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Commutative Harmonic Analysis

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The present article is organized around four themes: 1. the theorem of Fejér, 2. the theorem of Riesz–Fischer, 3. boundary values of analytic functions, 4. Riesz products and lacunary trigonometric series. This does not cover the whole field of the Hungarian contributions to commutative harmonic analysis. A final section includes a few spots on other beautiful matters. Sometimes references are given in the course of the text, for example at the end of the coming paragraph on Fejér. Other can be found at the end of the article.

1. The theorem of Fejér

On November 19, 1900 the Académie des Sciences in Paris noted that it had received a paper from Leopold FEJEV in Budapest with the title “Proof of the theorem that a bounded and integrable function is analytic in the sense of Euler”. On December 10 the Comptes Rendus published the famous note “On bounded and integrable fonctions”, in which Fejér sums, Fejér kernel and Fejér summation process appear for the first time, and where the famous Fejér theorem, which asserts that any decent function is the limit of its Fejér sums, is proved. The spelling error of November 19 in Fejér’s name (Fejev instead of Fejér) is not reproduced on December 10. It is replaced by an other one: the paper presented by Picard is attributed to Leopold TEJER. This is how Fejér’s name enters into history (*C.R. Acad. Sci. Paris,* **131** (1900), 825 and 984–987).

Fejér was a very young man, 20 years old, and unknown. But Fejér’s theorem became famous quickly. It was used for proving the completeness of the trigonometric system in Hurwitz’s work on the isoperimetric problem, extended by Lebesgue to Lebesgue-integrable fonctions (Fejér–Lebesgue theorem), generalized to other kernels useful in approximation theory (Ch. de la Vallée Poussin), applied in a simple and completely new proof of the Dirichlet–Jordan theorem (Hardy), made more precise by the notion of absolute summability (Hardy and Littlewood), and, above all, its simplicity made it accessible to any mathematics student {1}, {2}, {3}, {4, p. 245}, {5}, {6}. By the 1910's the Fejér theorem had already the status of a classical result, and no mathematician could ignore Fejér's name and its spelling — perhaps except for the place of the accent.

The change can be appreciated in comparing the first and the second edition of Ch. de la Vallée Poussin's Cours d'Analyse {7}. In the first edition (1903–1906) it follows the tradition of the great analysis textbooks of the time: it devotes little place to Fourier series and presents Dirichlet's convergence theorem in the tradition of Dirichlet and Jordan. In the second edition (1912) we can find the Fejér theorem, Hardy's method for deducing the Dirichlet–Jordan theorem from it, and also Fejér's example of a continuous functions whose Fourier series diverges at one point. The difference shows well enough the importance of the revolution which took place.

What was the nature of this revolution? In order to see this, let us go back to 1900.

In the 19th century the theory of analytic functions of one complex variable progressed by giant leaps. It had become "Theory of Functions" par excellence. The theory of functions of several real variables had developed through the investigation of partial differential equations arising in physics. On the other hand, the field of functions of one real variable revealed strange and disquieting creatures: continuous but nowhere differentiable functions (Weierstrass), continuous functions whose Fourier series diverges at one point (du Bois Reymond). Hermite wrote to Stieltjes that he "turned away with fright and horror from that lamentable ulcer: a continuous function with no derivative". Poincaré in l'Enseignement Mathématique complained that examples were not constructed any more in order to illustrate theorems and theories, but just for the purpose of showing that our predecessors were wrong. Gaston Darboux, who wrote an important memoir on discontinuous functions in 1875, turned to geometry quite prudently.

Between 1880 et 1900 there were only a few works on trigonometric series and they did not attract much attention. Fourier series did not appear as a reliable tool; there were too many strange things about them. Maybe there are continuous functions whose Fourier series diverge everywhere, just as there are nowhere differentiable continuous functions (this was still an open question in 1965, before Carleson's theorem {8}). It is even possible that the Fourier series converges but does not represent the function, as it is the case for Taylor series of C^∞-functions? This question seems to have been raised by Minkowski ([40, I, p. 24]).

Emile Picard's Traité d'analyse, which appeared in 1891, is quite instructive in this respect. The problem of finding a harmonic function in a domain when boundary values are given (the Dirichlet problem) is treated for the sphere, in the section of the book devoted to functions of several variables, before being treated for the circle. This is not because things are worse for the circle than for the sphere. But the solution for the circle belongs to the chapter "Fourier series" and this is not a subject to begin with {9}.

However, the pages devoted by Emile Picard to the Dirichlet problem on the circle are quite interesting; in 1900 Fejér knew them well and refers to them in his note. Picard presents the method of Schwarz (1872), which is based on the properties of the "Poisson kernel"

$$\frac{1-r^2}{1-2r\cos t+r^2} = 1 + 2\sum_1^\infty r^n \cos nt$$

As an application he shows that a continuous function on the circle whose Fourier series is

$$\sum_0^\infty (a_n \cos nt + b_n \sin nt)$$

can be expressed as a uniform limit of trigonometric polynomials of the form

$$\sum_{n\le N(r)} (a_n \cos nt + b_n \sin nt) r^n,$$

providing therefore a new proof of the Weierstrass approximation theorem.

In 1893, Ch. de la Vallée Poussin also used the Poisson kernel in order to establish the Parseval formula, which is essentially equivalent to the totality of the trigonometric system {10}. In 1901 Adolph Hurwitz stated the Parseval formula as a lemma to his solution of the isoperimetric problem by means of Fourier series, saying that he would prove it later {11}.

There were actually a few interesting results on Fourier series, but results and problems were not related to each other. The problem of Minkowski could have been solved easily using the method of Schwarz presented in

Picard's book, but Picard did not know about the problem and Minkowski was apparently unaware of this part of the works of Schwarz and Picard. Schwarz's method also yielded the formula obtained by Ch. de la Vallée Poussin, but de la Vallée Poussin was not aware of it. Certainly Hurwitz did not know the paper of de la Vallée Poussin: this is acknowledged in his article of 1903 {1}. Thus all these were isolated works on a marginal subject.

Fejér wrote at the beginning of his thesis that nothing essentially new appeared on Fourier series between 1885 and 1900. Though this is not completely true, still Fourier series appeared as a stagnant subject, out of fashion.

Fejér's discovery is that the averages of the partial sums

$$\sigma_n = \frac{1}{n}(S_0 + S_1 + \cdots + S_{n-1})$$

approximate the given function f at each point where $f(x+0)$ and $f(x-0)$ exist and $f(x) = \frac{1}{2}\big(f(x+0)+f(x-0)\big)$, and uniformly when f is continuous on the circle. Let us trace the circumstances of that discovery.

The idea of assigning a sum to a divergent series by means of some summation process was familiar to mathematicians. According to Lebesgue, d'Alembert already used the process of taking averages of partial sums for the series

$$\frac{1}{2} + \sum_1^\infty \cos nt \qquad (0 < t < 2\pi).$$

Summation processes became a significant topic of Abel's investigations, then of those of Poisson, Frobenius, Hölder, Cesàro and Borel. Abel proved the famous theorem asserting that

$$\lim_{r\uparrow 1} \sum_1^\infty a_n r^n = \sum_1^\infty a_n$$

whenever the right-hand side exists in the usual sense. Poisson considered the left hand side as the generalized sum of the series, whether or not the series converges. Frobenius generalized Abel's theorem by showing that

$$\lim_{r\uparrow 1} \sum_1^\infty a_n r^n = \lim_{n\to\infty} \frac{1}{n}(S_0 + S_1 + \cdots + S_{n-1})$$

whenever the right side exists. Hölder generalized this theorem of Frobenius by iterating the process of arithmetic means. Cesàro generalized another theorem of Abel, on multiplication of series, and introduced for this purpose the processes that we now denote by (C, k). Emile Borel defined new summation processes and applied them to the analytic continuation of functions defined by a Taylor series. Fejér knew all these works.

In order to solve the Dirichlet problem for the circle in the form

$$f(r\cos t, r\sin t) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos nt + b_n \sin nt) r^n$$

when the function prescribed at the boundary has the Fourier series

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos nt + b_n \sin nt)$$

one might think of using Abel's theorem. However, the example of P. du Bois Reymond excludes any hope to obtain a solution for an arbitrary continuous function in this way. Could one apply the theorem of Frobenius? This seems to have been the starting point of Fejér.

Fejér learned about this problem during the academic year 1899–1900 which he spent as a student at the University of Berlin. He obtained a solution in Budapest at the end of October. He observed quickly that the method he used was more important that the new solution of the Dirichlet problem. In his Comptes Rendus note, the solution of the Dirichlet problem appears as one of the consequences of his theorem. The proof is based on the kernel

$$K_n(x) = \frac{1 - \cos nx}{n(1 - \cos x)},$$

and it is strongly related to Schwarz's solution of the Dirichlet problem, which makes use of the Poisson kernel.

Fejér indeed did **not** introduce any new summation process for divergent series; on the contrary, he used the most evident of them. It was **not** he who introduced positive kernels in investigating Fourier series: the Poisson kernel was well known and its application to Fourier series could be found in Picard's treatise long before 1900. Fejér did **not** solve a difficult conjecture by sophisticated methods.

What he did is much more than that. He gave a clear, simple and powerful statement in a field where the strange and the bizarre prevailed before. By coupling Fourier series and summation processes he provided a convenient frame for both theories. Since that time, the summation process of Riemann (based on the second differences of the second integral), the method of Schwarz for the Dirichlet problem on the circle (that is, what Fourier already used for computing temperatures inside a heated body) and the newly introduced process of Weierstrass in order to study temperature as a function of time by means of the series

$$\sum (a_n \cos nx + b_n \sin nx) e^{-nt^2}$$

appeared as expressing the same principle, most simply presented in Fejér's note: on one hand, regularization of the function by means of a convenient kernel, on the other, a summation process for Fourier series. The role of positive kernels was emphasized, and developed in many works of Fejér himself later (see section 5). Cesàro processes of different orders appeared for different purposes — for Laplace series, the relevant process is $(C, 2)$ ([40, I no 22, 24, 28]; [40, II no 63]). The two pages long note of Fejér completely changed the position of trigonometric series in mathematics. It also gave impetus to the study of general summation methods.

