AN ARITHMETIC-GEOMETRIC METHOD IN THE STUDY OF THE SUBGROUPS OF THE MODULAR GROUP

By Ravi S. Kulkarni*

1. Introduction (1.1). Let $\Gamma \approx PSL_2(\mathbf{Z})$ denote the inhomogeneous modular group acting on the upper half plane $\mathbf{H}$ in the standard way:

$$(1.1.1) \qquad z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbf{Z}, ad - bc = 1.$$

We shall denote the element in (1.1.1) sometimes by its matrix form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with the understanding that A and $-A$ define the same element. Among the subgroups of Γ the *congruence subgroups* such as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a \equiv d \equiv 1(N), b \equiv c \equiv 0(N) \right\},$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0(N) \right\}, \qquad \Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| b \equiv 0(N) \right\}$$

$$\Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a \equiv d \equiv 1(N), c \equiv 0(N) \right\}$$

have been the objects of detailed studies due to their significance in the arithmetic of elliptic curves, integral quadratic forms, elliptic modular

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forms etc., cf. [H], [Fri], [FK] for the early accounts, and [An], [O], $[R]_{1,2,3}$, [S] for the modern accounts. In these studies as a means of potentially useful geometric visualization one often tries to construct fundamental domains. Thus there is an explicit fundamental domain for $\Gamma^0(p)$ where p is a prime, cf. [S] p. 88, and for $\Gamma_0(p)$ in [Z], p. 377, cf. also Fricke, [Fri] Chapter 3, p. 349. This is constructed out of certain tiles of the well-known modular tessellation $\mathcal{T}$. In a related work J. Nielsen, cf. [N], constructed a fundamental domain for the subgroup of the triangle group $\Delta_{2,3,p}$ which uniformizes the surface $\Gamma(p)\backslash\mathbf{H}$ *compactified by adjoining its cusps*. This is done for $p > 5$ for which 2 is either a primitive root or a square of a primitive root. Besides these constructions except for certain low values of N no general method for constructing the fundamental domains for $\Gamma(N)$ or $\Gamma_0(N)$ seems to be known.

(1.2). H. Rademacher, in connection with his work on modular forms and Dedekind sums, cf. $[R]_{2,3}$, was led to introduce another facet to the study of these congruence subgroups. He notes that $\Gamma \approx \mathbf{Z}_2 * \mathbf{Z}_3$ as an abstract group, and so by Kurosh's theorem any of its subgroups is a free product of a certain number of the copies $\mathbf{Z}_2's$, $\mathbf{Z}_3's$, and $\mathbf{Z}'s$. A system of generators S for a subgroup Φ of Γ is said to be *independent* if Φ is a free product of the cyclic subgroups $\langle x\rangle$, as x runs over S. In $[R]_1$ Rademacher asked for a construction of an independent system of generators for $\Gamma_0(N)$, and using the Reidemeister-Schreier process gave a procedure for such a construction in the case of $\Gamma_0(p)$, where p is a prime. See also Frasch, [F]. Only relatively recently this work has been extended by Chuman, cf. [C], for $\Gamma_0(N)$, where N is an arbitrary natural number. In $[R]_1$, [F], or [C] the problem is not related to the constructions of fundamental domains. Now it is well-known that once a hyperbolic polygon which is a fundamental domain for a group is constructed such that its translates by the group form a locally finite tessellation then its side-pairing transformations form a system of generators for the group. However this procedure in general may not lead to an independent system of generators. In fact it is easy to see that the side-pairing transformations for the fundamental domains for $\Gamma^0(p)$ or $\Gamma_0(p)$ referred to in (1.1) do *not* form an independent system of generators.

(1.3). A motivation for this paper was to construct fundamental domains for $\Gamma(N)$ and $\Gamma_0(N)$ so that the side-pairing transformations form an independent set of generators. In the process however we were led to a partly geometric, partly arithmetic method which applies to all subgroups of finite index in the modular group. The geometric part goes as follows. First it is essential to replace the standard modular tessellation by the *extended modular tessellation* $\mathcal{T}^*$ corresponding to the extended modular group Γ^* which is generated by Γ and $z \mapsto -\bar{z}$. *A special polygon* is a convex polygon of finite hyperbolic area in $\mathbf{H}$ which is made up of the tiles of $\mathcal{T}^*$ and which satisfies certain other conditions. Each of its sides is either a Γ-translate of the complete hyperbolic geodesic joining 0 to ∞ or else it is a geodesic segment joining a fixed point of an elliptic element of order 3 to a cusp. Moreover there are certain *admissible* procedures for the side-pairing, cf. Section 2 for details. In Section 3 we prove

THEOREM. *A special polygon is a fundamental domain for the subgroup generated by the side-pairing transformations and these transformations form an independent set of generators for the subgroup. Conversely every subgroup of finite index in* Γ *admits a special polygon as a fundamental domain.*

(1.4). The arithmetic part of the method goes as follows. A special polygon has one vertex at ∞ and its other vertices which lie in $\mathbf{R} \cup \{\infty\}$ are rational numbers which form *a generalized Farey sequence*—or *a g.F.S.* for short—in the sense that for any two consecutive vertices $\frac{a}{c}$, $\frac{b}{d}$ (reduced fractions) we have $|ad - bc| = 1$. Moreover the side-pairing of the special polygon imposes a certain additional structure on this *g.F.S.*. This motivates the notion of a *Farey symbol* which is a generalized Farey sequence with a certain extra structure, cf. Section 5 for details. The special polygons are in a natural 1-1 correspondence with the Farey symbols. A procedure for a construction of the fundamental domains for $\Gamma(N)$ and $\Gamma_0(N)$ is given in terms of appropriate Farey symbols. This program is carried out in Sections 12–13, and Appendices 2–4.

(1.5). We now indicate the contents of the paper in more detail. Besides a special polygon and a Farey symbol we introduce two other graph-theoretic objects: *tree diagrams* and *bipartite cuboid graphs* in Section 4. The relationships among these various notions may be readily discerned from the diagram in (5.4). An important geometric result is, cf. Theorem (4.2), *The bipartite cuboid graphs are in a* 1-1 *correspon-*

dence with the conjugacy classes of subgroups of finite index in Γ. An earlier version of this result appears as a special case of a more general result on finitely generated noncocompact Fuchsian groups, cf. [K]. In a completely different context related to computing volumes of the moduli spaces of Riemann surfaces this result has come up in some work of R. Penner, cf. [P].

In Theorem (6.1) there is a procedure for obtaining an independent set of generators for a subgroup given by a Farey symbol. Section 7 contains procedures to compute the geometric invariants such as the genus, the number and widths of cusps etc. for a subgroup given by its Farey symbol or tree diagram. We also give there a very short proof of a theorem of Millington, cf. [Mi]. The geometric interpretation of a finite continued fraction is given in Section 8. It implies that if a *g.F.S.* contains a given rational number x, then it must also necessarily contain some other rational numbers which can be determined from the convergents of x. This property of a *g.F.S.* is very useful in constructing a Farey symbol for an arithmetically or otherwise given subgroup.

(1.6). In Sections 9 to 11 we study conjugations in the extended modular group Γ^* and its subgroups. This study was largely motivated by some of the remarkable regularities we observed in the Farey symbols for $\Gamma_0(N)$ when N is a prime and their lack of it when N is not a prime. A geometric reason for this lies in the following fact: let $\Gamma_0^*(N)$ denote $\langle\Gamma_0(N), z \mapsto -\bar{z}\rangle$. Then the boundary of $\Gamma_0^*(N)\backslash\mathbf{H}$, when N is a prime, has a relatively simple structure. Let Φ^* be an arbitrary subgroup of finite index in Γ^* containing a reflection. In Γ^* there are two conjugacy classes of reflections, cf. (9.6), whereas in a general Φ^* there are usually several. Roughly speaking they are classified by the boundary components of $\Phi^*\backslash\mathbf{H}$, cf. (10.5) for a precise statement. An interesting fact is that there are precisely six possible geometric structures on a neighborhood of a boundary component of $\Phi^*\backslash\mathbf{H}$, cf. (10.6). Let $\Phi = \Phi^* \cap \Gamma$ which is subgroup of index 2 in Φ^*. There is a canonical reflection θ on $\Phi\backslash\mathbf{H}$ and $\Phi\backslash\mathbf{H}/\langle\theta\rangle \approx \Phi^*\backslash\mathbf{H}$ in a natural way. In Section 11 we have investigated the question whether it is possible to lift θ to a reflection on a suitable special polygon which is a fundamental domain for Φ. A necessary and sufficient condition is given in (11.3).

(1.7). The main arithmetic applications are in Sections 12, 13, and (14.11)–(14.14). In Sections 12 and 13 we note some special properties of $\Gamma^1(N)$ and $\Gamma_0(N)$ and show how they lead to a convenient procedure for constructing a Farey symbol, hence a fundamental domain, for $\Gamma^1(N)$, $\Gamma_0(N)$, and $\Gamma(N)$. In particular for $\Gamma_0(p)$, where p is a prime, it *is* possible to lift θ on $\Gamma_0^*(p)\backslash\mathbf{H}$ to a reflection on a special polygon which is a fundamental domain for $\Gamma_0(p)$, cf. Theorem (13.5). So the work of constructing a Farey symbol for $\Gamma_0(p)$ is reduced by 50%! The Appendix 3 which was constructed quite easily by hand lists Farey symbols for $\Gamma_0(p)$, where p is a prime less than 100. Such calculations could be extended quite substantially especially using a computer. The work in Section 14 was motivated by a desire to recover (13.5) for $\Gamma_0(N)$ for a general N. We first use the work in Section 10 and briefly outline a procedure for obtaining fundamental domains for an arbitrary subgroup of finite index in Γ^*. Two modifications of the notion of a special polygon, namely a *special *-polygon* and a *weak *-polygon* are developed and the arithmetic procedures are suitably modified. Roughly speaking, two consecutive vertices $\frac{a}{c}$, $\frac{b}{d}$ (reduced fractions) in a **-g.F.S.* satisfy $|ad - bc| = 1$, *or* 2, and the extra structure to make it into a *-Farey symbol allows for conjugations in Γ^* and encodes information about corners with angles $\frac{\pi}{2}$ or $\frac{\pi}{3}$ in the boundary. Every subgroup of finite index in Γ^* (except Γ^* itself and one other subgroup of index 2 in Γ^*) admits a special *-polygon as a fundamental domain. It turns out that *for* $N \geq 4$, *the boundaries of both* $\Gamma^{1*}(N)\backslash\mathbf{H}$ *and* $\Gamma_0^*(N)\backslash\mathbf{H}$ *contain no corners*. This fact substantially allows to recover (13.5) for $\Gamma_0(N)$, and correspondingly also improves the result (12.2) on $\Gamma^1(N)$ and $\Gamma(N)$ partially.

It should be pointed out that the procedures for constructing Farey symbols or *-Farey symbols involve some trial and error. The property of a *g.F.S.* mentioned in (1.5) above somewhat reduces this trial and error. However it would be of interest to develop better arithmetic algorithms to further reduce or eliminate this trial and error.

Still, when N is a prime the Farey symbols for $\Gamma_0(N)$ exhibit certain remarkable properties, cf. Appendix 3. At the beginning of (A3.7) we have given some empirical rules for constructing the Farey symbols for $\Gamma_0(p)$, p a prime. For $p < 100$ we observed that *the congruences which need to be satisfied for constructing these Farey symbols can actually be lifted to equalities in natural numbers*. This may well be true for all primes and there may be some explanation for this in "elementary" number theory. But as yet we are missing such an explanation, nor do we have an arithmetic justification whether the empirical rules mentioned above will always work for all primes.

(1.8). The geometric interpretation of a finite continued fraction given in Section 8 may be useful in other contexts. It is different from the usual "Euclidean" interpretation as in e.g. [St], Chapter 7. The interpretation may be extended to the case of an arbitrary real number. In that form it comes close to the interpretation due to Artin, cf. [A], which is beautifully explained by Series, cf. [Se].

(1.9). As mentioned above, except for two exceptions, a subgroup of finite index in Γ^* admits a special *-polygon as a fundamental domain. This implies an elementary structural property, cf. (14.7), which seems to have gone unnoticed in the vast literature on this subject: *except for two exceptions, a subgroup of finite index in* Γ^* *is isomorphic to a free product of finitely many copies of* $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}$, $\mathbf{Z}_2^2$ *and* S_3. (Here S_3 denotes the symmetric group on three letters.) A system of generators adapted to this structure is called *quasi-independent*, cf. (14.5) for a precise definition. This notion is a reasonable substitute for the Rademacher's notion of independent system of generators of a group which in general cannot exist for subgroups of Γ^*. The side-pairing transformations of a special *-polygon for a subgroup of Γ^* form a quasi-independent system of generators for the subgroup; if a special *-polygon does not contain a corner with an angle $\frac{\pi}{2}$ or $\frac{\pi}{3}$ in its boundary then the system is actually independent. In particular for $N \geq 4$, both $\Gamma_0^*(N)$ and $\Gamma^{1*}(N)$ admit independent system of generators, cf. (14.14).

(1.10). The work of Rademacher mentioned earlier is based on the Reidemeister-Schreier method which is a very general method of finding a presentation of a subgroup of a finitely presented group. In the present context of the modular group the method based on Farey symbols is much more efficient. Roughly speaking in the Reidemeister-Schreier method one starts with the computation of coset representatives. In terms of fundamental domains the coset representatives are in 1-1 correspondence with the tiles of the modular tessellation contained in the fundamental domain of the subgroup in question. The number of vertices of a special polygon is about one-third of the number of tiles, cf. (7.2), and moreover since these vertices form a *g.F.S.* the implied arithmetic properties substantially reduce the work. Moreover the additional symmetries which may be "liftable" to a Farey symbol may further reduce the work. For example using the symmetries coming from $z \mapsto z + 1$ and $z \mapsto 1 - \bar{z}$ which normalize $\Gamma(N)$ the work for $\Gamma(N)$ is further reduced by a factor of $2N$. At the same time the method gives independent or quasi-independent generators of a subgroup *in a matrix form*.

