describe. We use the notation and terminology of [**1**], which contains details of non-standard methods, and of Loeb measures, that are used.

Let G and ν be obtained as above. Let $V(S)$ be a superstructure which contains $C(X)$, and let ${}^*V(S)$ be an ω_1-saturated non-standard universe. Then ${}^*\nu$ is a finitely additive mapping from *(Baire (X)) into ${}^*[0, \|G\|]$. By Loeb's theorem ([**1**], theorem 3·1), we can construct the Loeb-measure space $({}^*X, L({}^*(\text{Baire}\,(X))), L({}^*\nu))$.

Now let j be the standard part map from ${}^*X$ to X. Arguments similar to those given above show that j is $L({}^*(\text{Baire}\,(X)))$-to-Baire measurable (this was shown by Henson [**3**]), and that if π is the image of $L({}^*\nu)$ under j then $\int f d\pi = F(f)$ for each f in $C(X)$.

Comparing the two arguments, we see that $R = \{{}^*A : A \in \text{Baire}\,(X)\}$, a proper subset of *(Baire (X)) and that λ is the restriction of ${}^*\nu$ to R, μ the restriction of $L({}^*\nu)$ to $\sigma(R)$.

REFERENCES

[**1**] N. J. CUTLAND. Nonstandard measure theory and its applications. *Bull. London Math. Soc.* **15** (1983), 529–589.

[**2**] D. J. H. GARLING. A 'short' proof of the Riesz representation theorem. *Proc. Cambridge Philos. Soc.* **73** (1973), 459–460.

[**3**] C. W. HENSON. Analytic sets, Baire sets and the standard part map. *Canad. J. Math.* **31** (1979), 663–672.

[**4**] P. LOEB. A functional approach to non-standard measure theory. *Contemporary Mathematics* **26** (1984), 251–261.