I shall restrict myself to a very few examples of applications and continuations of Fejér's theorem, that I chose because they involve Hungarian mathematicians. Others can be found in my joint book with Pierre-Gilles Lemarié-Rieusset, Fourier series and wavelets ([83, part I, chapter 7]).

The Fejér kernel can be written as

$$K_n(x) = \sum_{-n}^{n} \left(1 - \frac{|m|}{n}\right) e^{imx}.$$

A linear combination of K_n with positive coefficients yields a positive function whose Fourier coefficients c_m (in the complex form) are even ($c_m = c_{-m}$), positive (≥ 0), convex for $m \geq 0$ and decreasing to 0 as $m \to \infty$. Conversely each sequence (c_m) of this form is a sequence of Fourier coefficients of a positive and integrable function. This is a theorem of W. H. Young (1913), rediscovered many times, and very useful $\{12\}$. The analogue for functions $c(t)$ ($t \in \mathbb{R}$) is important in probability theory: if $c(t)$ is even, positive, convex on $\mathbb{R}^+$ and decreasing to 0 at infinity, it is Fourier transform of a positive and integrable function — therefore, a characteristic function if moreover $c(0) = 1$. This was pointed out by György Pólya and such functions $c(t)$ are called Pólya functions $\{13\}$.

Another kind of linear combinations of Fejér kernels was introduced by Hardy (1910) {14}, used by Fejér (1913) ([40, I, 715–718]) and later on by Ch. de la Vallée Poussin (1919) {15} in his lectures on the approximation of functions of a real variable, namely

$$(N+1)K_{(N+1)n}(x) - NK_{Nn}(x) \\ \left(= \sum_{|m| \le Nn} e^{imx} + \sum_{Nn<|m|<(N+1)n} \frac{(N+1)n - |m|}{n} e^{imx} \right),$$

The convolution with a function $f(x)$ reads

$$S_{Nn}(x) + \sum_{Nn<|m|<(N+1)n} \frac{(N+1)n - |m|}{n} c_m e^{imx}$$

and the last sum is controlled when we assume an extra condition on the coefficients. As a consequence, the statement of the Fejér theorem stays valid when Fejér sums are replaced by partial sums, under specific conditions on the coefficients. Hardy's condition is $c_m = O\big(\frac{1}{|m|}\big)(m \to \infty)$. The Fejér condition is

$$(F) \qquad \sum |m|\,|c_m|^2 < \infty.$$

Here is an application: if a function $f(z) = \sum_0^\infty c_m z^m$ is holomorphic in the disc $|z| < 1$ and continuous on $|z| \le 1$ and if the image of the disc has a finite area on the Riemann surface spanned by $f(z)$ (that is, (F) holds), then the Taylor series converges uniformly on the closed disc.

This is the case in particular when $f(x)$ yields a conformal mapping of the disc $|z| < 1$ on the interior of a simple Jordan curve. This theorem was used by Harald Bohr and Gyula Pál and later by Raphaël Salem in order to prove that for every **real** continuous function g on $\mathbb{T}$ there is a homeomorphism h of $\mathbb{T}$ such that the Fourier series of $g\big(f(t)\big)$ $(t \in \mathbb{T})$ converges uniformly {16}, {17}, {18}.

Only in the 1970's was the result extended to complex functions g, by purely real methods {19}, {20}.

Processes of summation raised a number of publications in relation with Dirichlet series and Fourier series between 1910 and 1940. One of the main contributors was Marcel Riesz. In particular, M. Riesz extended the theorem of Fejér by showing that it stays valid when the process of arithmetic means

(Cesàro process of order 1, or $(C, 1)$) is replaced by Cesàro process (C, α) of any positive order $(\alpha > 0)$ {21} [158, pp. 62–64].

The effect of Fejér's theorem on the theory of Fourier series was instantaneous. Actually this effect did not decrease along time. Books on harmonic analysis or Fourier series give always Fejér's theorem a special status. In Katznelson's *Introduction to Harmonic Analysis* [86] the first chapter is devoted to Fejér's theorem in the frame of Banach-valued continuous functions when the Banach space under consideration is invariant under translation and the translation is continuous (a "homogeneous Banach space"). In the book of Zygmund *Trigonometric series* [203] only a very few statements are given their author's name: one of them is "The Theorem of Fejér".

2. The theorem of Riesz–Fischer

Another and even more important revolution occurred at the beginning of the century: it was the Lebesgue theory of measure and integral. Though the Lebesgue integral of bounded functions on a bounded interval was already introduced in 1901 {22} and Lebesgue extended it to unbounded functions in 1902 {23}, the first exposition of Lebesgue's integral as we know it now (for unbounded as well as bounded fonctions) appeared first in his 1906 book "Leçons sur les séries trigonométriques" {24}. At the same time, 1906, Fatou defended his thesis on "Séries trigonométriques et séries de Taylor" {25}.

From the very beginning there was a strong linkage between the Lebesgue integral and Fourier series.

However, the intimate relation between the two subjects appeared with the Riesz–Fischer theorem on Fourier coefficients of L^2 functions, that is, the isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ through the Fourier formulas, called by Frigyes Riesz in a pleasant way "billet aller-retour permanent entre deux espaces à une infinité de dimensions" ([156, I, p. 327]).

The essential tool used by both authors in a more or less explicit way was the completeness of the space L^2. Actually the terms were not yet defined: a rather long sentence was needed in order to express that L^2 is complete. Now the theorem "L^p is complete" is a three-words statement, but the substance of the method elaborated independently by Frederic Riesz and Ernst Fischer is to be found in the definitions of a complete metric space and an L^p space ($1 \leq p < \infty$). The superiority of Lebesgue's over Riemann's integral and its role to come in functional analysis relies on this three-words statement. The history of the Riesz–Fischer theorem deserves to be related. Here it is.

Frederic Riesz was visiting Göttingen at the beginning of the year 1907, while Ernst Fischer was in Brno (Brünn). Both of them, independently, discovered the following fact: given a sequence $(a_n) \in \ell^2$, there exists a function $f \in L^2$ whose Fourier coefficients are the a_n. Both of them stated the result for general orthonormal systems. Riesz communicated his result in a lecture at Göttingen on February 26, and Fischer in a lecture at Brünn on March 5. Riesz immediately published the theorem in two forms: an article in the Göttingen Nachrichten, presented by Hilbert on March 9, and a note in the Comptes-Rendus de l'Académie des Sciences de Paris, presented by Picard on March 11, published on March 18. The motivation of Riesz was the Hilbert treatment of linear integral equation by means of an orthonormal system. However he proved the theorem first for the trigonometric system, then extended it for an orthonormal system of functions defined on an interval, then to a general orthonormal system. The last step is realized in another note aux Comptes-Rendus, presented on April 2, published on April 8.

As soon as he read Riesz's note of March 18, Ernst Fischer reacted by sending his own contribution to the Comptes-Rendus. In a first note, dated May 13, he recognized the priority of Riesz in publishing the result, but pointed out that he had got it and lectured on it in Brno (Brünn) already on March 5. The result was the same but Fischer's approach was more direct and stated the main point (L^2 is complete) in an explicit (though not so concentrated) form. In a second note, dated May 27, Fischer introduced best approximation in L^2 and announced that he would develop this theory later in the frame of a kind of geometry of functions ("en m'appuyant sur une espèce de géométrie des fonctions").

It was Riesz's turn to react. The Comptes rendus of June 24 published a note of Frédéric Riesz with the title "Sur une espèce de géométrie analytique des systèmes de fonctions sommables". First, he reinforces his priority by mentioning his lecture at Göttingen on February 26. Then he enlarges the question. "After the fundamental result obtained by several geometers during the past last years and based for the main part on those of Mr Fejér, the idea of representing a function by its Fourier constants became very familiar. The set of the summable functions could be represented in this way on a subset of the space with infinitely many dimensions. What is this subspace? Until now we are not able to answer".

In modern notations, what can we say of $A(\mathbb{Z}) = FL^1(\mathbb{T})$? Then Riesz explains what happens when L^1 is replaced by L^2, namely $FL^2(\mathbb{T}) = \ell^2(\mathbb{Z})$. ({26}, {27}, {28}, {29}, {30}).

Frédéric Riesz went back to this subject in a later article when he extended Fischer's theorem (L^2 is complète) in the now classical form known as Riesz's theorem: L^p is complete ($p \geq 1$) ([156, I, C. 10, 441–489]). This is still more fundamental than the theorem of Riesz–Fischer and its goes far beyond harmonic analysis.

Fischer and Riesz worked independently and got the same result about the same time at the beginning of 1907. According to the alphabetical order, it would be justified, as some authors do (Nina Bari in her treatise on Trigonometric series), to say "the theorem of Fischer and Riesz". However most authors and textbooks (and Frédéric Riesz in the first place) call it the theorem of Riesz and Fischer. Anyhow it is the most important theorem relating functions and Fourier coefficients. Let us try to see it in its true perspective.

Given $f \in L^2(T)$ and its Fourier coefficients C_n, the formula

$$\sum |c_n|^2 = \int |f|^2,$$

called the Parseval formula, was known. Fatou had given the appropriate framework, namely the totality of the trigonometric system in L^2, and also credit to Parseval for the formula, which can be found in a memoir, dated 1806, at a time when nobody was able to really prove anything like totality of the trigonometric system. Prior to Fatou, the Parseval formula was called by Hurwitz "Fundamentalsatz der Fourierschen Konstanten" and Fischer had just published "Zwei neue Beweise für der Fundamentalsatz der Fourierschen Konstanten", but the proofs were given only for bounded integrable functions. Fatou was very near to the Fischer–Riesz theorem, but he missed it ({25}, {31}, {32}).