(1.11). There are possible uses and generalizations of the present work in other contexts. The method of (3.2) and (6.1) and their later modifications in Section 14 may be used in various construction problems. For example it may be used to construct noncongruence subgroups of small indices starting with 7. (This was suggested to us by C. Moreno.) Also it may be easily modified to generate subgroups of infinite index in a concrete way which exhibit interesting function-theoretic properties. The method should partially extend in the case of Hecke groups $\approx \mathbf{Z}_2 * \mathbf{Z}_p$, where p is a prime. For example it is easy to see that the Theorem (4.2) generalizes to cover this case if one replaces the cuboid graphs by the graphs in which every vertex has valence 1 or p. The geometric part essentially generalizes to all Fuchsian or Kleinian groups with cofinite volume and precisely one cusp and the arithmetic part would be expected to carry over to the cases such as the Hecke groups and the Bianchi groups.

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2. Special polygons (2.1). In the upper half plane model $\mathbf{H}$ of the hyperbolic plane let $\mathfrak{D}^*$ be the hyperbolic triangle with vertices at $i = \sqrt{-1}$, $\rho = \exp(\frac{\pi i}{3})$, and ∞. The extended modular group Γ^* as defined in (1.3) is the group generated by reflections in the edges of $\mathfrak{D}^*$. This follows easily from the fact that the union of $\mathfrak{D}^*$ and its image under $z \mapsto -\bar{z}$ is the standard fundamental domain for Γ. Moreover Γ^* may also be identified with the group $PSL_2^*(\mathbf{Z})$ of 2×2 integer matrices with determinant 1 or -1, modulo its center $\langle -I \rangle$ where I denotes the identity matrix. Under this identification an element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1, i.e. an element of Γ, acts as in (1.1.1), whereas with determinant -1 it acts by

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}, \qquad a, b, c, d \in \mathbf{R}. \tag{2.1.1}$$

The Γ^*-translates of $\mathfrak{D}^*$ define the *extended modular tessellation* $\mathcal{T}^*$ of $\mathbf{H}$. The boundary of $\mathbf{H}$ consists of the real axis $\mathbf{R}$ and ∞. Each complete geodesic of $\mathbf{H}$ has two distinct endpoints on the boundary. The elements in the Γ^*-orbit of i will be called the *even vertices* of $\mathcal{T}^*$ and those in the Γ^*-orbit of ρ will be called the *odd vertices* of $\mathcal{T}^*$. The Γ^*-orbit of ∞ consists of rational numbers and they are the *cusps* of $\mathcal{T}^*$. The elements in the Γ^*-orbit of the edge joining i to ∞, resp. the edge joining ρ to ∞ will be called the *even edges* resp. the *odd edges* of $\mathcal{T}^*$. Each of these edges has infinite hyperbolic length. Each of the edges in the Γ^*-orbit of the edge joining i to ρ has finite hyperbolic length. These edges will be called the *f-edges* of $\mathcal{T}^*$.

(2.2). The following properties of $\mathcal{T}^*$ are presumably known. But since we do not know a reference we briefly sketch the details. The reader should note that the following procedure involves only rational numbers and so it may be readily implemented on a computer to draw fairly accurate pictures of finite portions of $\mathcal{T}^*$. Throughout we agree to consider ∞ as $\frac{1}{0}$, and write rational numbers in the reduced form $\frac{a}{b}$ with $b > 0$. An integer n is to be regarded as $\frac{n}{1}$.

PROPOSITION. i) *The even edges come in pairs, each pair forming a complete hyperbolic geodesic. These geodesics are precisely the ones with end-points* $\frac{a}{c}$, $\frac{b}{d}$ *satisfying* $|ad - bc| = 1$. *Each of these geodesics contains an even vertex.* Γ *acts transitively on these geodesics and the stabilizer subgroup of* Γ *preserving any one of these geodesics is isomorphic to* $\mathbf{Z}_2$ *which fixes the even vertex.*

ii) *A pair of odd edges and a pair of f-edges form a complete hyperbolic geodesic. The geodesics obtained in this way are precisely the* *ones which have endpoints* $\frac{a}{c}$, $\frac{b}{d}$ *satisfying* $|ad - bc| = 2$. *Each of these geodesics contains an even vertex and a pair of odd vertices. Again* Γ *acts transitively on these geodesics and the stabilizer subgroup of* Γ *preserving any one of these geodesics is isomorphic to* $\mathbf{Z}_2$ *which fixes the even vertex.*

iii) *If* $\frac{a}{c}$, $\frac{b}{d}$ *are the end-points of a geodesic of type* ii) *then* $a \equiv b$ (2) *and* $c \equiv d$ (2).

(*Note*: For the computation of "$ad - bc$," as remarked above ∞ is to be regarded as $\frac{1}{0}$. Thus the half-lines $x = n$, $n \in \mathbf{Z}$, $y > 0$, consist of two even edges, and the half-lines $x = n + \frac{1}{2}$, $n \in \mathbf{Z}$, $y > 0$, consist of two f-edges and two odd edges.)

Proof. First notice that under the action of Γ or Γ^* every pair of rational numbers $\{\frac{a}{c}, \frac{b}{d}\}$ is equivalent to exactly one pair of the form $\{\infty, \frac{b'}{d'}\}$ where $d' = |ad - bc|$ and $0 \leq b' < d'$, $(b', d') = 1$. In particular $|ad - bc|$ is a complete invariant of such pairs when its value is 1 or 2. So the first part follows as it is true for the geodesic joining $\infty = \frac{1}{0}$ to $0 = \frac{0}{1}$. Similarly the second part follows as it is true for the geodesic joining $-1 = \frac{-1}{1}$ to $1 = \frac{1}{1}$ or also for the geodesic joining ∞ to $\frac{1}{2}$. As for the third part notice that it holds for the pair $\{-1, 1\}$ (or $\{\infty, \frac{1}{2}\}$), and a simple computation shows that it holds for the translates of this pair by Γ. *q.e.d.*

(2.3). For simplicity the complete geodesics which are unions of two even edges will be called the *even lines*, and the complete geodesics which are unions of two f-edges and two odd edges will be called the *odd lines*.

(2.4). We now define a *special polygon* which is a certain type of a convex hyperbolic polygon with certain rules for side-pairing. Let P be a convex hyperbolic polygon with boundary ∂P which is a union of even and odd edges. The following is assumed.

S_1) The even edges in ∂P come in pairs, each pair forming an even line.

S_2) The odd edges in ∂P come in pairs. The edges in each pair meet at an odd vertex making an internal angle $\frac{2\pi}{3}$.

A *side-pairing* is an involution on edges so that no edge is carried into itself and the following rules hold.

S_3) An odd edge e is paired to the odd edge f which makes an internal angle $\frac{2\pi}{3}$ with e.

S_4) Let e, f be two even edges in ∂P forming an even line. Then either e is paired to f, or else e, f form a *free side* of P and this free side is paired with another such free side of P.

The odd edges, the even edges which form a complete geodesic and which are paired, and the free sides as defined above will simply be called *the sides* of P. The points of intersection of the adjacent sides including those on $\partial\mathbf{H}$ are called the *vertices* of P. We finally assume

S_5) 0 and ∞ are two of the vertices of P.

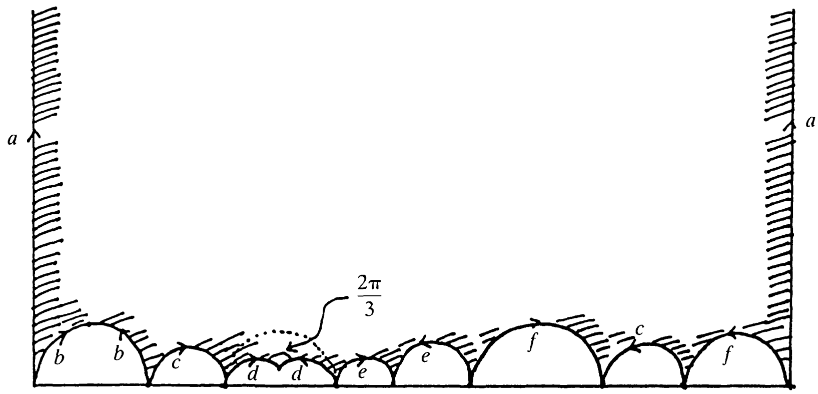
A *special polygon* is a convex hyperbolic polygon satisfying S_1 – S_5.

Notice for emphasis that a special polygon does not contain any f-edge in its boundary. Indeed every side of P has at least one endpoint which is a cusp-vertex. So in fact S_2 is a consequence of this fact and the convexity of the polygon.

As an example for S_1) and S_4) we have the even line joining 0 to ∞ for which the quantity $|ad - bc| = 1$. We consider i as a vertex of P iff we are in case S_1), i.e. the two even edges meeting in i are paired. As an example for S_2) we have the union of odd edges joining ∞ to ρ and ρ to 0. Here ρ will always be a vertex of P. Also notice that the cusp-vertices on these odd edges still satisfy $|ad - bc| = 1$. Thus in any case neglecting the vertices of P which lie in $\mathbf{H}$ we see that any two consecutive cusp-vertices of P always satisfy $|ad - bc| = 1$.

(2.5). Let P be a special polygon. Note that P has a canonical orientation induced from that of $\mathbf{H}$ and so it induces a canonical "counterclockwise" orientation on each of its sides. In view of the assertion about the stabilizers in the above proposition it follows that *if e, f are two sides which are paired then there is a unique element in Γ which carries e into f in an orientation-reversing manner*. The elements of Γ obtained this way will be called *the side-pairing transformations* of P. Also the subgroup of Γ generated by the side-pairing transformation will be denoted by Φ_P.

(2.6). A special polygon has a shape as indicated in the following figure.



3. Admissible fundamental domains (3.1). Let Φ be a subgroup of finite index in Γ. In this paper we shall consider only those fundamental domains for Φ which are convex hyperbolic polygons of finite hyperbolic area consisting of the tiles of the extended modular tessellation $\mathcal{T}^*$, cf. (2.1). It is wellknown that for such a fundamental domain there are elements of Φ which identify its sides and these side-pairing transformations generate Φ. Such a fundamental domain will be called *admissible* if its side-pairing transformations form an independent set of generators.

(3.2) THEOREM. *Let P be a special polygon, and Φ_P the associated subgroup of* Γ *as defined in* (2.5). *Then P is an admissible fundamental domain for Φ_P. Moreover Φ_P is free if and only if P has only free sides.*

Proof. By the standard arguments, as in for example [Si] Chapter 3, Section 9, Theorem 1, we see that P is a fundamental domain for Φ_P. So the space obtained by identifying the sides of P by the side-pairing transformations associated to P, cf. (2.5), has a complete hyperbolic metric (with singularities corresponding to the branch points). So the conditions for the application of the well-known theorem of Poincarè on fundamental polygons as developed by Maskit, cf. [M], are fulfilled. This allows one to see a complete set of relations among the generators given by the side-pairing transformations. In our case these relations are of the form $x^3 = 1$ corresponding to the side-pairing of the type S_3, and $x^2 = 1$ corresponding to the first alternative in the side-pairing of type S_4, cf. (2.4). These relations appear if and only if P has vertices in $\mathbf{H}$, i.e. P has a nonfree side. This precisely means that the side-pairing transformations are independent. *q.e.d.*

(3.3) THEOREM. *Every subgroup Φ of finite index admits an admissible fundamental domain which is a special polygon P so that $\Phi = \Phi_P$.*

Proof. Let Φ be a subgroup of finite index in Γ. First we describe the nature of all the hyperbolic polygons which are fundamental domains for Φ and which are made up of the tiles of $\mathcal{T}^*$. Let S_Φ denote $\Phi\backslash\mathbf{H}$, and let $p : \mathbf{H} \to S_\Phi$ denote the canonical projection. Since Φ preserves $\mathcal{T}^*$ we have an induced tessellation $\mathcal{T}^*{}_\Phi$ of S_Φ. The p-images of the even vertices, . . . , f-edges, cf. (2.1), in S_Φ will again be called the even vertices etc. in S_Φ. Notice however that there are two types of even vertices, (resp. odd vertices) in S_Φ, namely *type* 1: those which are incident to a single f-edge, and *type* 2: those which are incident to two f-edges (resp. three f-edges). Now if P is a hyperbolic polygon which is a fundamental domain for Φ then $p(P) = S_\Phi$, p is injective on the interior of P and it identifies the sides of P in pairs. Conversely let A be a subset of the edges of S_Φ, and let P_A be the space obtained by cutting S_Φ along A. If P_A is connected and simply connected then developing P_A isometrically along the tiles of $\mathcal{T}^*$ we obtain a polygon P, and some translate of P by an element of Γ serves as a fundamental domain for Φ. (Notice that P itself may *not* be a fundamental domain for Φ although it is surely a fundamental domain for some conjugate of Φ in Γ. This uses the fact that Γ is the full group of orientation-preserving symmetries of $\mathcal{T}^*$.)

Let $\mathscr{E}_f$ denote the union of f-edges in S_Φ. We consider $\mathscr{E}_f$ as a graph whose vertex set consists of the even vertices and odd vertices in S_Φ and the edge set consists of the f-edges in S_Φ. Since the union of f-edges in $\mathbf{H}$ is a connected set so is $\mathscr{E}_f$. Let T be a maximal tree in $\mathscr{E}_f$. A vertex of valence 1 will be called a *terminal vertex*. Notice that the even vertices and odd vertices of type 1 are necessarily terminal in $\mathscr{E}_f$ and so also in T. However some of the even vertices and odd vertices of type 2 may also be terminal in T although they are not terminal in $\mathscr{E}_f$. Let A be the union of all the even edges in S_Φ incident with the terminal even vertices in T together with all the odd edges which are incident to the odd vertices of type 1 in S_Φ. Let P_A be S_Φ cut along A.

We claim that P_A is connected and simply connected. Indeed first note that all f-edges in $\mathbf{H}$ project to a single edge, again called the f-edge, in the space $\Gamma^*\backslash\mathbf{H}$. The space $\Gamma^*\backslash\mathbf{H}$ can be retracted continuously to the f-edge it contains. So there is a Γ^*-equivariant retraction of $\mathbf{H}$ onto the union of f-edges. Passing to the quotient by Φ we see that S_Φ retracts continuously onto $\mathscr{E}_f$. Now each edge in A after cutting contributes two edges on the boundary of P_A and so $S_\Phi - A = P_A - \partial P$ is homeomorphic to int P_A.

Assertion. It suffices to show that int P_A is connected and simply connected.

Proof. Each component of the boundary of P_A is either an even line or a union of a pair of odd edges. So each of these components is isometric to $\mathbf{R}$. Clearly P_A is connected iff int P_A is connected. Now let C be any smooth Jordan curve in P_A. Since C has finite length, a component of $C \cap \partial P_A$ can only be a compact arc possibly reduced to a point. It follows that we can homotop C into int P_A. So P_A is simply connected iff int P_A is simply connected. *q.e.d. of the Assertion.*

Now the space int $P_A = S_\Phi - A$ continuously retracts to $\mathscr{E}_f$ − (the terminal vertices in T) = U say. So it suffices to show that U is connected and simply connected. Let e be an edge in $\mathscr{E}_f$ which is not in T. It connects an even vertex say v which is (necessarily) of type 2 to some odd vertex say w. Since v is of valence 2 in $\mathscr{E}_f$ and e is not in T it follows that v is a terminal vertex of T. On the other hand w is *not* a terminal vertex for otherwise $\Phi = \Gamma$ and $\mathscr{E}_f = T$ which cannot happen since e is in $\mathscr{E}_f$ but not in T. It follows that if we remove v exactly one circuit in $\mathscr{E}_f$ is broken. So U can be continuously retracted into T, hence U—and so also P_A—is connected and simply connected.

Since P_A is connected and simply connected it may be developed isometrically into $\mathbf{H}$. In fact once one tile of P_A is developed into a tile of $\mathscr{T}^*$, P_A develops uniquely into a hyperbolic polygon P which is a union of the tiles in $\mathscr{T}^*$. We shall show that some Γ-translate of P is a special polygon which is an admissible fundamental domain for Φ. One needs to consider 3 cases depending on the nature of the terminal vertices in T. In the following v denotes a terminal vertex in T.

Case 1 (*v is an even vertex of type* 1). There is a unique f-edge incident to v in S_Φ. Corresponding to it we obtain a pair of even edges in ∂P which forms an even line. These edges form two sides of P which are paired.

Case 2 (*v is an even vertex of type* 2). There is a pair of f-edges incident to v. Correspondingly there is an even line incident to v in S_Φ. Correspondingly we have two free sides of P which are paired.

Case 3 (*v is an odd vertex of type* 1). There is a unique odd edge incident to v in S_Φ. Corresponding to it we obtain a pair of odd edges in ∂P which make an internal angle $\frac{2\pi}{3}$. These edges form two sides of P which are paired.

It now follows that the conditions S_1–S_4 for a special polygon, cf. (2.5), are satisfied by P. Notice moreover that P has a shape of a hyperbolic polygon P_0 bounded by finitely many complete hyperbolic geodesics possibly together with a finite number of hyperbolic triangles with angles $0, 0, \frac{2\pi}{3}$ which are attached externally to P_0 along a complete geodesic. In particular P is convex. As observed in the remarks at the beginning of the proof some Γ-translate P_1 of P is a fundamental domain for Φ. So $\mathbf{H}$ is tessellated by Φ-translates of P_1. Also every tile of $\mathcal{T}^*$ is contained in some Φ-translate of P_1. Let P_2 be that Φ-translate of P_1 which contains the tile $\mathcal{D}^*$, i.e. the hyperbolic triangle with vertices at $i = \sqrt{-1}$, $\rho = \exp(\frac{\pi i}{3})$, and ∞. Now the boundary of P_2 by construction does not contain any f-edge. It follows that P_2 must contain the tile which is the hyperbolic triangle with vertices at $i = \sqrt{-1}$, ρ and 0 as well. So 0 and ∞ are among the vertices of P_2. This proves that S_5 also holds, and so P_2 is a special polygon which is an admissible fundamental domain for Φ. This finishes the proof. *q.e.d.*

(3.4). The following is an interesting property of an admissible fundamental domain. In counting the *sides* of a fundamental domain we follow the convention, already introduced in (2.4), that if an even line is contained in the boundary of the fundamental domain and the even edges contained in this even line are paired then this line counts as *two* sides of the fundamental domain.

PROPOSITION. *Let Φ be a subgroup of finite index in Γ. Among all fundamental polygons for Φ whose Φ-translates form a locally finite tessellation of $\mathbf{H}$ an admissible fundamental domain has the least number of sides. If Φ is isomorphic to a free product of a copies of $\mathbf{Z}_2$, b copies of $\mathbf{Z}_3$, and r copies of $\mathbf{Z}$ then this least number is $2(r + a + b)$.*

Proof. If P is a fundamental polygon for Φ whose Φ-translates form a locally finite tessellation of $\mathbf{H}$ then by [B], Theorem 9.2.7, the side-pairing transformations generate Φ. If Φ has the free product decomposition as stated in the theorem then by Grushko's theorem the least number of generators for Φ is $(r + a + b)$. So P has at least twice this number of sides. Finally if P is admissible then its side-pairing transformations are independent so it has exactly $2(r + a + b)$ sides.

q.e.d.

4. Conjugacy classes of subgroups—A graph-theoretic method

(4.1). *A bipartite cuboid graph* is a finite graph whose vertex set is divided into two disjoint subsets V_0 and V_1 such that

i) every vertex in V_0 has valence 1 or 2,

ii) every vertex in V_1 has valence 1 or 3,

iii) there is a prescribed cyclic order on the edges incident at each vertex of valence 3 in V_1,

iv) every edge joins a vertex in V_0 with a vertex in V_1.

An isomorphism of bipartite cuboid graphs is of course an isomorphism of the underlying graphs preserving the cyclic orders on the edges of each vertex of valence 3.

(4.2) THEOREM. *The conjugacy classes of subgroups of finite index in* Γ *are in* 1-1 *correspondence with the isomorphism classes of bipartite cuboid graphs.*

Proof. Recall from (2.1) the hyperbolic triangle $\mathfrak{D}^*$ with vertices at $i = \sqrt{-1}$, $\rho = \exp(\frac{\pi i}{3})$, and ∞. Let S be an orientable surface. *A modular tessellation* $\mathcal{T}_S^*$ on S is a homeomorphism with the space obtained as a union of finitely many copies $\mathfrak{D}_i^*$ of $\mathfrak{D}^*$ where each even edge, odd edge, f-edge is isometrically glued to another even edge, odd edge, f-edge respectively so that S is locally modelled on $\mathbf{H}$, or $\mathbf{Z}_2\backslash\mathbf{H}$, or $\mathbf{Z}_3\backslash\mathbf{H}$ where $\mathbf{Z}_2$, $\mathbf{Z}_3$ act on $\mathbf{H}$ by a rotation around a fixed point through an angle π or $\frac{2\pi}{3}$ respectively. Then S is a complete 2-dimensional hyperbolic orbifold in the sense of Thurston, cf. [T], Chapter 13. We may obtain S from a special polygon by the process described in the proof of Theorem (3.3) and so it is of the form $\Phi\backslash\mathbf{H}$ where Φ is a subgroup of finite index in Γ. Since Γ is the full group of orientation-preserving isometries of $\mathbf{H}$ preserving $\mathcal{T}^*$, $(S, \mathcal{T}_S^*)$ determines Φ upto conjugacy in Γ, and so the tessellation-preserving isometry classes of spaces $(S, \mathcal{T}_S^*)$ are in 1-1 correspondence with the conjugacy classes of subgroups of finite index in Γ.

So to prove the theorem it suffices to set up a 1-1 correspondence of the tessellation-preserving isometry classes of the spaces $(S, \mathcal{T}_S^*)$ with the isomorphism classes of the bipartite cuboid graphs. Given $(S, \mathcal{T}_S^*)$ let $\mathcal{E}_{f,S}$ denote the union of the f-edges in S. Then $\mathcal{E}_{f,S}$ has a natural structure of a bipartite cuboid graph by taking V_0 resp. V_1 to be the set of even vertices resp. odd vertices, and the cyclic order on the edges incident at a vertex of valence 3 being the one induced from the orientation of S. Conversely let G be a bipartite cuboid graph. Let $\mathcal{D}_e^*$, $\mathcal{D}_e^{*\prime}$ be the two sets of copies of $\mathcal{D}^*$ each indexed by the edges e of G. Attach $\mathcal{D}_e^*$ to $\mathcal{D}_e^{*\prime}$ isometrically along e so that a vertex in V_0 in one copy of e is attached to a vertex in V_0 in the other copy of e. We thus obtain $\mathcal{D}_e^{*\prime\prime}$ which is isometric to a hyperbolic triangle with angles 0, 0, $\frac{2\pi}{3}$. There is a canonical isometry of $\mathcal{D}_e^{*\prime\prime}$ with the hyperbolic triangle with vertices 0, ∞, and ρ. Using this isometry we can equip $\mathcal{D}_e^{*\prime\prime}$ with a canonical orientation and in turn the "counterclockwise" orientation on its boundary edges. If two distinct edges e and f share an even vertex then attach $\mathcal{D}_e^{*\prime\prime}$ to $\mathcal{D}_f^{*\prime\prime}$ isometrically along the complete geodesics made up of even edges in the orientation reversing way. If e, f share an odd vertex v then there is a third edge g also sharing v. By symmetry we may suppose that the cyclic order is e, f, g. Orient these edges so that they "emanate" from v. In $\mathcal{D}_e^{*\prime\prime}$ using its orientation we can then uniquely determine an odd edge k which makes the angle $+\frac{\pi}{3}$ with e. Similarly there is a unique choice of an odd edge l in $\mathcal{D}_f^{*\prime\prime}$ which makes the angle $-\frac{\pi}{3}$ with f. We attach $\mathcal{D}_e^{*\prime\prime}$ to $\mathcal{D}_f^{*\prime\prime}$ so that k is isometrically identified with l. If e has a terminal even (resp. odd) vertex v we identify the even (resp. odd) edges of $\mathcal{D}_e^{*\prime\prime}$ incident at v in an orientation-reversing way. This procedure thus defines a $(S, \mathcal{T}_S^*)$ which is canonically attached to G. It is easy to see that these maps set up a desired 1-1 correspondence among the isomorphism classes of bipartite cuboid graphs and the tessellation-preserving isometry classes of the spaces $(S, \mathcal{T}_S^*)$. *q.e.d.*

(4.3) Tree diagrams. We shall now describe a variant of the above graph-theoretic construction which is useful in practice in the constructions of subgroups of finite index in Γ. *A cuboid tree diagram*, or *a tree diagram* for short is a finite tree T with at least one edge such that

i) all the internal vertices are of valence 3,

ii) there is a prescribed cyclic order on the edges incident at each internal vertex,

iii) the terminal vertices are partitioned into two possibly empty subsets R and B where the vertices in R (resp. B) are called red (resp. blue) vertices,

iv) there is an involution σ on R.

T can be embedded in the plane so that the cyclic order on the edges at each internal vertex coincides with the one induced by the orientation of the plane. Any two such embeddings are in fact isotopic to each other. An isomorphism of two tree diagrams is defined in the obvious way and amounts to an isotopy class of planar trees satisfying i). So a tree diagram can be best represented on paper without explicitly indicating the cyclic order; the red (resp. blue) vertices are represented by a small hollow (resp. shaded) circles; and distinct red vertices related by σ are given the same numerical label, it being understood that the unlabelled vertices are fixed by σ and different pairs of distinct red vertices related by σ carry different labels. In this form they are drawn in Appendix 1.

(4.4). The correspondences among the special polygons, the bipartite cuboid graphs, and the tree diagrams are as follows.

The correspondence between the bipartite cuboid graphs, and the tree diagrams: Let T be a tree diagram. Identifying v with $\sigma(v)$ one obtains a graph G. On all edges joining two internal vertices or an internal vertex with a blue vertex introduce a new vertex of valence 2. These new vertices and the red vertices consitute V_0. The vertices of valence 3 and the blue vertices constitute V_1. The cyclic orders on the vertices in V_1 are defined by ii). This turns G into a bipartite cuboid graph.

Conversely let G be a bipartite cuboid graph. If its cycle-rank (= the first Betti number) is r we can choose r vertices of valence 2 in V_0 so that cutting G along these vertices we obtain a tree T. Corresponding to these r cuts we have $2r$ terminal vertices in T. These $2r$ vertices and the terminal vertices of valence 1 in V_0 constitute the red vertices, and the terminal vertices in V_1 constitute the blue vertices. Set up the involution σ as fixing the terminal vertices of valence 1 in V_0 and interchanging the two vertices obtained by each one of the r cuts. We agree not to count the remaining vertices of valence 2 in V_0 as vertices. Finally the cyclic order on the edges incident at the vertices of valence

3 in T is the same as that in G. This turns T into a tree diagram. Notice that T depends on the choices of the r cuts.

It is clear that we have a well-defined finite-to one map from the isomorphism classes of tree diagrams onto those of bipartite cuboid graphs.