The Parseval formula can be extended to the case of other exponents in the form of two inequalities: when $p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left(\sum |c_n|^q\right)^{1/q} \leq \int \left(|f|^p\right)^{1/p}$$

and

$$\int \left(|f|^q\right)^{1/q} \leq \left(\sum |c_n|^p\right)^{1/p}.$$

These are the Hausdorff–Young inequalities (Young 1912 when q is an even integer, Hausdorff 1923 in the general case). Therefore

$$FL^p(\mathbb{T}) \subset \ell^q(\mathbb{Z}), \quad F\ell^p(\mathbb{Z}) \subset L^q(\mathbb{T}).$$

Frédéric Riesz generalized these inegalities by considering a bounded orthonormal system instead of the trigonometric system (1923). After a conversation with his collegue Alfréd Haar in Szeged he observed that his result can be expressed in a purely algebraic form, namely

$$\left(\sum_1^N |y_n|^q\right)^{\frac{1}{q}} \leq M^{(2-p)/p}\left(\sum_1^N |x_n|^p\right)^{\frac{1}{p}}$$

when $(y_1, y_2, \ldots, y_N)$ is obtained from $(x_1, x_2, \ldots, x_N)$ through an orthogonal matrix with entries majorized by M in absolute value. These inequalities are easy consequences of the convexity theorem of Marcel Riesz (1924), generalized with a simplified proof by his student G. O. Thorin in 1948, which now forms a part of the large theory of interpolation of linear operators ([158, C13], and other references in [203, chapter XII]).

The above inclusions, involving FL^p and $F\ell^p$, are strict except when $q = p = 2$. Frédéric Riesz pointed out in 1907 that no characterisation was known of $FL^1(\mathbb{T})$, in terms of usual sequence spaces. We know that all spaces $FL^p(\mathbb{T})$ $(1 \leq q \leq \infty)$ as well as $FC(\mathbb{T})$ (Fourier coefficients of continuous functions) are intrisically intricate except for the case $q = 2$. The same is true for the spaces $F\ell^p(\mathbb{Z})$. Actually I wrote a whole book on $F\ell^1(\mathbb{Z})$ [82] and it is an inexhaustible subject.

Let me relate a personal experience. When I got interested in the subject it was not clear whether or not $c \in FL^1(\mathbb{T})$ implies $|c| \in FL^1(\mathbb{T})$, and $f \in F\ell^1(\mathbb{Z})$ implies $|f| \in F\ell^1(\mathbb{Z})$. Now we know, by Katznelson's theorem $\{33\}$, that only analytic function operate, and consequently the answer is negative. The first proof of this was given by means of explicit constructions, namely a positive function on $(0, \pi)$ such that its expansion as a sine series is absolutely convergent and its expansion as a cosine series is not absolutely convergent, and a positive sequence (a_n) such that $\sum a_n \sin nx$ is a Fourier–Lebesgue series and $\sum a_n \cos nx$ is not. And I was led to such constructions by an article of Béla Sz. Nagy in which he proved the following theorem: given a positive and decreasing function g on $(0,\pi)$, and

$$a_n = \frac{2}{\pi}\int_0^\pi g(x)\cos nx\,dx, \quad b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin nx\,dx$$

(assuming $g \in L^1$ for the computation of the a_n, and $xg \in L^1$ for the b_n), then, given $0 < \gamma < 1$,

$$x^{\gamma-1}g(x) \in L^1(0,\pi) \Longleftrightarrow \sum_1^\infty n^{-\gamma}|a_n| < \infty \Longleftrightarrow \sum_1^\infty n^{-\gamma}|b_n| < \infty,$$

while, for $\gamma = 1$,

$$g \in L^1(0,\pi) \Longleftrightarrow \sum_1^\infty |a_n| < \infty,$$

$$g(x)\log x \in L^1(0,\pi) \Longleftrightarrow \sum_1^\infty |b_n| < \infty \qquad \{34\}.$$

This nice theorem extends results of Young, Pólya and Zygmund concerning the case $\gamma = 1$ (Young and Pólya are mentioned above), and actually I got aware of these previous results through the article of B. Sz.-Nagy.

I shall not dwell on $FL^1(\mathbb{T})$ (the question raised by F. Riesz) nor on $FC(\mathbb{T})$. However, I have to explain their relation with the Riesz–Fischer theorem.

If $\sum |c_n|^2 < \infty$ $(n \in \mathbb{Z})$, then $\sum c_n e^{inx}$ is a Fourier–Lebesgue series. Is there a better condition on the modulus $|c_n|$ with the same conclusion? The answer is negative, and the clearest proof is given by a theorem of Paley and Zygmund on random trigonometric series: if $\sum |c_n|^2 = \infty$, almost surely $\sum \pm c_n e^{inx}$ is not a Fourier–Lebesgue series; on the other hand, if $\sum |c_n|^2 < \infty$, the random series represents almost surely a function belonging to all $L^p(\mathbb{T})$ $(1 \le p < \infty$; but **not** $L^\infty)$.

If $\sum c_n e^{inx}$ $(n \in \mathbb{Z})$ represents a continuous function, then $\sum |c_n|^2 < \infty$ (Parseval). Is there a better conclusion? The answer is negative if the following sense: given any $(d_n) \in \ell^2(\mathbb{Z})$ there exist (c_n) such that $|c_n| \ge |d_n|$ and $\sum c_n e^{inx}$ represents a continuous function. The original proof by De Leeuw, Kahane and Katznelson used randomization; another proof by Nazarov avoids it {35}, {36}.

The question raised by F. Riesz in 1907 has a rather disappointing answer not only for $L^1(\mathbb{Z})$, but for $C(\mathbb{T})$ and all $L^p(\mathbb{T})$ except $L^2(\mathbb{T})$: the spaces of Fourier coefficients are not easily described and in particular they are not stable by the operation of taking absolute values. Simple characterizations of Fourier coefficients exist only for very few classes of functions: analytic, C^∞, Sobolev spaces, Schwartz distributions.

The success of the modern theory of wavelets comes from the fact that many properties of functions can be recognized easily on the coefficients of their wavelet expansion. There is a simple description of many function spaces in terms of their wavelet transforms. Let me recall that a wavelet system is an orthonormal system in $L^2(\mathbb{R})$ of the form

$$2^{\frac{j}{2}}\Psi(2^j x - k) \qquad (j \in \mathbb{Z}, k \in \mathbb{Z})$$

where Ψ is a "good" function, for example $\Psi \in S(\mathbb{R})$, the Schwartz space of C^∞ rapidly decreasing functions.

The wavelet theory is recent and explosive since 1985. However the paradigm of wavelets is rather old and it deserves attention. It is the Haar system, corresponding to $\Psi(x) = 1$ on $\left(0, \frac{1}{2}\right), \Psi(x) = -1$ on $\left(\frac{1}{2}, 0\right)$ and $\Psi(x) = 0$ elsewhere. Haar considered only the Haar wavelets on $(0,1)$ and had to add the constant 1 in order to have a complete orthonormal system in $L^2(0,1)$. The article of A. Haar was published by Mathematische Annalen in 1910, and his modern impact on harmonic analysis can not be overestimated ([62, 331–371]).

The history of wavelets is very interesting example of interactions between physicists, engineers, and mathematicians [83]. The Hungarian physicist Dénes Gábor, who was awarded the Nobel Prize, is one of the important figures of this history; his fundamental contribution can be found in his article "Theory of communication" (1946) {37}.

3. Boundary values of analytic functions

The Taylor series of an analytic function in the unit disc of the complex plane,

$$\sum_0^\infty c_n z^n$$

can be considered as a one-parameter family of trigonometric series by writing $z = re^{it}$ ($0 < r < 1$, $0 \le t \le 2\pi$). The real and imaginary parts of the Taylor series are conjugate trigonometric series.

Conversely, given two conjugate trigonometric series S and $\widetilde{S}$, $S + i\widetilde{S}$ can be written formally as

$$\sum_{0}^{\infty} c_n e^{int},$$

There is therefore an intimate relation between conjugate trigonometric series and boundary values of analytic functions inside the unit disc.

The subject was explored and developed after 1906 by Fatou, Young and Hardy. In the following decades three Hungarian names emerge: Frédéric and Marcel Riesz, and Gábor Szegő. The book of Henry Helson, Harmonic analysis [70] is in a large part devoted to their works and their consequences.

"Über (die) Randwerte einer analytischen Funktion" is the common title of three important papers of F. and M. Riesz (1916), G. Szegő (1921) and F. Riesz (1923) and also the matter of a joint article of F. Riesz and G. Szegő (1920) ([156, D 4]). The first paper is the only one that Frédéric and Marcel Riesz signed together ([156, D 1]). The second and the third come from an exchange of letters between F. Riesz and G. Szegő, exploited separately by both of them after their joint article ([173, 21–6], [156, D 1]). The notation H^p for Hardy spaces, that is, spaces of analytic functions in the unit disc such that

$$\int_0^{2\pi} \left| f(re^{it}) \right|^p dt = O(1) \qquad (r \to 1)$$

was introduced by F. Riesz in the third paper. I use $H^p(\mathbb{T})$ for the boundary value.

Helson explains the meaning of the F. and M. Riesz theorem in the following way:

1. "Theorem of F. and M. Riesz". If μ is a mesure of analytic type, meaning that its Fourier coefficients of order n vanish when $n < 0$, then μ is absolutely continuous. Equivalently, if F is analytic in the unit disc and $\int \left| F(re^{it}) \right| dt = O(1), F$ is the Poisson then integral of a function in $H^1(\mathbb{T})$ (here $H^1(\mathbb{T})$ denotes the subspace of $L^1(\mathbb{T})$ consisting of functions with vanishing coefficients of negative order).