The correspondence between the special polygons and the tree diagrams: Let P be a special polygon and T the union of all the f-edges in P. We agree not to count the even vertices in int P as vertices. The even vertices resp. odd vertices in ∂P constitute the red resp. blue vertices. The involution on the red vertices is given by the side-pairing datum in P. Finally the cyclic order on the edges incident to the vertices of valence 3 is induced by the orientation of P. This turns T into a tree diagram.

Conversely let T be a tree diagram. On all edges joining two internal vertices or an internal vertex with a blue vertex introduce a new vertex of valence 2. Equip T with a metric in a standard way so that each edge has the same length equal to the length of an f-edge (which is equal to $\frac{1}{2}\ln 3$). T must have at least one red vertex or at least one blue vertex. Suppose it has a red vertex v. Isometrically develop the unique edge containing v onto the f-edge joining i to ρ. Then T itself develops isometrically and uniquely along the f-edges in $\mathcal{T}^*$ so that the cyclic orders on the edges incident at the vertices of valence 3 in T match with the ones induced by the orientation of $\mathbf{H}$. At the image of a red vertex v in this development assign the even line passing through that even vertex. These even edges are paired if the vertex v is fixed by the involution σ. Otherwise this complete geodesic will be considered as a free side. It will be paired with the other free side constructed at $\sigma(v)$. Similarly at the image of a blue vertex incident to the (unique) edge say e assign those two 3-edges which make an angle $\frac{\pi}{3}$ with the image of e. These odd edges are paired. It is easy to see that these even sides, odd sides, and free sides together with their pairing defines a special polygon.

It is fairly clear that we have a well-defined finite-to-one map from special polygons onto the isomorphism classes of tree diagrams.

5. Farey symbols (5.1). *A generalized Farey sequence* is an expression of the form

$$\{\infty, x_0, x_1, \ldots, x_n, \infty\} \tag{5.1.1}$$

where

i) x_0 and x_n are integers, and some $x_t = 0$,

ii) $x_i = \frac{a_i}{b_i}$ are rational numbers in their reduced forms and ordered according to their magnitudes, such that

$$|a_i b_{i+1} - b_i a_{i+1}| = 1, \qquad i = 1, 2, \ldots, n-1. \tag{5.1.2}$$

We shall abbreviate the expression *generalized Farey sequence* to *g.F.S.*. It will be convenient to set $x_{-1} = x_{n+1} = \infty$, and consider the x_i's as forming a cyclic order. Moreover we recall that $\infty = \frac{1}{0}$.

(5.2). Recall that classically the finite sequence of rationals between 0 and 1 and with denominators at most n is called the n-th Farey sequence and it has the property (5.1.2), cf. [HW], Chapter 3. The importance of the notion of a *g.F.S.* for us is that the vertices of a special polygon, lying in $\mathbf{R} \cup \{\infty\}$, i.e. neglecting those in $\mathbf{H}$ form a *g.F.S.*, cf. Section 2. On the other hand if we start with a *g.F.S.* and take its convex hull in $\mathbf{H}$ we obtain a convex hyperbolic polygon which is a union of finitely many tiles of $\mathcal{I}$. It is clear that the *g.F.S.*s are in 1-1 correspondence with such polygons which moreover contain 0 and ∞ as their vertices.

(5.3). We shall equip a *g.F.S.* with an extra structure which is an abstract analogue of encoding the side-pairing information of a special polygon. First suppose P is a special polygon and (5.1.1) is the *g.F.S.* formed by its vertices in $\mathbf{R} \cup \{\infty\}$. If the complete hyperbolic geodesic joining x_i to x_{i+1}, $i = -1, 0, 1, \ldots, n+1$ consists of two sides of P which are paired then we indicate this information by

$$\underbrace{x_i \quad x_{i+1}}_{\circ}. \tag{5.2.1}$$

We shall call $\underbrace{x_i \quad x_{i+1}}_{\circ}$ *an even interval* of the *g.F.S.*. If x_i and x_{i+1} are the endpoints of two odd edges which are two sides of P and which are paired then we indicate this information by

$$\underbrace{x_i \quad x_{i+1}}_{\bullet}. \tag{5.2.2}$$

We shall call $\underbrace{x_i \quad x_{i+1}}_{\bullet}$ *an odd interval* of the *g.F.S.*. If x_i and x_{i+1} are the endpoints of a free side e of P and $x_{i'}$ and $x_{i'+1}$ are the endpoints of the free side of P paired to e then we indicate this information by

$$\underbrace{x_i \quad x_{i+1}}_{a}, \qquad \underbrace{x_{i'} \quad x_{i'+1}}_{a}. \tag{5.2.3}$$

Here a is a numerical symbol. If the a's occur at all they will be numbered from 1 to some positive integer r, it being understood that different pairs of associated free sides carry different numerical symbols. Of course the specific numerical values for the labels have no significance. We shall call each of $\underbrace{x_i \quad x_{i+1}}_{a}$ and $\underbrace{x_{i'} \quad x_{i'+1}}_{a}$ *a free interval* of the *g.F.S.*.

A *g.F.S.* (5.1.1), without any reference to P, adorned with an extra structure on each consecutive pair of $x_i's$ of the type (5.2.1)–(5.2.3), will be called *a Farey symbol*. Thus a typical Farey symbol may look like

$$\begin{array}{ccccccccccccccccc} \{\infty & & x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & \infty\}. \\ & 1 & & \circ & & 2 & & 1 & & \bullet & & 3 & & 2 & & 3 & \end{array}$$

(5.4). Conversely given a Farey symbol whose underlying *g.F.S.* is given by (5.1.1) we can construct a special polygon as follows. Let P_0 be the hyperbolic convex hull of the $x_i's$, $i = -1, \ldots, n+1$. Suppose we have an odd interval $\underbrace{x_i \quad x_{i+1}}_{\bullet}$ in our symbol. Then the complete hyperbolic geodesic joining x_i to x_{i+1} together with two odd edges situated outside P_0 form a hyperbolic triangle with angles $0, 0, \frac{2\pi}{3}$. Adjoining such triangles for each odd interval in the Farey symbol one obtains a convex hyperbolic polygon P. The side-pairing is defined by reversing the process described above. Thus in view of (4.5) we have

PROPOSITION. *The set of Farey symbols is in a natural* 1-1 *correspondence with the set of special polygons. In particular a Farey symbol* *determines a subgroup of finite index in* Γ, *and every subgroup of finite index in* Γ *arises in this way. The map*

$$\{\textit{Farey symbols}\} \to \{\textit{subgroups of finite index in } \Gamma\}$$

is finite-to-one.

(5.5). The following diagram summarizes the various relationships among the objects introduced in Section 4 and Section 5.

$$\begin{tikzcd}
 & \{\textit{special polygons}\} \arrow[r] \arrow[dl] & \{\textit{Farey Symbols}\} \arrow[d] \\
\{\textit{tree diagrams}\} \arrow[d] & & \{\textit{the subgroups of finite index in } \Gamma\} \arrow[d] \\
\{\textit{bipartite cuboid graphs}\} \arrow[rr] & & \{\textit{the conjugacy classes of the subgroups of finite index} \text{ in } \Gamma\}
\end{tikzcd}$$

All the arrows here are finite-to-one surjective maps whereas the top and bottom horizontal arrows are also one-to-one.

(5.6). In the definition of a *g.F.S.*, cf. (5.1), the first and the last finite element on the real axis are integers. This imposes certain restrictions on the other elements. We note two such restrictions in the next two propositions. Another restriction is noted in Section 8, cf. (8.7).

PROPOSITION. *Let* $\mathcal{F}$ *be a g.F.S. in the form* (5.1.1). *Let* k *be an integer, and* b_i*'s the denominators of those* x_i*'s which lie in* $[k, k + 1)$. *Then* b_i*'s determine* x_i*'s uniquely.*

Proof. Consider the diophantine equation $sb_i - tb_{i+1} = 1$, to be solved for s and t in integers. Clearly $s = a_{i+1}$, $t = a_i$, where a_i's are the numerators of the x_i's is one solution. Any other solution has the form $s = a_{i+1} + lb_{i+1}$, $t = a_i + lb_i$, where l is an integer. It is clear that for any $l \neq 0$ the corresponding $\frac{s}{b_{i+1}}$, $\frac{t}{b_i}$ lie outside the given interval $[k, k + 1)$. *q.e.d.*

(5.7) PROPOSITION. *Let* x *be an element of a g.F.S. which is not an integer, and let* k *be an integer such that* $k < x < k + 1$. *Let* $y < x < z$ *be the three successive terms in the g.F.S.. Then* $k \leq y < x < z \leq k + 1$. *Moreover let* $y = \frac{e}{f}$, $x = \frac{b}{d}$, $z = \frac{a}{c}$ *with* c, d, f *positive. Then* $a + e = nb$, $c + f = nd$ *where* $n = af - ce$ *is a positive integer.*

Proof. Distinct even lines do not intersect. So since k, $k + 1$ are endpoints of an even line it is clear that y, z lie in the interval $[k, k + 1]$. Now note that since c, d, f are positive $ad - bc$, $fb - ed$, $af - ce$ are positive integers, and moreover since x, z and x, y are endpoints of even lines we also have $ad - bc = fb - ed = 1$. In a matrix form these two equations become

$$\begin{pmatrix} a & -c \\ -e & f \end{pmatrix}\begin{pmatrix} d \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{or}$$

$$(af - ce)\begin{pmatrix} d \\ b \end{pmatrix} = \begin{pmatrix} f & c \\ e & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives $a + e = nb$, $c + f = nd$ where $n = af - ce$ is a positive integer. *q.e.d.*

(The integer n in the above proposition has the following geometric interpretation. Let P be the convex hull of the *g.F.S.* in $\mathbf{H}$. Then n is the number of tiles of $\mathcal{I}$ which are contained in P and which are incident with x, cf. (7.4).)

6. Independent generators.

(6.1) THEOREM. *Let σ be a Farey symbol whose underlying g.F.S. is given by* (5.1.1). *Let Φ_σ be the subgroup of finite index in Γ determined by σ. Let $x_i = \frac{a_i}{b_i}$ (reduced forms) with b_i positive. Let the number of even (resp. odd) intervals be a (resp. b) and the number of pairs of free intervals be r.*

i) *For each even interval* $\underbrace{x_i \quad x_{i+1}}_{\circ}$, $i = i_1, i_2, \ldots, i_a$ *in σ consider*

$$A_i = \begin{pmatrix} a_{i+1}b_{i+1} + a_ib_i & -a_i^2 - a_{i+1}^2 \\ b_i^2 + b_{i+1}^2 & -a_{i+1}b_{i+1} - a_ib_i \end{pmatrix},$$

ii) *For each odd interval* $\underbrace{x_j \quad x_{j+1}}_{\bullet}$, $j = j_1, j_2, \ldots, j_b$ *in σ consider*

$$B_j = \begin{pmatrix} a_{j+1}b_{j+1} + a_j b_{j+1} + a_j b_j & -a_j^2 - a_j a_{j+1} - a_{j+1}^2 \\ b_j^2 + b_j b_{j+1} + b_{j+1}^2 & -a_{j+1}b_{j+1} - a_{j+1}b_j - a_j b_j \end{pmatrix},$$

iii) *For each pair of free intervals* $\underbrace{x_k \quad x_{k+1}}_{a}$, $\underbrace{x_{k'} \quad x_{k'+1}}_{a}$, $k = k_1, k_2, \ldots, k_r$ *in* σ *consider*

$$C_k = \begin{pmatrix} a_{k'+1}b_{k+1} + a_{k'}b_k & -a_{k'}a_k - a_{k'+1}a_{k+1} \\ b_{k'}b_k + b_{k'+1}b_{k+1} & -a_{k+1}b_{k'+1} - a_k b_{k'} \end{pmatrix},$$

Then the $a + b + r$ *elements* A_i, B_j, C_k *form an independent set of generators of* Φ_σ.

Proof. Let $x = \frac{b}{d}$, $y = \frac{a}{c}$ (reduced forms) with a, c positive be endpoints of an even line. If $x < y$ then $ad - bc = 1$. Notice that the unique element of Γ which carries ∞ to y and 0 to x is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whereas the one carrying ∞ to x and 0 to y is

$$\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}.$$

Also if $x = \frac{b}{d}$, $y = \frac{a}{c}$ and $u = \frac{q}{s}$, $v = \frac{p}{r}$ are endpoints of even lines satisfying $ad - bc = 1$ and $ps - qr = 1$ then the element in Γ which carries x into u and y into v is QM^{-1} where

$$Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad \text{and} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.1.1}$$

In case i) we need the element of Γ interchanging x_i and x_{i+1}. By the above remark this element is

$$\begin{pmatrix} a_i & -a_{i+1} \\ b_i & -b_{i+1} \end{pmatrix} \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix}^{-1}. \tag{6.1.2}$$

Similarly in case iii) the element carrying x_k to $x_{k'+1}$ and x_{k+1} to $x_{k'}$ is

$$(6.1.3)\qquad \begin{pmatrix} a_{k'} & -a_{k'+1} \\ b_{k'} & -b_{k'+1} \end{pmatrix} \begin{pmatrix} a_{k+1} & a_k \\ b_{k+1} & b_k \end{pmatrix}^{-1}.$$

The case ii) is a bit more tricky. Let $x = \frac{b}{d}$, $y = \frac{a}{c}$ be a pair of consecutive rationals in a *g.F.S.* with positive denominators. Let $z = \frac{a+b}{c+d}$. It is easy to see that the hyperbolic triangle with vertices x, y, z contains a unique fixed point of an element of order 3 in Γ. (Indeed this fixed point is the point of intersection of the odd lines joining $\frac{b}{d}$ to $\frac{2a+b}{2c+d}$, and $\frac{a}{c}$ to $\frac{a+2b}{c+2d}$.) The element of Γ fixing this point and permuting x, z, y *in this cyclic order*, is given by

$$(6.1.4)\qquad \begin{pmatrix} b & -a-b \\ d & -c-d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

Applying this to $x = x_j$, $y = x_{j+1}$ we obtain the elements $B_j's$. The cases where one of the $x_i's$ is ∞ are easily taken care of. The conclusion in the theorem follows from the correspondence between the special polygons and Farey symbols, and Theorem (3.2). *q.e.d.*

7. Geometric invariants of a subgroup (7.1). Let Φ be a subgroup of finite index in Γ. Then we have a diagram of branched coverings

$$\begin{tikzcd} \mathbf{H} \arrow[r] \arrow[dr] & \Phi\backslash\mathbf{H} \arrow[d] \\ & \Gamma\backslash\mathbf{H} \end{tikzcd}$$

The geometric invariants of Φ include

i) e_2 = the number of branch points of $\mathbf{H} \to \Phi\backslash\mathbf{H}$ of order 2,
ii) e_3 = the number of branch points of $\mathbf{H} \to \Phi\backslash\mathbf{H}$ of order 3,
iii) $d = (\Gamma : \Phi)$ = the degree of the branched covering $\Gamma\backslash\mathbf{H} \to \Phi\backslash\mathbf{H}$,
iv) g = the genus of $\Phi\backslash\mathbf{H}$,
v) t = the number of cusps of Φ,
vi) $w(C_k)$ = the width of the k-th cusp, $k = 1, 2, \ldots, t$,
vii) r = the rank of $\pi_1(\Phi\backslash\mathbf{H})$.