2. If $f \in H^1(\mathbb{T})$ and f vanishes on a set of positive measure, then $f = 0$. What F. and M. Riesz stated and proved combines the two statements.

Let g be an analytic function in the disc, of bounded variation on $\mathbb{T}$. It was already known that g must be continuous. Thus g maps $\mathbb{T}$ into a continuous and rectifiable curve in the plane. The arc length along the curve determines a measure on the curve. The original theorem asserts that the image under g of a Lebesgue null set on $\mathbb{T}$ is a null set on the curve and vice-versa. This is the equivalent to statements 1 and 2.

The starting point of F. and M. Riesz was Fatou's theorem on boundary values of bounded analytic functions — each bounded analytic function in the unit disc has non-tangential limits almost everywhere on the circle.

The starting point of the article of G. Szegő and F. Riesz was the theorem of Fejér (1916) which states that each positive trigonometric polynomial on the circle is of the form $|a_0 + a_1 e^{it} + \cdots + a_m e^{imt}|^2$. Szegő showed that the square moduli of the functions in $H^2(\mathbb{T})$ are characterized by the property that they are positive and their logarithm is Lebesgue integrable (except for the zero function). Riesz extended this to all $H^p(\mathbb{T})$ by means the following decomposition theorem: every function in $H^p(p > 0)$ can be written in the form of a product gh, where h is bounded, g has no zero in the open unit disc, and $g \in H^p$.

This decomposition plays an essential role in the study of invariant subspaces of $H^2(\mathbb{T})$. According to Arne Beurling, g is an outer and h an inner function, if moreover the boundary values of h have modulus 1 almost everywhere. Outer functions satisfy

$$\int \log |g| = \log \left| \int g \right| > -\infty$$

and this characterizes them. Inner functions can, in turn, be represented as the products of a Blaschke factor and a singular factor:

$$h(z) = \prod \left(\frac{z - z_j}{1 - z\bar{z}_j} \right) \exp \left(- \int \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$

where the z_j are inside the unit disc and $\Pi |z_j| > 0$, and $d\mu$ is a positive measure on $\mathbb{T}$. Beurling's theorem, generalized by Helson, is as follows. Let M be an invariant subspace of $L^2(\mathbb{T})$ ($e^{it} M \subset M$). Either M consists precisely of all functions of $L^2(\mathbb{T})$ that are supported on some measurable subset of $\mathbb{T}$, or it is $qH^2(\mathbb{T})$ for some inner function q. (The first kind of subspace is a Wiener subspace, the second a Beurling subspace). It is worth mentioning that already in 1907 Carathéodory and Fejér introduced the so-called Blaschke factors, in the case of a finite number of zeros z_j, as solutions of an extremal problem ([40, I, 19]).

An earlier theorem of Szegő (1920–21) answers an extremal problem, namely to compute the infimum of $\int |1+P|^2\omega$ when ω is a non negative integrable function on $\mathbb{T}$, and P ranges over all trigonometric polynomials of the form $P(x) = a_1 e^{ix} + \cdots + a_n e^{inx}$. The Szegő theorem is

$$\inf_P \int |1+P|^2\omega = \exp \int \log \omega.$$

The interpretation when $\log \omega$ is not integrable, that is, the right hand side is zero, is obvious but very important, and it is of constant use in prediction theory ([173, 20–3, 21–1]).

The subject of H^p spaces was renewed in the 60s and 70s from the probabilistic point of view by Austin, Gundy, Burkholder and Silverstein. The H^p martingales associated with Brownian motion give an elegant way to recover the classical result on H^p spaces of analytic functions. Boundary values of analytic functions is both an old and a modern subject {38}, {39}, {40}, {41}, {42}, {43}, {44}.

Together with H^p spaces of analytic functions, the subject of conjugate fonctions was in the air in the 1920s. Is $L^1(\mathbb{T})$ stable under conjugacy? The answer is easily seen to be negative (consider a cosine series with convex coefficients) and Kolmogorov in 1924 {45} and Zygmund in 1929 {46} gave substitutes involving either "weak L^1" or the space $L \log L$. Is $L^p(\mathbb{T})$ stable under conjugacy when $1 < p < \infty$? Now the answer is positive and it is due to Marcel Riesz (a Comptes rendus note in 1924 and a developed article in 1927) ([158, 29, 33]). The theorem is also expressed by inequalities

$$\int |\widetilde{f}|^p \leq A_p \int |f|^p.$$

It is a deep result, and many proofs of it were given (a pretty proof was published in 1990 in the Comptes rendus by S. Pichorides {47}). Marcel Riesz himself gave two proofs. One is long and direct, the other applies the convexity theorem that he found exactly at that time (1927) ([158, 32]) as a tool for interpolating the inequality between $p = 2$ (obvious case) and $p =$ even integer (amenable case).

The convexity theorem, in the form given by Thorin (1948) is still more important than the Hausdorff–Young and Marcel Riesz inequalities. But undoubtebly Marcel Riesz had these applications in mind when he discovered and proved his convexity theorem.

4. Riesz product and Sidon sets

As a consequence of the F. and M. Riesz theorem the following holds: let f be a function of bounded variation on the circle, together with $\widetilde{f}$, the conjugate function; then the Fourier coefficients of f are $o\big(\frac{1}{n}\big)(n \to \infty)$.

In 1918, two years after the communication of F. and M. Riesz at the fourth Congress of Scandinavian Mathematicians, Frédéric Riesz published a paper on the following question: given a **continuous** function of bounded variation in the circle, are its coefficients necessarily $o\big(\frac{1}{n}\big)$ $(n \to \infty)$?

He provided a negative answer with the example

$$f(x) = -x + \lim_{m\to\infty} \int_0^x (1+\cos x)(1+\cos 4x)\dots(1+\cos 4^{m-1}x)\,dx.$$

Nowadays we are more familiar with the language of measures and we can say that the derivative of $f(x)+x$ is the first example of a positive continuous measure whose Fourier coefficients do not tend to zero. It can be written as a "Riesz product"

$$\mu = \Pi_1^\infty(1+\cos 4^n x).$$

The partial products are positive and can be written in the form

$$1 + \sum \cos 4^n x + R,$$

R containing only frequencies of the form $\pm 4^{n_1} \pm 4^{n_2} \cdots \pm 4^{n_j}$ $(1 \le n_1 \le n_2 \le \cdots \le n_j,\ j \ge 2)$. Their L^1-norm is bounded (actually, constant). Taking a weak limit of a subsequence defines μ as a positive measure, with a Fourier expansion of the type just described: $\widehat{\mu}(4^n) = \frac{1}{2}$ $(n = 1, 2, \dots)$.

There are other ways to answer F. Riesz's question and they appeared in the 1920's in the investigations of A. Rajchman on sets of uniqueness. Maybe the simplest is to consider the probability measure carried by the standard triadic Cantor set, whose Fourier transform is

$$\widehat{v}(u) = \Pi_1^\infty \cos(2\pi 3^{-n} u);$$

it is easy to see that $\widehat{v}(3^m)$ tends to a non-zero limit as $m \to \infty$.

The construction of F. Riesz is versatile and proved very useful in harmonic analysis. Given a positive sequence of integers λ_n such that $\lambda_{n+1}/\lambda_n \ge q \ge 3$ $(n = 1, 2, \dots)$ a sequence (a_n) such that $0 < a_n \le 1$ and a real sequence φ_n, the Riesz product corresponding to these data is

$$R\big((\lambda_n), (a_n), (\varphi_n)\big) = \Pi_1^\infty\big(1 + a_n \cos(\lambda_n x + \varphi_n)\big).$$

It defines a positive and continuous measure whose Fourier expansion reads

$$1+\sum_{1}^{\infty} a_n \cos(\lambda_n x+\varphi_n)+R,$$

R containing only frequencies different from 0 and from the λ_n. Such a measure is either absolutely continuous (when $(a_n)\in\ell^2$) or purely singular (when $(a_n)\notin\ell^2$). When the sequence λ_n is fixed, the measure depends only on the sequence $c=(c_n)$ with $c_n=a_n e^{i\varphi_n}$ and it can be written as

$$\mu_c=\Pi_1^\infty\big(1+Re(c_n e^{i\lambda_n x})\big).$$

It is known that, given c and c', μ_c and $\mu_{c'}$ are either equivalent (same null-sets) or orthogonal (carried by disjoint Borel sets). When all c_n are in a compact subset of the open unit disc, the equivalence condition is $c-c'\in\ell^2$ (Peyrière, Brown and Moran). In the general case it is still unknown and it is an active field of research {48}, {49}, {50}, {51}.

The lacunary condition on the λ_n implies that the factors of the Riesz product are pretty independent. The study of Riesz products anticipated the theory of multiplicative processes and the random measures that they generate. Actually random Riesz products ((λ_n) and (a_n) fixed, φ_n random with the usual distribution) are parts of both theories {52}.

In 1924 Simon Sidon went back to the original question: given a continuous function of bounded variation on the circle, what can we say about its Fourier coefficients? Sidon proved that they satisfy the equivalent conditions $\lim_{n\to\infty}\frac{1}{n}\sum_1^n|ka_k|^2=0$ and $\lim_{n\to\infty}\frac{1}{n}\sum_1^n|ka_k|=0$.

Using an expression introduced by Mihály Fekete in 1916, the sequence (ka_k) "quasi-converges" to 0. Actually Sidon's condition are necessary and sufficient for the continuity of the function whose Fourier coefficients are a_k, when bounded variation is assumed; this is Wiener's condition for the continuity of measures. Though Wiener found it independently, part of the result belongs to Sidon.

Riesz products in the modern acceptation were introduced by Sidon in his study of lacunary trigonometric series. In the 1920's Hadamard lacunary trigonometric series, meaning series of the form

$$(*)\qquad \sum_{1}^{\infty} a_n\cos(\lambda_n x+\varphi_n),\quad \lambda_{n+1}/\lambda_n\ge q>1\quad (n=1,2,\dots),$$

became a popular subject, with contributions of Kolmogorov, Banach and Zygmund in particular. The contribution of Sidon proved crucial. First he proved that if $(*)$ is the Fourier series of a function bounded from above (or from below), then $\sum_1^\infty |a_n| < \infty$ (1927). Then he identified the sequences $\big(\hat{f}(\lambda_n)\big)\,\big(f \in L^1(\mathbb{T})\big)$ with the sequences tending to zero; in brief,

$$FL^1 \mid \wedge = c_0(\wedge), \quad \wedge = \{\lambda_n\} \quad (1932).$$

The two results are strongly related to each other. Nowadays Sidon sets are defined as subsets $\wedge$ of $\mathbb{Z}$ satisfying one of the equivalent relations:

$$\sup_x \sum_{\lambda \in \wedge} Re(a_\lambda e^{i\lambda x}) \geq c \sum_{\lambda \in \wedge} |a_\lambda|$$

for some $c > 0$ and all finite sequences $(a_\lambda)(\lambda \in \wedge)$,

$$FL^1 \mid \wedge = c_0(\wedge)$$

already mentioned, and

$$FM \mid \wedge = \ell^\infty(\wedge),$$

where M is the space of complex measures on the circle. The relation of the last condition with the Riesz product is clear: if $\lambda = (\pm\lambda_n)$ with $\lambda_{n+1}/\lambda_n \geq q > 3$, given any sequence (c_n) in the unit disc, the measure that I denoted by μ_c satisfies $\widehat{\mu}_c(\lambda_n) = \frac{1}{2}c_n$. From this fact it is easy to deduce Sidon's theorems of 1927 and 1932.

Sidon sets were investigated vigorously since the end of the 1950's. A crucial step was accomplished in 1970 by S. Drury when he proved that the union of the two Sidon sets is a Sidon set. New characterizations were given by G. Pisier and J. Bourgain in the 1980's. Meaningfully the 1985 paper of Bourgain is entitled: "Sidon sets and Riesz products". The subject is a crossing point between harmonic analysis, combinatorics and probability theory {53}, {54}, {55}, {56}.

There are many other properties of Hadamard lacunary trigonometric series. Let me mention two of them, involving Hungarian mathematicians.

If the coefficients of a Hadamard lacunary trigonometric series are real and tend to zero, the series converges at some point. This is theorem of Zygmund. What can we say of the sequence (λ_n) $(1 \leq \lambda_1 < \lambda_2 \ldots)$ such that every series

$$\sum_1^\infty a_n \cos(\lambda_n x + \varphi_n), \quad \lim_{1 \to \infty} a_n = 0,$$

converges at some point x? This question was considered by Erdős in 1966, and the subject seems difficult. There are necessary conditions, and sufficient conditions that look like the necessary conditions, and sufficient conditions for $\wedge$ to be a Sidon set, but it is not known whether the Sidon implies the Zygmund properties, nor the reverse {57}.

Suppose now that $f(t)$ is a continuous function and

$$f(t) = \sum_{1}^{\infty} a_n \cos(\lambda_n x + \varphi_n), \quad (t \in \mathbb{R}),$$

(λ_n) being a Hadamard sequence. Then there is an intimate relation between the local properties of the function and the order of magnitude of the coefficients. Very precise results were obtained by Géza Freud in 1962 and 1966. Here are two of them {58}, {59}.

1. Assume $a_n = 1/\lambda_n$. Then

a) $f(t+h) - f(t) = O\left(|h| \log \dfrac{1}{|h|}\right)$ $(h \to 0)$ everywhere

b) $\overline{\lim\limits_{h\to 0}} \big(f(t+h) - f(t)\big) / \left(|h| \log \dfrac{1}{|h|}\right) > 0$ quasi everywhere

c) $\overline{\lim\limits_{h\to 0}} \big(f(t+h) - f(t)\big) / \left(|h| \sqrt{\log \dfrac{1}{|h|} \log\log \dfrac{1}{|h|}}\right) < \infty$ almost everywhere

d) $f(t+h) - f(t) = O\big(|h|\big)$ $(h \to 0)$ on a dense t-set

e) f is nowhere differentiable.

This last statement, e), is due to Hardy, and the function under consideration is called a Hardy–Weierstrass function. The theorem expresses that the run of the function is as fast as possible (taking a) into account) when t belongs to some set of the second category of Baire (it is the meaning of "quasi everywhere"), and that is as slow as possible (taking e) into account) on some dense set, with an average behaviour ("almost everywhere") inbetween. Such a behaviour was discovered later for the Brownian motion, with $|h|^{\frac{1}{2}}$ instead of $|h|$, and it is very much in the spirit of the modern investigations on multifractal analysis.

2. Assume

$$\int_{-\pi}^{\pi} \left| \frac{f(t+h)-f(t)}{h} \right| dh < \infty$$

for **some** t. Then $\sum |a_n| < \infty$.

Assume $f(t+h) - f(t) = O\big(|h|^\alpha\big)$ $(h \to 0)$ for **some** t $(0 < \alpha < 1)$. Then $a_n = O(\lambda_n^{-\alpha})$ $(n \to \infty)$ and $f \in \wedge_\alpha$, the Lipschitz–Hölder class of order α.

The last statement was discovered independently by Masako Sato–Izumi and there are several variations about in a paper of Izumi–Izumi–Kahane {60}. It is one the very rare properties of Hadamard lacunary trigonometric series where the Hadamard condition proves necessary and sufficient.

5. Miscellaneous*

I insisted on the importance of the theorem that Fejér published in 1900. Fejér's works and personality go far beyong this first and essential result. On the other hand, Fourier analysis in Hungary involved many first-class mathematicians and many more topic than what I considered in the preceding sections. Let me try to fill some of the large gaps of this review.

Let me begin with a theorem of Ferenc Lukács (1919).

Suppose $f \in L^1(\mathbb{T})$ and $\int_0^h \big| f(x+t) - f(x-t) - D_x \big|\, dt = O(h)$. Then

$$D_x = -\lim_{n\to\infty} \pi \frac{\widetilde{S}_n(x)}{\log n}$$

where $\widetilde{S}_n(.)$ denotes the n-th partial sum of the conjugate Fourier series of f.

This is theorem B in the original paper of Lukács {61}. Theorem A is a particular case of special interest, when it is assumed that both $f(x+0)$ and $f(x-0)$ exist, and $D_x = f(x+0) - f(x-0)$. Theorem C is a consequence, namely that $\widetilde{S}_n(x) = o(\log n) \quad (n \to \infty)$. almost everywhere.

*Most of the references of this section are given in the text. A few others are listed at the References section of the paper.

The paper was published in 1920, but the actual date is 1918. F. Lukács was a young man, who died on November 30, 1918, and this is his legacy. It is worth seeing the context.

Theorem A answers in a complete way a question that Fejér asked in 1913: to determine the jump of a function at a point through its Fourier expansion. Fejér himself indicated different ways and gave a complete answer in the case of a function satisfying the Dirichlet conditions. Both Fejér's and Lukács papers are entitled: "Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe" ([40, I p. 718], {61}). In between several articles appeared in Hungarian journals and Hungarian language, by Fejér again (1913), Pál Csillag (1918) and S. Sidon (1918) ([40, I p. 744], {62}, {63}). The formula of Lukács was discovered by Fejér when Dirichlet's conditions are satisfied, then proved by Csillag for functions of bounded variation, and reproved by Sidon in a simpler way for the same functions. Both Csillag and Sidon were stimulated by the questions F. Riesz asked at the end of this paper of 1918 when he introduced the Riesz product:

"Gibt es eine stetige, nach 2π periodische Funktion von beschränkter Schwankung, für welche die Folgen na_n, nb_n konvergieren und wenigstens einer der beiden Grenzwerte von Null verschieden ist? Gibt es eine unstetige Funktion von beschränkter Schwankung, für welche $a_n = o\left(\frac{1}{n}\right)$ und $b_n = o\left(\frac{1}{n}\right)$?" ($a_n$ et b_n are the cosine and sine coefficients). The formula answers both questions in a negative way.

Though quite different, Lukács's theorem originated from the same area as Riesz products and Sidon sets. Theorem A is well known, theorems B and C deserve to be known also.

The thesis of Marcel Riesz was published in 1910, in Hungarian, in the *Mathematikai és Physikai Lapok.* The main result was already published in the Comptes-rendus in October 1907 {64}, when M. Riesz was not yet 21. It answers a question asked by Fejér, namely, to give a condition on the coefficients of a trigonometric series $\sum(a_n \cos nx + b_n \sin nx)$ such that, if this condition is satisfied and if the arithmetic means $\sigma_n(x)$ of the partial sums converge to 0 everywhere, it is necessarily the null series. The condition given by M. Riesz is

$$\sum_1^\infty \frac{|a_n| + |b_n|}{n^2} \quad < \infty.$$

Moreover, the assumptions $|a_n| + |b_n| = o(1)$ and $\sigma_n(x) = o(1)$ except on a countable compact (= reductible) set imply the same conclusion. As a test of the sharpness of the result one can consider the series

$$\sum_{2}^{\infty} \frac{n \sin nx}{\log n}, \quad \frac{1}{2} + \sum_{1}^{\infty} \cos nx$$

the first is $(C, 1)$ summable to 0 everywhere (last example in the thesis) and the second everywhere except 0 (mod 2π) (example given by Fejér). There are very good results but the subject was not explored later anymore neither by Riesz nor by other Hungarian mathematicians (see a comment on {64} in the references).