The invariants e_2, e_3, d, and r can be described purely group-theoretically. Indeed $d = (\Gamma : \Phi)$, and e_2, e_3 are the number of conjugacy classes of subgroups of order 2 and 3 respectively in Φ. Also $\Phi\backslash\mathbf{H}$ is a noncompact surface so its fundamental group is free. The invariant r is the rank of $\pi_1(\Phi\backslash\mathbf{H})$ as a free group. It is also the rank of the free factor of Φ. From the topology of surfaces we know that

$$r = 2g + t - 1. \tag{7.1.1}$$

And the Riemann-Hurwitz formula relates these invariants by

$$d = 3e_2 + 4e_3 + 6r - 6 = 3e_2 + 4e_3 + 12g + 6t - 12. \tag{7.1.2}$$

Recall that the usual definition of *the width of a cusp* is as follows. For the cusp ∞, its stabilizer is a cyclic subgroup generated by some translation $z \mapsto z + b$, for some integer b. This b is nonzero since Φ is of finite index, hence it may as well be taken to be positive. This positive b is the width of the cusp ∞. For any other cusp there is an element of Γ conjugating it to ∞. Using this conjugation its width can be defined similarly. From the point of view of hyperbolic geometry a cusp of Φ corresponds to a "spike" or a "puncture" of $\Phi\backslash\mathbf{H}$. For the sake of brevity we shall refer to these punctures as the *cusps* of $\Phi\backslash\mathbf{H}$. To say that the cusp-width of ∞ is b amounts to the fact that a small annular neighborhood of the corresponding cusp on $\Phi\backslash\mathbf{H}$, after cutting along a piece of an even line running into it, can be isometrically developed in a strip of the form $x = k$ to $x = b + k$. From this description it is seen that in a special polygon for Φ containing ∞ there are b copies of the standard fundamental domain of Γ incident with the cusp ∞, and hence there are $2b$ tiles of $\mathcal{T}^*$ incident there. So alternately the width of a cusp of Φ may be defined as *half the total number of the tiles of* $\mathcal{T}^*$ *in a special polygon* P *for* Φ *incident with the vertices of* P *which are equivalent under* Φ *and which define the given cusp*.

It may be remarked that all these invariants are invariants of the conjugacy class of Φ. Moreover it may be noted that these invariants are actually *topological* invariants. Perhaps the simplest selfcontained way to see this for the number of cusps of Φ is to observe that it is the same as the number of *ends* of $\Phi\backslash\mathbf{H}$ in the very general sense of Freudenthal, cf. [Fr]. Using surface topology it is of course the same number t which makes (7.1.1) right, or again the same number so that $\Phi\backslash\mathbf{H}$ is homeomorphic to a closed orientable surface with t points removed. A cusp has a neighborhood homeomorphic to an annulus. The width of a cusp of Φ is the same as *the local degree* of $\Gamma\backslash\mathbf{H} \to \Phi\backslash\mathbf{H}$ at the corresponding cusp—it is the number of times the image of a simple closed nonnullhomotopic curve in an annular neighborhood of the puncture in $\Phi\backslash\mathbf{H}$ wraps around the unique cusp in $\Gamma\backslash\mathbf{H}$. With this formulation it is possible to define the *end-width of an end* of any finitely generated subgroup of any finitely generated Fuchsian group.

We shall see below how to read these invariants when the subgroup is given by a Farey symbol or if its conjugacy class is given by a tree diagram.

(7.2). Let σ be a Farey symbol, P_σ the corresponding special polygon and Φ_σ the corresponding subgroup. We assume that the underlying *g.F.S.* is given by (5.1.1). Write S_σ for $\Phi_\sigma\backslash\mathbf{H}$. From the way S_σ is obtained from P_σ we see that e_2 = the number of even intervals in σ, and e_3 = the number of odd intervals in σ. Again from the way S_σ is obtained from P_σ we see that r = half the number of free intervals in σ. We note the following simple formula for the index.

Proposition. *Let σ be a Farey symbol with underlying g.F.S. $\{\infty, x_0, x_1, \ldots, x_n, \infty\}$, and Φ_σ the corresponding subgroup. If σ contains e_3 odd intervals then the index d of Φ_σ in Γ is given by $d = 3n + e_3$.*

Proof. The number of free intervals in σ is $n + 2 - e_2 - e_3$. So by (7.1.2)

$$d = 3e_2 + 4e_3 + 6\left\{\frac{1}{2}(n + 2 - e_2 - e_3) - 1\right\} = 3n + e_3. \quad q.e.d.$$

(7.3). The number of cusps of S_σ is the same as the number of x_i's obtained after the sides are identified according to the pairing in P_σ. To read this number from σ consider the equivalence relation generated by the following relation. If x_i, x_{i+1} are the endpoints of an even interval or an odd interval then $x_i \sim x_{i+1}$. If x_k, x_{k+1} are the endpoints of a free interval and $x_{k'}$, $x_{k'+1}$ are the endpoints of the associated free interval then $x_k \sim x_{k'+1}$ and $x_{k+1} \sim x_{k'}$. So t is the number of the equivalence classes of this relation. In view of (7.1.1) we can also read g.

(7.4). To read the widths of cusps from σ is a bit more tricky. Let $C_1, C_2, \ldots, C_t$ be the cusps of S_σ which we identify with the equivalence classes of $x_i's$ defined above. We consider x_i, $i = -1, 0, 1, \ldots, n + 1$ in a cyclic order and write $x_i = \frac{a_i}{b_i}$ in reduced form with the convention $\infty = \frac{1}{0}$ and $b_i's$ are always taken to be nonnegative. Let

$$d(x_i) = |a_{i-1}b_{i+1} - a_{i+1}b_{i-1}|, \qquad i = -1, 0, 1, \ldots, n + 1.$$

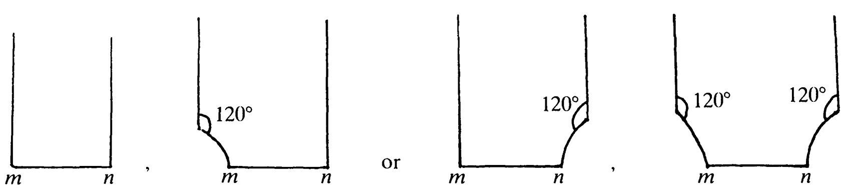
We define $w(x_i)$ *the width at* x_i to be $d(x_i)$, *resp.* $d(x_i) + \frac{1}{2}$, *resp.* $d(x_i) + 1$ according as x_i is incident to 0, resp. 1, resp. 2 odd intervals.

PROPOSITION. *The width of the k-th cusp* C_k *is given by*

$$w(C_k) = \Sigma w(x_i)$$

where x_i *runs over the elements in the equivalence class* C_k.

Proof. The width of a cusp is clearly Γ-invariant, as is the value of $d(x_i)$, cf. (2.2), so we may assume that x_i is the cusp ∞. Then necessarily $x_{i-1} = m$ and $x_{i+1} = n$ for some integers m and n. Then according as x_i is incident to 0, 1, or 2 odd intervals the associated special polygon looks like



This gives the widths $w(x_i)$ as claimed. *q.e.d.*

(7.5). As a variant to the above approach we shall now start with a tree diagram T and briefly show how to read the geometric invariants of a group belonging to the corresponding conjugacy class. Let R resp. B be the set of red resp. blue terminal vertices of T and σ the involution on R. Clearly

$$e_2 = \textit{the number of fixed points of } \sigma,$$

$$e_3 = \textit{the number of points in } B,$$

$$r = \frac{1}{2}\,[\textit{the number of elements in } R - \{\textit{the fixed points of } \sigma\}].$$

This also determines d by the Riemann-Hurwitz formula as noted in (7.2). To compute the other invariants consider T as actually embedded in the f-edges of $\mathcal{T}^*$ as explained in (4.4), and P the associated special polygon. Then the terminal vertices of T lie in ∂P. This gives a cyclic order to the terminal vertices of T—say $v_0, v_1, \ldots, v_{s-1}$ are these vertices in cyclic order. Let π_i be the shortest path of edges from v_i to v_{i+1} (i counted mod s.) Consider the equivalence relation generated by the following relation on the set $\{\pi_i\}$. If v_i is either a blue vertex or a vertex fixed by σ then $\pi_{i-1} \sim \pi_i$. If v_i is paired to v_j by σ then let $\pi_{i-1} \sim \pi_{j+1}$, and $\pi_{i+1} \sim \pi_{j-1}$. Then

$$t = \textit{the number of classes of this equivalence relation.}$$

By (7.2.1) we can also read g. We may identify a cusp with an equivalence class of $\pi_i's$. To read the cusp-widths attach the weight $\frac{1}{2}$ (resp. 1) to an edge of T if it is incident to a red vertex (resp. otherwise). To each π_i attach the weight $w(\pi_i)$ = the sum of the weights of the edges in π_i. Finally the width of a cusp is the sum of $w(\pi_i)$ where π_i runs over the equivalence class defining the cusp. The justification of this statement follows easily by relating $\pi_i's$ to the components of int $P - T$, and it is left to the reader.

(7.6). As an application we give a new simple proof of the following theorem of Millington, cf. [Mi]. This theorem was proved by Millington by using permutations.

THEOREM (Millington). *Let $a \geq 0$, $b \geq 0$, $d \geq 0$, $g \geq 0$, $t \geq 1$ be integers s.t.*

$$d = 3a + 4b + 12g + 6t - 12 \geq 1.$$

Then Γ admits a subgroup of index d, genus g, and with t cusps and a (resp. b) conjugacy classes of elliptics of order 2 (*resp.* 3).

Proof. Let $r = 2g + t - 1$. Consider a $g.F.S.$ in the form (5.1.1), where $n = a + b + 2r - 2$. There are $n + 2$ intervals in this $g.F.S.$. Declare the first a intervals to be the even intervals, and next b intervals to be the odd intervals. The next $2(t - 1)$ intervals are declared to be free intervals. They are divided into $t - 1$ pairs each pair consisting of consecutive intervals which are paired. There now remain $4g$ intervals. These are also declared to be free intervals and the pairing is defined in the usual $aba^{-1}b^{-1}$-fashion. It is obvious that the corresponding subgroup defined by this Farey symbol has the invariants as stated.

q.e.d.

8. A geometric interpretation of continued fractions (8.1). Let x be a rational number, and let

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}}} \tag{8.1.1}$$

be its continued fraction expansion. Here $a_0 = [x]$, $a_1 = [\frac{1}{x-a_0}]$ etc. We shall abbreviate (8.1.1) to $x = [a_0; a_1, a_2, \ldots, a_k]$. If x is not an integer the a_i's, $i \geq 1$, are positive, and $a_k \geq 2$. We define *the depth of* x by

$$a_1 + a_2 + \cdots + a_k, \tag{8.1.2}$$

and denote it by $\Delta(x)$. If x is an integer then $\Delta(x) = 0$. Clearly $\Delta(x)$ depends only on the congruence class of x mod 1. In the following we shall give an interpretation of the a_i's and $\Delta(x)$ in terms of the modular tessellation. It implies an interesting property of a *g.F.S.*.

(8.2). The even lines tile **H** into ideal triangles. Let this tessellation be denoted by $\mathcal{I}$.

PROPOSITION. *Let x be a rational number in* $(0, 1)$. *Then there exists a hyperbolic polygon which is a union of finitely many of tiles of* $\mathcal{I}$, *whose boundary contains x and ∞ as vertices, and which is contained in the vertical strip bounded by the geodesics joining* 0 *to* ∞ *and* 1 *to* ∞.

Proof. Let $x = \frac{a}{b}$, where b is positive. Let $\mathcal{F}_b$ be the classical b-th Farey sequence, cf. [HW], Chapter 3. The hyperbolic convex hull of the points in $\mathcal{F}_b$ together with ∞ is a polygon with the desired properties. *q.e.d.*

The vertices of any polygon given by the proposition form a *g.F.S.* of the form

$$\{\infty, 0 = x_0, x_1, \ldots, x_n = 1, \infty\} \tag{8.2.1}$$

where one of the x_i's is x. Since no two even lines intersect it is clear that the intersection of any such polygons is again a polygon of the same type and so there exists a unique polygon $P_0(x)$ which is the intersection of all such polygons. Interpreted in terms of the *g.F.S.*'s this means that *among all the g.F.S.'s of the form* (8.2.1) *containing x there is a unique minimal one.* It consists of the vertices of $P_0(x)$.