However, the subject belongs to an important stream of real and harmonic analysis, the Riemann theory of trigonometric series and the Cantor theory of uniqueness, and it was cultived later by Russian and Polish mathematicians, Menšov, Bari, Rajchman and Zygmund. Moreover, the thesis as a whole is a masterpiece of exposition of old and new ideas, not unlike the historical models, Riemann's and Cantor's theses on trigonometric series. The thesis of M. Riesz was translated into English by J. Horváth and published in English in the Collected papers of M. Riesz in 1988. It is available now to a large international public and it is a delightful piece of reading. It connects the Riemann process of summation, the sets of uniqueness introduced by Cantor, the process of summation of Fejér, the Lebesgue integral, the summability processes used in Taylor series and a critical examination of Hadamard's statement on analytic continuation, with also a series of examples among which I have mentioned just one [158].

A permanent question of interest in Fourier analysis is the relation between properties of functions and behaviour of Fourier coefficients. For instance, Marcel Riesz in his thesis asked for a characterisation of Fourier coefficients of continuous functions (in a simple form which involves only the coefficients and not a variable x); the specific question is still unsolved and very likely unsolvable. We already discussed the question after the Riesz–Fischer theorem. Let us write again c_m $(m \in \mathbb{Z})$ for the Fourier coefficients of a function f. A theorem of S. Bernstein (1914) asserts that the assumption $f \in \operatorname{Lip}\alpha$, $\alpha > \frac{1}{2}$ implies $\sum |c_n| < \infty$. Here $f \in \operatorname{Lip}\alpha$ means

$$\sup_{x,h} \Big(|h|^{-\alpha} \big| f(x+h) - f(x) \big| \Big) < \infty.$$

The subject was renewed by Ottó Szász (see [172]), then, using Szász's results, by Hardy and Littlewood (1928). Here is the first result of Szász

(1922): the assumption

$$(L_2) \qquad \int_0^{2\pi} \left| f(x+2t) + f(x-2t) - 2f(x) \right|^2 dx = O(t^{2\alpha}) \quad (t \downarrow 0)$$

with $0 < \alpha < 1$, implies $\sum |c_m|^k < \infty$ when $k > \frac{2}{2\alpha+1}$, but not when $k = \frac{2}{2\alpha+1}$. The best way to see it is to observe that

$$(L_2) \Longleftrightarrow \sum_{|m| \le n} m^4 |c_m|^2 = O(n^{4-2\alpha}) \quad (n \to \infty)$$

as Szász does at the end of this 1928 paper. The latter is devoted to the consequences of a more general assumption, namely

$$(L_p) \qquad \int_0^{2\pi} \left| f(x+2t) + f(x-2t) - 2f(x) \right|^p dx = O(t^{p\alpha}) \quad (t \downarrow 0)$$

with $p > 1$ and $-\frac{1}{p} < \alpha < 1$. In particular, when $1 < p \le 2$ and $0 < \alpha < 1$, L_p implies $\sum |c_m|^k < \infty$ when $k > \frac{p}{p(1+\alpha)-1}$, but not necessarily when $k = \frac{p}{p(1+\alpha)-1}$. The same statement appears as Theorem 8 in the article of Hardy and Littlewood devoted to the $\mathrm{Lip}\,(\alpha, p)$ classes (essentially, classes of functions that satisfy condition (L_p)), with reference to the results and observations of Szász (see collected papers of G. H. Hardy, vol III, pp. 631–632).

Late in life (1948) L. Fejér gave an expository talk on singular and positive kernels, and he made use of a pleasant image: at a first look, “elles se ressemblent comme les pingouins”, they look all the same, like penguins ([40, II, pp. 750]). Of course, Fejér then shows how different these creatures can be. It is worth visiting some pieces of the Fejér collection of positive trigonometric polynomials, the Fejér personal zoo. The favorite, born in 1900, is the Fejér kernel

$$K_n(x) = 1 + 2\left(1 - \frac{1}{n}\right)\cos x + \cdots + \frac{2}{n}\cos{(n-1)x}.$$

Then comes the integrated Dirichlet kernel (divided by 2 as usual)

$$F_n(x) = \sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{n}\sin nx,$$

positive on $]0,\pi[$ and uniformly bounded with respect to x and n, that serves as a basic cell in order to construct continuous functions whose Fourier series diverge at 0 ([40, I p. 258], 1909). Not far from those we meet the Lukács polynomials

$$L_n(x) = \sin x + \left(1 - \frac{1}{n}\right) \sin 2x + \cdots + \frac{1}{n} \sin nx,$$

also positive on $]0,\pi[$ ([40, II. p. 230]), which a little more should be said. As we saw, Lukács died in 1918, and he never published that result, but he mentioned it to Fejér, with a direct proof. Fejér included this "Satz von Lukács" in an extended article in 1928, with a new and interesting proof, using generating functions of the form

$$\sum_{n=1}^{\infty} \frac{nL_n(x)}{\sin x}\, r^{n-1} \quad \left(= \frac{1}{(1-r)^2} \frac{1}{1 - 2r\cos x + r^2} \right).$$

The method of generating functions was used again in order to derive properties of the Cesàro means of order > 1 of the Dirichlet kernels, as presented in the 1948 survey ([40, II p. 742–4]). The most striking result, going back to 1932, is that the Cesàro means of order 3

$$K_n^{(3)}(x) = \tbinom{n+3}{3}^{-1} \left(\tbinom{n+3}{3} + 2\tbinom{n+2}{3} \cos x + 2\tbinom{n+1}{3} \cos 2x + \cdots + 2\tbinom{3}{3} \cos nx \right)$$

are not only positive on $]0,\pi[$, but also decreasing on that interval. The zoo contains other interesting creatures, but I'll stop the visit here.

The properties of $K_n^{(3)}(x)$ were used immediately by Fejér in order to compare the graph of a function and those of the Cesàro means of order 3 of its Fourier series, that is, the convolutions

$$S_n^{(3)} = f * K_n^{(3)}.$$

For example, if f is a continuous, odd, and concave on $[0,\pi]$, the same holds for $S_n^{(3)}$ ([40, II p. 480], 1933). After Fejér, the comparison between the graph of a function and those of approximating polynomials became a Hungarian speciality, with important contribution of Pál Turán and Pólya in particular. Here are some of them.

Though different shapes of functions were considered, let me stick to functions f of the above type (odd, $f(0) = f(\pi) = 0$, concave on $[0,\pi]$). In

1935, Turán proved a spectacular result for the Cesàro means $S_n^{(k)}$ of these functions, namely

$$f(x) \geq S_n^{(1)}(x) \geq S_n^{(2)}(x) \geq S_n^{(3)}(x) \ldots (0 \leq x \leq \pi),$$

implying that the best approximation is provided by the $S_n^{(1)}$, the Fejér sums (*Journal of the London Mathematical Society,* **10** (1935), pp. 277–280). For such functions again, Turán proved in 1938 that the pointwise approximation by the $S_n^{(k)}$ improves when n increases:

$$f(x) \geq S_{n+1}^{(k)}(x) \geq S_n^{(k)}(x) \quad (0 \leq x \leq \pi;\ k \geq 2,\ n \geq 0),$$

(Theorem 3, *Proceedings of the Cambridge Philosophical Society,* **34** (1938), pp. 134–143); the same holds when $k = 1$ and f satisfies the symmetry condition.

$$f(\pi - x) = f(x) \quad (0 \leq x \leq \pi)$$

(Theorem 4, ibidem). The situation is different for ordinary partial sums $S_n^{(0)}$ which are positive on $]0, \pi[$ (L. Koschmieder, *Monatshefte für Mathematik und Physik,* **39** (1932), pp. 321–344). What Turán proved is

$$0 \leq S_n^{(0)}(x) \leq 2f(x) \quad (0 \leq x \leq \pi)$$

and 2 is the best constant (*Journal of the London Mathematical Society,* **13** (1938), pp. 278–282).

In 1958, G. Pólya and I. I. Schoenberg returned to the general topic introduced by Fejér: comparing the shapes of approximating trigonometric polynomials and the shape of the function. They considered the de la Vallée Poussin kernels

$$V_n(x) = c_n \left(\cos \frac{x}{2}\right)^{2n} \quad \left(\int_{-\pi}^{\pi} V_n(x) \frac{dx}{2\pi} = 1\right)$$

and the convolutions $f * V_n$. They observed that, compared to the Cesàro means $S_n^{(k)}$, "the de la Vallée Poussin means possess such shape-preserving properties to a much higher degree, thanks to their variation diminishing character" (*Pacific Journal of Mathematics,* **8** (1958), pp. 295–334). Among the numerous results of Pólya and Schoenberg, one is related to the simple situation mentioned above (odd functions, concave on $[0, \pi]$); then the de la Vallée Poussin sums enjoy the property that Fejér pointed out for $S_n^{(3)}$, namely

$$0 \leq (f * V_n)(x) \leq f(x) \quad (0 \leq x \leq \pi)$$

and $f * V_n$ is concave on $[0,\pi]$. For general oscillating shapes, they were able to prove results of the same nature.

At the end of the thirties multiple Fourier series became a topic of interest, with new techniques developed by Marcinkiewicz and Zygmund {65}. The history involves Fejér again. Going back to 1913, Hardy and Littlewood proved that

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n+1}\left(S_0^2(x) + S_1^2(x) + \cdots + S_n^2(x)\right) = f^2(x)$$

when $f \in L^1(T)$ and f is continuous at x, $S_n(= S_n^{(0)})$ denoting the n-th partial sum of the Fourier series. As a consequence

$$\lim_{n\to\infty} \frac{1}{n+1}\left(S_0(x) - f(x)\right)^2 + \cdots + \left(S_n(x) - f(x)\right)^2 = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n+1}\left(\left|S_0(x) - f(x)\right| + \cdots + \left|S_n(x) - f(x)\right|\right) = 0$$

the "strong summability" theorem. The proof of $(*)$ is not easy. Fejér had the idea of deriving it from a statement on double Fourier series ([40, II p. 725], 1938): if $f \in L^1(\mathbb{T})^2$ and f is continuous at (x,y) the "square partial sums" $S_{n,n}(x,y)$ converge to $f(x,y)$ through the Cesàro process of order 3 $(C,3)$. This relies on the positivity of the corresponding kernel, one more creature of the Fejér zoo. Actually Géza Grünwald extended the result by proving $(C,1)$ summability almost everywhere, without the continuity condition (*Proceedings of the Cambridge Philosophical Society,* **35** (1939), pp. 343–350) and it was extended later to (C,α) summability $(\alpha > 0)$ by J. Herriot (*Transactions of the American Mathematical Society,* **53** (1952)).

The strong summability theorem of Hardy and Littlewood can be applied in order to give conditions for a cosine series to be a Fourier–Lebesgue series, as the Fejér theorem is applied in order to get the Young convexity condition mentionel in section 1. This was done by S. Sidon (Hinreichende Bedingungen für den Fourier Charakter einer trigonometrische Reihe, *Journal London Math. Soc.,* **14** (1939), 158–160) and generalized by S. A. Teljakowskii (On a sufficient condition of Sidon for integrability of trigonometric series (in Russian), *Mat. Zametki,* **14** (1973), 317–328) and more recently by F. Móricz (Sidon type inequalities, *Publ. Inst. Math. Szeged,* **48** (1990), 101–109).

Summability processes can be considered from a quantative point of view: given a process and a function space, how far does the process converge to functions that belong to the space? Given a process, is there an optimal speed of convergence, and a maximal function space for which the process converges that fast? The last question was asked by Jean Favard in 1947 (Colloque d' Analyse Harmonique, CNRS, Nancy); the maximum function space is called the "saturation class" of the process. The first question goes back to the beginning of the century and the works of S. Bernstein, Dunham Jackson and others.

Both questions were investigated in Hungary. The first main contribution is due to Béla Sz. Nagy (Sur une classe générale de procédés de sommation pour les séries de Fourier, *Hungarica Acta Mathematica,* **1**, 3 (1948), 14–52). For quite general processes of summation Sz. Nagy obtained very precise estimates on the order of approximation on one hand, on "Lebesgue constants" on the other. The particular processes given by fractional integration were investigated by M. Mikolás and related to some one-sided generalized limits, denoted by $f\langle x\pm 0\rangle$ (Application d'une nouvelle méthode de sommation aux séries trigonométriques et de Dirichlet, *Acta. Math. Acad. Sci. Hungar.,* **11** (1960), 317–344). For the strong summability process $h(p)$ $(p \leq 1)$ which associates with a given function f and its Fourier sums $S_k(f,.)$ the pointwise approximations

$$h_n(f,x,p) = \frac{1}{n+1}\left(\sum_{k=0}^{n} \bigl| f(x) - s_k(f,x)\bigr|^p\right)^{\frac{1}{p}},$$

the pioniering work was that of György Alexits (*Acta. Sci. Math. Szeged,* **26** (1965), 211–224): if f belongs to the Hölder class $\wedge_\alpha$ and $\alpha < \frac{1}{p}$, then $h_n(f,x,p) = O(n^{-\alpha})$ $(n \to \infty)$ uniformly with respect to x, and that is false for $\alpha = \frac{1}{p}$. In 1969, Géza Freud investigated the boundary case and the saturation problem, and obtained very precise results: given $h(p)$, the best speed of convergence is $n^{-1/p}$, that is, $h_n(f,x,p) = O(n^{-1/p})$ uniformly in x when $f \in X_p$, the saturation class (containing non-constant functions), while $h_n(f,x,p) = o(n^{-1/p})$ implies that f is a constant. The space X_p is included in $\wedge_{\frac{1}{p}}$ and consists of functions f satisfying $f(x+h) - f(x) = o\bigl(|h|^{1/p}\bigr)$ a.e. The space X_2 is defined by

$$X_2 = \left\{ f \,:\, \int \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right|^2 dt < \infty \right\}$$

(*Acta Math. Acad. Sci. Hungar.,* **20** (1969), pp. 275–279). More information about the topic can be found in the article of L. Leindler in *Journal of Approximation Theory,* **46** (1986) pp. 58–64.

Let us turn back to the favorite animals of Fejér, the positive (≥ 0) trigonometric polynomials. The fundamental theorem expresses that they are nothing but the square moduli of polynomials in e^{ix}:

$$f(x) = \left| P(e^{ix}) \right|^2.$$

It was guessed by Fejér, proved by F. Riesz, proved again in several ways by J. Egerváry and Fejér himself (see [40, I. p. 843], 1915). From this representation Fejér derived a number of inequalities. Here are the simplest. If

$$f(x) = 1 + \lambda_1 \cos x + \mu_1 \sin x + \cdots + \lambda_n \cos nx + \mu_n \sin nx \geq 0,$$

then $f(0) \leq n+1$, with equality if and only if f is the Fejér kernel of order n. If moreover f is even ($\mu_1 = \cdots = \mu_n = 0$), then

$$|\lambda_1| \leq 2 \cos \frac{\pi}{n+2}$$

([40, I. p. 869]). This last inequality plays a role in the modern theory of operators in Hilbert spaces. It can be translated into the following statement (V. Haagerup, P. de la Harpe, Procedings of the *American Mathematical Society,* **115** (1992), pp. 371–379). If T is a linear operator in a Hilbert space H such that $\|T\| \leq 1$ and $T^n = 0$, its "numerical radius"

$$\omega_2(T) = \sup \left\{ \left| \langle Tx, x \rangle \, : \, x \in H, \|x\| = 1 \right\}$$

satisfies $\omega_2(T) \leq \cos \frac{\pi}{n-1}$. Inequalities of this type are a topic of current interest (see for example C. Badea and G. Cassier, Constrained von Neumann Inequalities, *Adv. Math.,* **166** (2002), 260–297).

After such a random walk through the legacy of Fejér a natural question is: why stop here? We visited only a few spots, leaving aside many more places of interest. Instead of filling the gaps between the first sections of this survey, I created more and more gaps — as it is natural for a plane Brownian motion. The best I can do is to invite the reader to a promenade inside journals and books, to discover by himself the Hungarian jewels in harmonic analysis, of which I uncovered only a small part.

References

[40] Fejér, Lipót, *Összegyűjtött Munkái = Gesammelte Arbeiten,* ed. Pál Turán, Akadémiai Kiadó (Budapest, 1970).

[62] Haar, Alfréd, *Összegyűjtött Munkái = Gesammelte Arbeiten,* ed. Béla Sz.-Nagy, Akadémiai Kiadó (Budapest, 1959).

[70] Helson, Henry, *Harmonic Analysis,* Addison-Wesley (Reading, MA, 1983).

[82] Kahane, Jean-Pierre, *Séries de Fourier absolument convergentes,* Ergebnisse der Mathematik und Ihrer Grenzgebiete, Band 50, Springer-Verlag (Berlin–Heidelberg–New York, 1970).

[83] Kahane, Jean-Pierre and Lemarié-Rieusset, Pierre Gilles, *Séries de Fourier et Ondelettes,* Cassini (Paris, 1998). *Fourier Series and Wavelets,* Gordon and Breach (New York, 1995).

[86] Katznelson, Yitzhak, *An Introduction to Harmonic Analysis,* Dover (New York, 1976).

[156] Riesz, Frigyes, *Összegyűjtött Munkái = Œuvres complètes = Gesammelte Arbeiten,* ed. Á. Császár, Akadémiai Kiadó (Budapest, 1960).

[158] Riesz, Marcel, *Collected Papers,* ed. L. Gårding and L. Hörmander, Springer-Verlag (Berlin–Heidelberg–New York, 1988).

[172] Szász, Ottó, *Collected Mathematical Papers,* ed. H. D. Lipsich, University of Cincinnati (1955).

[173] Szegő, Gábor, *Collected Papers,* ed. R. Askey, Birkhäuser (Boston–Basel–Stuttgart, 1982).

[203] Zygmund, Antoni, *Trigonometric Series,* Cambridge University Press (London–New York, 1959).

References, Section 1

{1} A. Hurwitz, Über die Fouriershen Konstanten integrierbarer Funktionen, *Math. Ann.,* **57** (1903), 425–446 = Math. Werke, Band I, S. 555–576, Basel (Birkäuser), 1932.

{2} H. Lebesgue, Sur une condition de convergence des séries de Fourier, *Comptes Rendus Acad. Sci. Paris,* **140** (1905), 1378–1381 = Oeuvres sci., vol III, p. 95–98, Genève (Enseign. Math.), 1972.

{3} H. Lebesgue, Recherche sur la convergence des séries de Fourier, *Math. Ann.,* **61** (1905), 251–280 = Oeuvres sci., vol III, p. 181–210, Genève (Enseign. Math.), 1972.

{4} C. de la Vallée Poussin, Sur l'approximation des fonctions d'une variable réelle et de leurs dérivées par des polynômes et des suites limitées de Fourier, *Acad. Royale Belgique, Bull. Classe Sci.,* 1908, 193–254.

{5} G. H. Hardy, *On the Summability of Fourier's series*, Proceed. London Math. Soc., (2), 12 (1913), 365-372 = Collected papers, vol. III p. 118-125, Oxford (Clarendon Press), 1969.

{6} G. H. Hardy, Sur la série de Fourier d'une fonction à carré sommable, *Comptes Rendus Acad. Sci. Paris,* **156** (1913) 1307–1309 = G. H. Hardy, Collected papers, vol. III, p. 118–125, Oxford (Clarendon Press), 1969.