(8.3) PROPOSITION. *Let x_0 be a rational number in* $(0, 1)$. *Then there exist uniquely determined rationals u_0 and v_0 in* $[0, 1]$, $u_0 < x_0 < v_0$, *with the following properties.*

i) u_0, x_0, v_0, *form the vertices of a tile of $\mathcal{I}$. Call this tile* $\tau_0(x_0)$.

ii) *Any even line incident with x_0 has its other endpoint lying either in* $[u_0, x_0)$ *or in* $(x_0, v_0]$.

iii) *The tile $\tau_0(x_0)$ is contained in* $P_0(x_0)$.

Proof. First consider the situation at ∞. The even lines incident to ∞ are the vertical half lines $x = n$, $y > 0$, where n is an integer. The endpoints of these lines are $\mathbf{Z} \cup \{\infty\}$ which have ∞ as the unique accumulation point. Now noting that distinct even lines do not intersect and translating the situation at x_0 we see that the endpoints of the even lines incident to x_0 have x_0 as their unique accumulation point and they are all contained in $[0, 1]$. Let u_0 be the smallest and v_0 the largest of these endpoints. So the property ii) is clear.

Now let y_0 and z_0 be any endpoints of some even lines incident to x_0 such that $y_0 < x_0 < z_0$. We claim that there exists a convex hyperbolic polygon Q satisfying i) it is a union of finitely many tiles of $\mathcal{I}$, and ii) it has the even lines joining y_0 to x_0 and z_0 to x_0 as sides. Indeed let A be an element of Γ which carries x_0 to ∞. Then $Ay_0 = l$ and $Az_0 = m$ must be integers. Moreover since A preserves the orientation of the circle $\mathbf{R} \cup \{\infty\}$ we see that $m < l$. If Q' is the convex hull of $\{\infty, m, m + 1, \ldots, l, \infty\}$ in $\mathbf{H}$ then clearly $Q = A^{-1}(Q')$ fulfils our requirements. Since A is determined up to a left-multiplication by an element of the form $\phi_n : z \mapsto z + n$ where n is an integer it is clear that Q' is determined also up to a translation by ϕ_n. In any case Q is determined uniquely. Notice that among all hyperbolic polygons Q'' satisfying i) and ii) our construction produces the smallest one which is contained in all such Q''. It has an additional interesting property that every vertex of Q is an endpoint of some even line incident to x_0. Let u_0' be the smallest and v_0' the largest of the vertices of Q. Then clearly $u_0' \leq y_0 < x_0 < z_0 \leq v_0'$ and u_0', x_0, v_0', form the vertices of a tile of $\mathcal{I}$. If we apply this construction in particular to $y_0 = u_0$ and $z_0 = v_0$ we see that $u_0' = u_0$ and $v_0' = v_0$. This proves i).

Finally let P be any polygon containing x_0 as a vertex as in the proposition in (8.2). Let y_0, z_0 be the vertices of P adjacent to x_0, such that $y_0 < x_0 < z_0$ and Q the corresponding polygon as above. Then Q is contained in P. So the tile $\tau_0(x_0)$ which has vertices u_0, x_0, v_0, is also contained in P. Since P is arbitrary we see that this tile is contained in $P_0(x_0)$. This proves iii). *q.e.d.*

(8.4). The above proposition implies an interesting property of $\mathcal{I}$ which for emphasis we note explicitly. Let x be a rational number which is *not* an integer. Let k be an integer such that $k < x < k + 1$. Then *there exists a unique tile of $\mathcal{I}$ with vertices u, x, v such that $u < x < v$. Moreover we must have $k \leq u < x < v \leq k + 1$.* We note a procedure to determine u and v. Let $x = \frac{a}{c}$ in a reduced form. First suppose that $k = 0$, i.e. x lies in $(0, 1)$. We take a and c positive. Consider the linear equation $-aX + cY = 1$. This line has a positive slope $\frac{a}{c}$ and a positive X-intercept $\frac{1}{c}$. Let (d, b) be the uniquely determined point with integer coordinates on this line which lies in the first quadrant and which is closest to the X-axis. Then $v = \frac{b}{d}$ and $u = \frac{a-b}{c-d}$. In the general case apply this procedure to $x - k$. If u' and v' are the solutions for $x - k$ then $u = u' + k$, $v = v' + k$ are the solutions for x. An elementary justification of this procedure may be left to the reader. An alternate procedure in terms of the convergents of x follows from the proof of (8.6) below.

(8.5). Let $x = [0; a_1, a_2, \ldots, a_k]$ be a rational number in $(0, 1)$. Set

$$y_i = [0; a_1, a_2, \ldots, a_i], \qquad 1 \leq i \leq k$$

be the convergents of x. So $y_k = x$. It is convenient to put

$$p_{-1} = 1, \qquad q_{-1} = 0, \qquad p_0 = 0, \qquad q_0 = 1,$$

$$\text{(8.5.1)} \quad p_i = a_i p_{i-1} + p_{i-2}, \qquad q_i = a_i q_{i-1} + q_{i-2}, \qquad 1 \leq i \leq k.$$

$$y_{-1} = \infty, \qquad y_0 = 0.$$

Then as is well-known $y_i = \frac{p_i}{q_i}$ (reduced fractions), where we regard 0 as $\frac{0}{1}$ and ∞ as $\frac{1}{0}$. Moreover

(8.5.2) $$p_i q_{i-1} - q_i p_{i-1} = (-1)^{i-1},$$

(8.5.3) $$p_i q_{i-2} - q_i p_{i-2} = (-1)^i a_i,$$

and

(8.5.4) $$0 < y_2 < y_4 < \cdots < x < \cdots < y_3 < y_1.$$

(8.6) PROPOSITION. *In the notation of* (8.5) *all* y_i*'s are among the vertices of* $P_0(x)$.

Proof. Write $x = \frac{a}{b}$, and let $u_0 = \frac{q}{s}$, and $v_0 = \frac{p}{r}$ be the rationals in $(0, 1)$ satisfying $u_0 < x < v_0$ and other properties listed in (8.3) where x_0 in (8.3) is now replaced by x. We also take b, s, r to be positive. Now from (8.4) we also have $x = y_k = \frac{p_k}{q_k}$ and y_k, y_{k-1} are end-vertices of an even line. So by property ii) in Proposition (8.3) y_{k-1} lies in $[u_0, x)$ or in $(x, v_0]$. For definiteness suppose that y_{k-1} lies in $(x, v_0]$. We claim that $y_{k-1} = v_0$. Indeed we have

$$|x - v_0| = \frac{1}{br} \geq |x - y_{k-1}| = \frac{1}{bq_{k-1}}.$$

So $r \leq q_{k-1}$. By (8.5.1) $q_{k-1} < q_k$. On the other hand both $\{p, r\}$ and $\{p_{k-1}, q_{k-1}\}$ are solutions in $\{\xi, \eta\}$ of $b\xi - a\eta = 1$. Since both r, *and* q_{k-1} lie in $(0, q_k)$ it is easy to see that they must be equal. Thus $v_0 = y_{k-1}$, and so y_{k-1} is a vertex of $P_0(x)$. Now it follows that $P_0(y_{k-1}) \subset P_0(x)$. So y_{k-2} is a vertex of $P_0(x)$. Continuing in this manner we see that all y_i's, $i \geq 1$, are vertices of $P_0(x)$. On the other hand y_0 and y_{-1} are clearly among the vertices of $P_0(x)$. *q.e.d.*

(8.7) PROPOSITION. *Between* y_{i-2} *and* y_i, $i \geq 1$, *there are* $a_i - 1$ *vertices of* $P_0(x)$.

Proof. By (8.5.2) y_{i-1} is joined to both y_i and y_{i-2} by even lines, say l_1 and l_2. If m is the number of tiles of $\mathcal{I}$ which are incident with y_{i-1} and lie in the circular sector made up by l_1 and l_2 then there are $m - 1$ vertices of $P_0(x)$ lying between y_{i-2} and y_i. By the argument in (7.4) there are $2|p_i q_{i-2} - q_i p_{i-2}|$ tiles of $\mathcal{T}^*$ which are coming into this circular sector. Since at each vertex of a tile of $\mathcal{I}$ there are two tiles of

$\mathcal{T}^*$ it follows that there are $|p_iq_{i-2} - q_ip_{i-2}|$ tiles of $\mathcal{I}$ in this sector which are incident with y_{i-1}. By (8.5.3) this number equals a_i. *q.e.d.*

(8.8) COROLLARY. *The number of vertices of $P_0(x)$ is precisely $\Delta(x) + 2$, and so the number of tiles of $\mathcal{I}$ in $P_0(x)$ is $\Delta(x)$.*

Proof. The y_i's are $k + 2$ vertices of $P_0(x)$, and in view of (8.4.4) in cyclic order they are

$$\infty, y_0 = 0, y_2, y_4, \ldots, y_k = x, \ldots, y_3, y_1 = \infty.$$

These form $k + 2$ intervals of which k have end-points of the form $\{y_i, y_{i-2}\}$. By (8.7) these k intervals contain $\Sigma_{i=1}^{k} (a_i - 1)$ vertices. The remaining two intervals are $\{\infty, 0\}$, and $\{y_k, y_{k-1}\}$. These end-points of each of these two intervals are also the end-points of an even line, so these intervals contain no other vertex of $P_0(x)$. So in all $P_0(x)$ has

$$k + 2 + \sum_{i=1}^{k} (a_i - 1) = \Delta(x) + 2$$

vertices. The last assertion follows easily by induction on the number of vertices. *q.e.d.*

(8.9) THEOREM. *Let x be any rational number. Let $x = [a_0; a_1, a_2, \ldots, a_k]$ be its continued fraction expansion and let $y_i = [0; a_1, a_2, \ldots, a_i]$, $1 \leq i \leq k$ be the convergents of $x - a_0$. Also set $y_{-1} = \infty$, $y_0 = 0$. Then any g.F.S. containing x must also contain the following $\Delta(x) + 2$ terms. We use the notation of* (8.3).

$$\infty, a_0 + y_i, \qquad i = 0, 1, 2, \ldots, k$$

and between $a_0 + y_i$ and $a_0 + y_{i-2}$, $i = 1, 2, \ldots, k$ there are $a_i - 1$ terms

$$a_0 + \frac{(a_i - 1)p_{i-1} + p_{i-2}}{(a_i - 1)q_{i-1} + q_{i-2}}, a_0 + \frac{(a_i - 2)p_{i-1} + p_{i-2}}{(a_i - 2)q_{i-1} + q_{i-2}}, \ldots,$$

$$a_0 + \frac{2p_{i-1} + p_{i-2}}{2q_{i-1} + q_{i-2}}, a_0 + \frac{p_{i-1} + p_{i-2}}{q_{i-1} + q_{i-2}}.$$

Proof. By the definition of a *g.F.S.*, cf. (5.1), its first and last term on the real axis are integers. Since the vertical geodesics at the integer points are parts of $\mathcal{I}$ it follows that no interval in a *g.F.S.* can contain an integer in its interior. So a *g.F.S.* must contain all the integers lying between its first and last term on the real axis. In particular if a *g.F.S.* $\mathcal{F}$ contains x then it must contain a_0 and $a_0 + 1$. For the remaining assertions by translating by $-a_0$ for simplicity we may assume that $a_0 = 0$. In the notation of (8.2) it is clear that $\mathcal{F}$ must contain all the vertices of $P_0(x)$. These are $\Delta(x)$ in number by (8.8). These include y_i's and by (8.7) between y_i *and* y_{i-2} there are $a_i - 1$ terms. Since the consecutive terms among the $a_i - 1$ terms written above together with y_i *and* y_{i-2} satisfy the $|ad - bc| = 1$-relation it is clear that these are precisely the terms one was looking for. *q.e.d.*

(8.10). The following interpretation of $\Delta(x)$ explains the terminology why it is called the *depth* of x. The f-edges determine a cubic tree which may be considered as the dual complex of the tiling $\mathcal{I}$. Let each f-edge be assigned the length $\frac{1}{2}$.

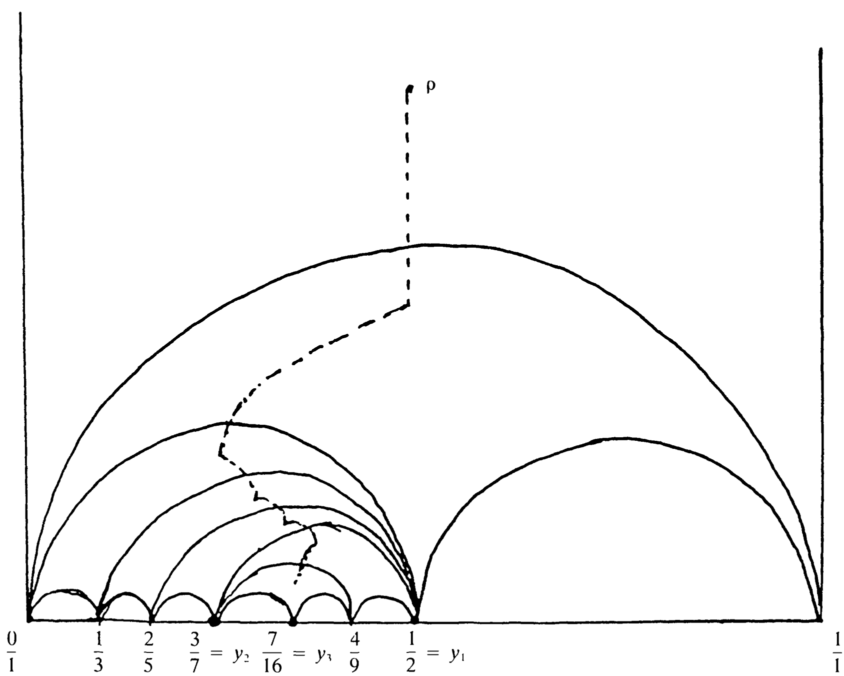
PROPOSITION. *Let x be rational number in* $(0, 1)$. *The shortest path in the cubic tree of f-edges from $\rho = e^{\pi i/3}$ leading into a tile of $\mathcal{I}$ incident with x has length* $\Delta(x) - 1$.