{7} C. de la Vallée Poussin, *Cours d'analyse infinitésimale*, Louvain (Oystruyst–Dieudonné)

{8} L. Carleson, On Convergence and Growth of Partial Sums of Fourier Series, *Acta. Math.,* **116** (1966), 135–157.

{9} E. Picard, *Traité d'analyse,* t. I, Paris (Gauthier-Villars), 1891.

{10} C. de la Vallée Poussin, Sur quelques applications de l'intégrale de Poisson, *Ann. Soc. Sci. Bruxelles,* **17** (1893), B, 18–34.

{11} A. Hurwitz, Sur les séries de Fourier, *Comptes Rendus Acad. Sci. Paris,* **132** (1901), 1473–1475 = Math. Werke, Band I, p. 506–508, Basel (Birkhäuser), 1932.

{12} W. H. Young, On the Fourier series of bounded functions, *Proc. London Math. Soc.,* **12** (1913), 41–70.

{13} G. Pólya, Remarks on characteristic functions, *Proc. Berkeley Symp. Math. Statistics and Probability,* Berkeley, 1949, 115–123.

{14} G. H. Hardy, Summability and convergence of slowly oscillating series, *Proc. London Math. Soc. (2),* **8** (1910), 301–320.

{15} C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle,* Paris (Gauthier-Villars), 1919.

{16} J. Pál, Sur des transformations de fonctions qui font converger leurs séries de Fourier, *Comptes Rendus Acad. Sci. Paris,* **158** (1914), 101–103.

{17} H. Bohr, Über einen Satz von J. Pál, *Acta Univ. Szeged, Sect. Sci. Math.,* **7** (1934–1935), 129–135 = Collected Math. Works, vol. III, E. 13, Kobenhavn (Dansk Mat. Forening), 1952.

{18} R. Salem, On a theorem of Bohr and Pál, *Bull. Amer. Math. Soc.,* **50** (1944), 579–580) = Oeuvres Math., p. 334–335, Références et errata, p. 5, Paris (Hermann), 1967.

{19} J.-P. Kahane et Y. Katznelson, *Séries de Fourier des fonctions bornées, Study in Pure Mathematics, to the memory of Paul Turan,* Basel, Académie des Sciences de Hongrie (1983), 395–410.

{20} A. A. Saakian, Integral modulus of continuity and Fourier coefficients under superposition of functions, *Mat. Sbornik,* **110** (152) (1979), 597–608.

{21} M. Riesz, Sur les séries de Dirichlet et les séries entières, *Comptes Rendus,* **149** (1909), 309–312.

References, Section 2

{22} H. Lebesgue, Sur une généralisation de l'intégrale définie, *C.R. Acad. Sc. Paris* (1901).

{23} H. Lebesgue, *Intégrale, longueur, aire,* Thèse (1902).

{24} H. Lebesgue, *Leçons sur les séries trigonométriques,* Paris, Gauthier-Villars (1906).

{25} P. Fatou, Séries trigonométriques et séries de Taylor, *Acta Math.,* **30** (1906), 335–400.

{26} F. Riesz, Sur les systèmes orthogonaux de fonctions, *C.R.A.S. Paris,* **144** (1907), 615–619 (Oeuvres C2).

{27} E. Fischer, Sur la convergence en moyenne, *C.R.A.S. Paris,* **144** (1907), 1022–1024.

{28} F. Riesz, Sur les systèmes orthogonaux de fonctions et l'équation de Fredholm, *C.R.A.S. Paris,* **144** (1907), 734–736, (oeuvres C3).

{29} E. Fischer, Application d'un théoreme sur la convergence en moyenne, *C.R.A.S. Paris,* **144** (1907), 1148–1151.

{30} F. Riesz, Sur une espèce de géométrie analytique des systèmes de fonctions sommables, *C.R.A.S. Paris,* **144** (1907), 1409–1411 (Oeuvres C4).

{31} A. Hurwitz, Über die Fourierschen Konstanten integrierbarer Funktionen, *Mathematische Annalen* **57** (1903), 425–446.

{32} E. Fischer, Zwei neue Beweise für der Fundamentalsatz der Fourierschen Konstanten, *Monatshefte für Mathematik und Physik,* **15** (1904), 69–92.

{33} Y. Katznelson, Sur les fonctions opérant sur l'algèbre des séries de Fourier absolument convergentes, *C.R.A.S. Paris,* **247** (1958), 404–406.

{34} B. Sz.-Nagy, Séries et intégrales des fonctions monotones non bornées, *Acta. Sci. Math. Szeged,* **13** (1949), 118–135.

{35} K. de Leeuw, J.-P. Kahane et Y. Katznelson, Sur les coefficients de Fourier des fonctions continues, *C.R.A.S. Paris,* **285** (1978), 1001–1003.

{36} F. Nazarov, The Bang solution of the coefficient problem, *Algebra i Analisis* (in Russian), **9** (1997), 2, *St. Petersburg Math. J.,* **9** (1998), 2, 407–419 (In English).

{37} D. Gabor, Theory of communication, *J. Inst. Elect. Engng.* **93** (1946), 249–457.

References, Section 3

{38} D. G. Austin, A sample function property of martingales, *Ann. Math. Statis.,* **37** (1966), 1396–1397.

{39} D. L. Burkholder, Martingales Transforms, *Ann. Math. Statis.,* **37** (1966), 1494–1504.

{40} D. L. Burkholder and R. F. Gundy, Extrapolation and interpolation of quasi-linear operators on martingales, *Acta. Math.,* **124** (1970), 249–304.

{41} D. L. Burkholder, R. F. Gundy and M. L. Silverstein, A maximal characterization of the class H^p, *Trans. Amer. Math. Soc.,* **157** (1971), 137–153.

{42} D. L. Burkholder, Brownian motion and classical analysis, *Proc. Symp. Pure Math.,* **31** (1977), 5–14.

{43} R. F. Gundy, *Inégalités pour martingales à un ou deux indices; l'espace H^p,* Ecole d'été Saint-Flour 1978, Lect. Notes in Math 774, Springer-Verlag (1980).

{44} F. Weisz, *Martingale Hardy Spaces and their application in Fourier Analysis,* Lecture Notes in Mathematica 1568, Springer (1994).

{45} A. Kolmogorov, Sur les fonctions harmoniques conjugées et les séries de Fourier, *Fondam. Math.,* **7** (1925), 23–28.

{46} A. Zygmund, Sur les fonctions conjuguées, *Fondam. Math.,* **13** (1929), 284–302; **18** (1932), 312.

{47} S. Pichorides, Une remarque sur le théorème de M. Riesz concernant la norme L^p de sommes partielles de séries de Taylor, *C.R.A.S. Paris I* **310** (1990), 549–551.

References, Section 4

{48} F. Parreau, Ergodicité et pureté des produits de Riesz, *Annales de l'Institut Fourier,* **40**, 2 (1990), pp. 391–405.

{49} J. Peyrière, Almost everywhere convergence of lacunary trigonometric series with respect to Riesz products, *Journal Australien Math. Society A,* **48** (1990), pp. 376–383.

{50} A. H. Fan, Quelques propriétés des produits de Riesz, *Bulletin Sciences Mathématiques 2e série,* **117** (1993), pp. 421–439.

{51} S. J. Kilmer and S. Saeki, On Riesz product measures: mutual absolute continuity and singularity, *Annales de l'Institut Fourier,* **38**, 2 (1988), pp. 63–93.

{52} A. H. Fan, Equivalence et orthogonalité de mesures aléatoires engendrées par martingales positives homogènes, *Studia Mathematica* (1991), pp. 249–266.

{53} S. Drury, Sur les ensembles de Sidon, *C.R.A.S. Paris,* **271** (1970), pp. 162–163.

{54} D. Rider, Randomly continuous fonctions and Sidon sets, *Duke Mathematical Journal,* **42** (1975), pp. 759–764.

{55} G. Pisier, Arithmetic characterizations of Sidon sets, *Bulletin American Mathematical Society,* **8** (1983), pp. 87–89.

{56} J. Bourgain, Sidon sets and Riesz products, *Annales de l'Institut Fourier,* **35**, 1 (1988), pp. 136–148.

{57} J.-P. Kahane, *Almost surely everywhere*, Paul Erdős and His Mathematics I, Budapest, 2003, pp. 333-343.

{58} G. Freud, Über trigonometrische Aproximation und Fouriersche Reichen, *Mathematischer Zeitschrift,* **78** (1962), pp. 252–262.

{59} G. Freud, On Fourier series with Hadamard gaps, *Studia Sc. Math. Hungarian I* (1966), pp. 87–96.

{60} M. Izumi, S. Izumi and J.-P. Kahane, Théorèmes élémentaires sur séries de Fourier lacunaires, *Journal d'Analyse Mathématique* **14** (1965), pp. 235–246.

References, Section 5

{61} F. Lukács, Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourier-Reihe, *J.f. reine und angew. Math,* **150** (1920), 107–112.

{62} P. Csillag, Über die Fourierschen Konstanten der Funktionen von beschränker Schwankung (Hungarian), *Math. és Phys. Lapok,* **27** (1918), 301–308.

{63} S. Sidon, *Bestimmung des Sprunges der Funktion aus der Fourier-Reihe* (Hungarian), ibidem, 309–311.

{64} M. Riesz, Sur les séries trigonométriques, *Comptes rendus Acad. Sci. Paris,* **145** (1907), 583–586.

{65} J. Marcinkiewicz and A. Zygmund, On the summability of double trigonometric series, *Fund. Math.,* **32** (1939), 122–132.

Comment on {64}. The note {64} is worth reading by those who know French. It is beautifully written and states the main results of the thesis. Surprisingly it is not reprinted in the Collected Works of M. Riesz. As a sign of bad luck it cannot be found in the 1907 issue of the Jahrbuch über die Fortschritte der Mathematik under the name of M. Riesz; it is attributed to F. Riesz.

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