Proof. We use the notation of (8.5). From the description in (8.8) we see that $P_0(x)$ is built as follows. We have $y_1 = \frac{1}{a_1}$. The convex hull of $0, \frac{1}{a_1}, \frac{1}{a_1 - 1}, \frac{1}{a_1 - 2}, \ldots, \frac{1}{2}, 1, \infty$ consists of a_1 tiles of $\mathcal{I}$ and is contained in $P_0(x)$. To this region there is attached along the even line connecting y_0 to y_1 the convex hull of y_0, y_1, y_2 and the a_2 vertices lying in (y_0, y_2). This region contains a_2 tiles of $\mathcal{I}$. And so on. It is easy to see that the path in the tree of f-edges starting from ρ and leading into a tile of $\mathcal{I}$ incident with x is the one which successively connects the "barycenters" of these $a_1 + a_2 + \cdots + a_k$ tiles and has length

$$\sum_{i=1}^{k} a_i - 1 = \Delta(x) - 1.$$

This path is the shortest one, for in "walking down" in a tree of f-edges no edge is traversed backwards, so by taking one "wrong" turn in the above path one will miss x altogether. *q.e.d.*

(8.11). The following example and the picture below fairly illustrate the above process. Let $x = \frac{7}{16} = [0; 2, 3, 2]$. A *g.F.S.* containing $\frac{7}{16}$ must contain the following $\Delta(x) + 2 = 9$ terms. These include ∞, 0, and the convergents $\frac{1}{2}$, $\frac{3}{7}$, and $\frac{7}{16}$. The interval $(\frac{7}{16}, \frac{1}{2})$ contains one extra vertex $\frac{4}{9}$. The interval $(0, \frac{3}{7})$ contains two extra vertices $\frac{2}{5}$ and $\frac{1}{3}$, and finally the interval $(\frac{1}{2}, \infty)$ contains one extra vertex namely 1. For the sake of clear visualization the following picture illustrates only the combinatorial pattern and is not drawn to the scale. The path mentioned in (8.10) is shown by a dotted line.



(8.12). We note some initial instances of the use of (8.10). Take the *g.F.S.* in the form (8.2.1). Then x_1 must be of the form $\frac{1}{k}$ for some natural number k. Now (8.2) says that if we choose $x_1 = \frac{1}{k}$ then we are forced to choose between $\frac{1}{k}$ and ∞ the $k - 1$ terms $\frac{1}{k-1}, \frac{1}{k-2}, \ldots, 1$. Similarly x_{n-1} must be of the form $\frac{l-1}{l}$ for some natural number l. By (8.2) if we choose $x_{n-1} = \frac{l-1}{l} = [0; 1, l - 1]$ then we have $y_1 = 1$ and $y_2 = \frac{l-1}{l}$, and we are forced to choose between $y_2 = \frac{l-1}{l}$ and $y_0 = 0$ the following $l - 2$ terms $\frac{l-2}{l-1}, \frac{l-3}{l-2}, \ldots, \frac{1}{2}$.

9. A classification of conjugations (9.1). The normalizers of subgroups of Γ in Γ^* or some even larger group often contains conjugations. For example $\Gamma(N)$, $\Gamma^1(N)$, $\Gamma_0(N)$ are all normalized by $z \mapsto -\bar{z}$, and $z \mapsto 1 - \bar{z}$. Such conjugations often provide some quite useful information. For example we shall make use of them to reduce the work by half in obtaining the fundamental domains for the above-mentioned congruence groups. In this section we note some generalities on conjugations which may be useful in other contexts as well. Recall that a *conjugation* on a Riemann surface is an antiholomorphic homeomorphism. If it has a fixed point then each component of the fixed point set is homeomorphic to a circle or an open interval. On a compact Riemann surface only the first possibility occurs. A conjugation with a fixed point will be called a *reflection*, whereas one without fixed points will be called a *glide reflection*.

(9.2). A conjugation on $\mathbf{H}$ is an isometry with respect to the hyperbolic metric. The full group of isometries of $\mathbf{H}$, denoted by $I(\mathbf{H})$, may be identified with $PSL_2^*(\mathbf{R})$, which is defined analogously to $PSL_2^*(\mathbf{Z})$ in (2.1) and whose action is given by (1.1.1) and (2.1.1). Its identity component $I_0(\mathbf{H})$ consists of the orientation-preserving isometries and may be identified with $PSL_2(\mathbf{R})$; the other component consists of orientation-reversing isometries and these are precisely the conjugations in $I(\mathbf{H})$. By abuse of notation we shall often again denote an element of $I(\mathbf{H})$ by a matrix A with the understanding that A and $-A$ define the same element.

(9.3). It would be natural to call a conjugation in $I(\mathbf{H})$ to be *hyperbolic*, *resp. elliptic*, *resp. parabolic* if its square, which belongs to $I_0(\mathbf{H})$, is such in the usual sense. However we note at once that with this definition

PROPOSITION. *A conjugation in $I(\mathbf{H})$ is either hyperbolic or else it is elliptic of order 2. A hyperbolic conjugation is a glide reflection, whereas an elliptic conjugation is a reflection. In $I(\mathbf{H})$ there is a unique conjugacy class of reflections whose representative may be taken as*

(9.3.1) $$z \mapsto -\bar{z}.$$

An elliptic conjugation is algebraically characterized as follows: a matrix A is an elliptic conjugation if and only if trace $A = 0$, *and* det $A = -1$.

Proof. If A is a conjugation then its determinant is -1. So its eigenvalues are real and distinct. In particular it is diagonalizable. So A^2 is also diagonalizable and so it cannot be parabolic. If A^2 is hyperbolic then it has no fixed point in $\mathbf{H}$. So A also has no fixed point in $\mathbf{H}$, and hence it is a glide reflection. If A^2 is elliptic then the eigenvalues of A are necessarily 1 and -1. So it is conjugate to

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which acts as in (9.3.1). In particular it is of order 2, and is clearly a reflection. The last assertion is clear. *q.e.d.*

(9.4). The following is a simple but basic and essentially topological fact. The author does not know a reference to it in the literature.

PROPOSITION. *Let X be a connected Riemann surface, $G = \langle\rho\rangle \approx Z_2$ a group generated by a reflection ρ on X, and $Y = G\backslash X$. Let A denote the fixed point set of G. Then Y is a surface with boundary which is orientable if and only if A separates X. In case A separates X then X is obtained from Y by "doubling" along the boundary. If A does not separate X then X is obtained from Y by taking the oriented double cover Y_1 of Y and identifying the boundary components of Y_1 in pairs.*

Proof. Let $p : X \to Y$ be the orbit-space projection. Notice that G acts freely on $X - A$. Cut X along A. If A separates X let C be one of the components of X cut along A. Clearly $C \cup \rho C$ is a connected surface without boundary. Since X is connected it follows that $X = C \cup \rho C$. In particular $X - A$ has two components. It is clear that X is obtained by doubling C along the boundary, and C is mapped homeomorphically by p onto Y. So Y is orientable and X may be considered as obtained from Y by doubling along the boundary. Now suppose that A does not separate X. Let Y_1 be X cut along A. Then $Y_1 - \partial Y_1$ may be identified with $X - A$. Moreover there is a natural projection $q : Y_1 \to Y$ so that $q|_{X-A} = p|_{X-A}$. Since A does not separate X it follows that $G\backslash\{X - A\} \approx \operatorname{int} Y$ is a connected nonorientable surface. So Y is a nonorientable surface with boundary. By construction each point on ∂Y_1 has a neighborhood which is mapped homeomorphically onto a neighborhood of a point in ∂Y by q, and for each point x in ∂Y there are exactly two points in ∂Y_1 mapped onto x by q. So each point in $Y =$

im q admits an evenly covered neighborhood. So q is a covering projection of degree 2 and we may consider the group of covering transformations as an extension of $G|_{X-A}$ from $X - A = Y_1 - (\partial Y_1)$ to Y_1. Now notice that a point on the boundary of any smooth surface has a well-defined "internal normal." So a tubular neighborhood of any component of the boundary of any smooth surface is always orientable. Since q is a covering from an orientable surface to a nonorientable one it follows that the inverse image under q of any component of the boundary of Y consists of two components each of which is mapped homeomorphically by q. It follows that the covering group of q permutes the boundary components of Y_1 in pairs. In other words we may consider X as obtained from Y by taking the oriented double cover Y_1 of Y and identifying the boundary components of Y_1 in pairs. *q.e.d.*

(9.5). The following is a similar general fact.

PROPOSITION. *Let X be $\Phi\backslash\mathbf{H}$ where Φ is a discrete subgroup of $I_0(\mathbf{H})$ with cofinite area. Let $\overline{X}$ be the Riemann surface obtained by adjoining its cusps. Let σ be a conjugation of X. Then σ extends to $\overline{X}$ as a conjugation.*

Proof. We may consider $\overline{X}$ as the end-compactification of X in the sense of Freudenthal, cf. [Fr]. So σ, as any homeomorphism of X, extends to $\overline{X}$. By a standard argument of the Riemann removable singularity theorem the extension is antiholomorphic at the finitely many adjoined points. *q.e.d.*

(9.6). After these generalities on conjugations we come to our case of main interest, namely the extended modular group Γ^*. Clearly Γ^* may be identified with a subgroup of $PSL_2^*(\mathbf{R})$, cf. (9.2). The single conjugacy class of reflections in $PSL_2^*(\mathbf{R})$ splits into two classes when restricted to Γ^*.

THEOREM. *There are two conjugacy classes of reflections in Γ^* whose representatives may be taken to be*

i) $z \mapsto -\bar{z}$, *and*
ii) $z \mapsto 1 - \bar{z}$.

In terms of matrices: if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a reflection in Γ^*, *then* A *is of type* i) (*resp.* ii)) *if and only if* A *is* (*resp. is not*) *congruent to the identity matrix mod* 2.

Proof. Let

$$J_u = \begin{pmatrix} -1 & u \\ 0 & 1 \end{pmatrix}, \qquad u = 0, 1.$$

So J_0 represents the conjugation of type i), and J_1 represents the conjugation of type ii). We have a canonical homomorphism $PSL_2^*(\mathbf{Z}) \to PGL_2(\mathbf{Z}_2)$ given by reduction mod 2. J_0 is in the kernel of this homomorphism and J_1 is not, so these elements are not conjugate in Γ^*.

Now let A as given above be any reflection in Γ^*. In the proof of (9.3) we observed that A has eigenvalues 1 and -1. Let (p, r) be a primitive integer vector which is an eigenvector for the eigenvalue -1, and let (q, s) be an integer vector with $ps - qr = 1$. It is easy to see in terms of a basis of Z^2 consisting of these integer vectors, i.e. taking a suitable Γ-conjugate A takes the form J_u for some integer u. But then

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} J_u \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = J_{u+2k}.$$

So A is conjugate to J_0 or J_1 as claimed. *q.e.d.*

(9.7). A reflection in Γ^* will be called *even* if it is conjugate to $z \mapsto -\bar{z}$ and *odd* if it is conjugate to $z \mapsto 1 - \bar{z}$.

10. Conjugations in a subgroup (10.1). Consider a subgroup Φ^* of Γ^*. Then Φ^* uniquely determines $\Phi = \Phi^* \cap \Gamma$. If Φ^* contains a conjugation then Φ is a subgroup of index 2 in Φ^*. There is a canonical conjugation θ defined on the surface $S_\Phi = \Phi\backslash\mathbf{H}$. It is defined as a "push-down" of any conjugation in Φ^*. Of course

$$\{\Phi\backslash\mathbf{H}\}/\langle\theta\rangle = S_\Phi/\langle\theta\rangle \approx \Phi^*\backslash\mathbf{H}.$$

We denote this surface by S_{Φ^*}. If Φ^* does not contain a reflection then S_{Φ^*} is a nonorientable surface (without boundary).

(10.2). For the remaining parts of this section we assume that Φ^* contains a reflection. Our aim is to obtain some geometric understanding of the conjugacy classes of reflections in Φ^*. By (9.4) S_{Φ^*} is a surface with boundary which is orientable if and only if the fixed point set separates S_{Φ^*}. Now a fixed point set of an even (resp. odd) reflection in Φ^* is an even line (resp. an odd) line and it projects either isometrically or at worst "folded in half" in ∂S_{Φ^*}. In the latter case there will be a "corner" in ∂S_{Φ^*}. In any case *it follows that no component of* ∂S_{Φ^*} *is a circle.* If a boundary component is an isometric projection of an even (resp. odd) line it will be called *an even* (*resp. odd*) *component* of ∂S_{Φ^*}.

(10.3). Now consider the possibilities of corners in ∂S_{Φ^*}. Obviously ∂S_{Φ^*} contains a corner precisely when the fixed lines of two reflections in Φ^* intersect. Since no two even lines intersect the only possibilities for the fixed lines of reflections in Φ^* are: one is an even line and the other an odd line, or both are odd lines. Thus there are two types.

Type 1. Suppose Φ^* contains an even reflection σ and an odd reflection ρ whose fixed lines intersect. Let s resp. r be the fixed lines of σ and ρ which intersect in a point t. Then Φ contains the element $\tau = \sigma \circ \rho$ of order 2 which fixes t. Also r and s cut orthogonally at t, so $s \cup r$ projects into a component of ∂S_{Φ^*} which has a corner with angle $\frac{\pi}{2}$ at the projection of t. Notice that σ and ρ are not conjugate even in Γ^* since one is even and the other is odd.

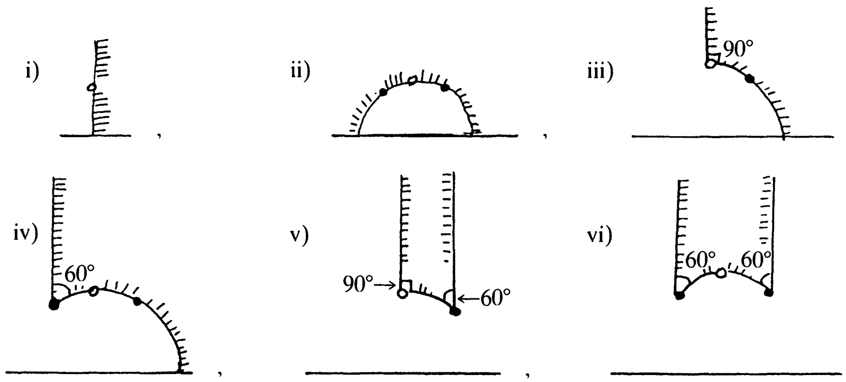
Type 2. Let Φ^* contain two odd reflections ρ_1 and ρ_2 whose fixed lines r_1 resp. r_2 intersect at a point t. Then Φ contains another odd reflection $\rho_3 = \rho_1 \circ \rho_2 \circ \rho_1$ whose fixed line r_3 also passes through t. The lines r_1, r_2, r_3 cut $\mathbf{H}$ into wedges each making an angle $\frac{\pi}{3}$ at t. This picture (locally) projects into a component of ∂S_{Φ^*} with a corner with angle $\frac{\pi}{3}$ at the projection of t. Notice that ρ_1, ρ_2, and ρ_3 are all conjugate to each other in Φ^* as indeed they are already conjugate in the isotropy subgroup of Φ^* at t.

(10.4). It is reasonable to ask: when does a component of ∂S_{Φ^*} contain two corners? The above discussion shows that if C is such a component then either C has two corners with angles $\frac{\pi}{2}$ and $\frac{\pi}{3}$ and thus consists of one even edge, one f-edge, and one odd edge, or else it has two corners each with angle $\frac{\pi}{3}$ and it consists of two f-edges and two odd edges. It is clear that in the first case the induced $\mathcal{T}^*$-tessellation of S_{Φ^*} contains only one tile and so S_{Φ^*} is isometric to a hyperbolic triangle with angles 0, $\frac{\pi}{2}$, and $\frac{\pi}{3}$ and in the second case the tessellation contains two tiles and so S_{Φ^*} is isometric to a hyperbolic triangle with angles 0, $\frac{\pi}{3}$, and $\frac{\pi}{3}$. It is easy to see that the first case occurs exactly for Γ^*, and the second case occurs exactly for a unique subgroup of index 2 in Γ^* which is generated by reflections in the edges of the hyperbolic triangle with vertices $e^{i\pi/3}$, $e^{i2\pi/3}$ and ∞. Let us denote the second group by Φ_0^*. It may be identified with $\langle \Phi_0, z \mapsto -\bar{z} \rangle$ where Φ_0 is the unique subgroup of index 2 in Γ.

(10.5) THEOREM. *Let Φ^* be a subgroup of Γ^* containing a reflection. Then S_Φ^* is a (possibly nonorientable) surface with nonempty boundary. Each component of ∂S_{Φ^*} is either an even line, or an odd line, or else it contains corners with angles $\frac{\pi}{2}$ or $\frac{\pi}{3}$. If $\Phi^* = \Gamma^*$ (resp. $= \Phi_0^*$) then ∂S_{Φ^*} has one component with two corners with angles $\frac{\pi}{2}$, and $\frac{\pi}{3}$ (resp. with angles $\frac{\pi}{3}$, and $\frac{\pi}{3}$). In all other cases a component of ∂S_{Φ^*} can contain at most one corner. In any case components of ∂S_{Φ^*} classify the conjugacy classes of reflections in Φ^* with the understanding that a component without a right-angled corner corresponds to a unique conjugacy class of reflections whereas one with a right-angled corner corresponds to two conjugacy classes in Φ^*.*

Proof. Everything except the last sentence is already noted above. The fixed lines of reflections in a conjugacy class clearly project into the same component of ∂S_{Φ^*}, and conversely the inverse image of a component of ∂S_{Φ^*} in $\mathbf{H}$ is a union of certain fixed lines of reflections in Φ^*. If a component does not contain any corner then its inverse image is a union of either all even lines or all nonintersecting odd lines, and it clearly corresponds to a conjugacy class of even resp. odd reflections in Φ^*. Suppose it contains two corners. Then as noted in (10.4) Φ^* is either Γ^* or Φ_0^*, and the assertion is easily checked in these cases. Otherwise the component contains exactly one corner with angle $\frac{\pi}{2}$ or exactly one corner with angle $\frac{\pi}{3}$. In the first case (resp. the second case) a component of the inverse image is a union of a pair of intersecting lines one even and the other odd (resp. three intersecting odd lines). From the discussion in (10.3) it follows that in the first case there are two conjugacy classes of reflections in Φ^* whereas in the second case there is only one such conjugacy class. *q.e.d.*

(10.6) *Remark.* One consequence of the above discussion is that there are precisely the following six possibilities for the geometric shapes of a neighborhood of a component of ∂S_{Φ^*}. (Here v) resp. vi) occurs only for Γ^* resp. Φ_0^*.)



11. Special polygons admitting a reflection (11.1). Let Φ^* be subgroup of finite index in Γ^* containing a reflection, and $\Phi = \Phi^* \cap \Gamma$. We use the notations used in (10.1). We know that S_Φ admits a canonical reflection θ. Also Φ admits a special polygon as an admissible fundamental domain. The question we wish to study in this section is: when is it possible to lift θ to a reflection symmetry of a suitable special polygon which is an admissible fundamental domain for Φ? It will turn out for example that $\Gamma_0(N)$ admits such a special polygon when N is a prime, but not necessarily so when N is not a prime, cf. (13.5).

(11.2). First let us formulate the question more precisely. Let P be a special polygon. We say that P *admits a reflection* if there is a reflection σ in Γ^* which leaves P invariant such that if e and f are two sides of P which are paired then so are $\sigma(e)$ and $\sigma(f)$. In this case if S is the surface obtained from P by glueing the sides by the side-pairing transformations then obviously σ induces a reflection on S. Now with Φ^* and Φ as in (11.1) we say that it is possible *to lift* θ to a reflection on a suitable special polygon P_Φ which is an admissible fundamental domain for Φ if P_Φ admits a reflection σ such that the induced reflection on S_Φ is θ.

(11.3) THEOREM. *Let Φ^* and Φ be as in* (11.1). *It is possible to lift θ to a reflection on a special polygon for Φ iff exactly one of the following possibilities occurs.* i) $\Phi^* = \Gamma^*$. ii) $\Phi^* = \Phi_0^*$, *cf.* (10.4). iii) ∂S_{Φ^*} *contains at most one odd component and no component with corners.* iv) ∂S_{Φ^*} *contains at most one component with a right-angled corner and no other component which is either odd or which contains corners.* v) ∂S_{Φ^*} *contains* *at most one component with a corner with angle $\frac{\pi}{3}$ and no other component which is either odd or which contains corners.*

Proof. If $\Phi^* = \Gamma^*$ consider the special polygon for Γ given by the triangle with vertices 0, $e^{i\pi/3}$, and ∞ with the reflection $z \mapsto \frac{1}{\bar{z}}$. If $\Phi^* = \Phi_0^*$ consider the special polygon for Φ_0 given by the quadrilateral with vertices 0, $e^{i\pi/3}$, ∞, and $e^{2i\pi/3}$ with the same reflection. Now let Φ^* be any other group different from Γ^* and Φ_0^*, and suppose Φ admits a special polygon P with a reflection σ which lifts θ. Let s be the fixed line of σ. Then s divides P into two mutually isometric parts and it projects into a component of ∂S_{Φ^*}. Let e and f be two sides of P which are paired and $\sigma(e) = f$ then since the pairing transformations are orientation-reversing on sides of P we see that e and f also project into a component of ∂S_{Φ^*}. (*Note*: In the following we use the expression *sides of P* in the technical sense of (2.4).) It is clear that ∂S_{Φ^*} is a union of the projections of such pairs e, f together with the projection of s. Note the three cases. a) If e and f are free sides, then the corresponding component of ∂S_{Φ^*} is necessarily even. b) If e and f are even edges then they project into a component of ∂S_{Φ^*} with a single corner with angle $\frac{\pi}{2}$. c) If e and f are odd edges then they project into a component of ∂S_{Φ^*} with a single corner with angle $\frac{\pi}{3}$.

Now if no component of ∂S_{Φ^*} contains a corner then b) and c) cannot occur. If moreover σ is even then all components of ∂S_{Φ^*} are even. If on the other hand σ is odd then s projects onto an odd component of ∂S_{Φ^*} and the other components are even. This is the possibility iii) in the theorem. Now suppose b) occurs for some pair of even edges e and f. Then $e \cup f$ is an even line, say r. Necessarily s intersects r and so s must be an odd line. Moreover s cannot intersect any other side of P since an odd line intersects precisely one even line. So if e_1, f_1 is another pair of sides of P which are paired and $\sigma(e_1) = f_1$ then e_1, f_1 must be free sides and they would contribute to an even component of ∂S_{Φ^*}. This leads to the possibility iv) in the theorem. Similarly the possibility v) arises from case c). This proves the "only if" part of the theorem.

As for the "if" part, omitting the simple cases i) and ii), we may suppose that we are in one of the three cases iii) to v) listed in the theorem. In particular by (10.5) every component of ∂S_{Φ^*} contains at most one corner and there is at most one such component. As in the proof of (3.3) let $\mathscr{E}_f^*$ denote the union of f-edges in S_Φ^*. Notice that there is at least one even vertex and none, or one, or exactly two f-edges on ∂S_{Φ^*} situated as described in (10.6). We choose a maximal tree T^* in $\mathscr{E}_f^*$ which contains these f-edges. Following the argument of (3.3) we can cut S_Φ^* open into a space which is isometric to a connected simply connected convex hyperbolic polygon P_1^* with the property that at most one component of ∂P_1^* is odd or contains a corner with angle $\frac{\pi}{2}$ resp. $\frac{\pi}{3}$ and one resp. two f-edges. (The condition that T^* contains the f-edges in ∂S_{Φ^*} ensures the last property.) Moreover except for the components of ∂P_1^* which correspond to the components of ∂S_{Φ^*} the other components may be subdivided suitably into "sides" and there is a natural pairing on these sides obeying the rules for the sidepairing of a special polygon.

Clearly the theorem is true for Φ^* iff it is true for any conjugate of Φ^* so we may argue for a suitable Γ-translate of P_1^* which allows to make special choices for its sides. (This is not strictly necessary but it simplifies the argument.) There are four cases corresponding to the first four cases of (10.6).

Case 1: *All components of* ∂P_1^* *are even.* Choose one, say C, which corresponds to a component of ∂S_{Φ^*}. We may assume that C is the even line joining 0 to ∞. Let P_2^* be the reflection of P_1^* across C, and let $P = P_1^* \cup P_2^*$. The partial sidepairing defined on P_1^* gives one on P_2^* and hence one on P, which can be extended to a total sidepairing by pairing each pair of components which corresponds to the same component of ∂S_{Φ^*}.

Case 2: ∂P_1^* *has an odd component* C. We may assume that C is the odd line joining ∞ to $\frac{1}{2}$. Then proceed as above.

Case 3: ∂P_1^* *has a component* C *containing a corner with angle* $\frac{\pi}{2}$. We may assume that C is a union of two segments C' and C'' where C' is a part of the odd line joining ∞ to $\frac{1}{2}$ and C'' is a part of the even line which is a semicircle joining 0 to 1. Let ∂P_2^* be the reflection of ∂P_1^* across C' and set $P = P_1^* \cup P_2^*$. Now proceed as above, noting further that P has this even line on the boundary, and in the side-pairing we pair the two even edges on this line.

Case 4: ∂P_1^* *has a component* C *containing a corner with angle* $\frac{\pi}{3}$. We may assume that C is a union of two segments C' and C'' where C' is a part of the odd line joining ∞ to $\frac{1}{2}$ and C'' is a part of the odd line which is a semicircle joining $\frac{1}{3}$ to 1. Proceed as in Case 3, noting that at the end we have two odd edges as part of ∂P forming an angle $\frac{2\pi}{3}$ which will be paired. *q.e.d.*

12. Balanced Farey symbols (12.1). Consider a *g.F.S.* of the form

$$\{\infty, 0 = x_0, x_1, \ldots, x_n = 1, \infty\}. \tag{12.1.1}$$

Let $x_i = \frac{a_i}{b_i}$ (reduced form) such that $a_1 = 0$, $b_1 = 1$, the rest of the a_i, b_i are positive, the leftmost ∞ is taken to be $\frac{1}{0}$, and the rightmost ∞ is taken to be $\frac{-1}{0}$. Let N be a natural number. We say that the *g.F.S.* (12.1.1) is *balanced for N* if for each $i = 0, 1, \ldots, n - 1$, there exists an $i^* = 0, 1, \ldots, n - 1$, such that either

$$b_i \equiv b_{i^*+1}(N), \qquad \text{and} \quad b_{i+1} \equiv -b_{i^*}(N), \tag{12.1.2}$$

or

$$b_i \equiv -b_{i^*+1}(N), \qquad \text{and} \quad b_{i-1} \equiv b_{i^*}(N). \tag{12.1.3}$$

Here n is even and (i, i^*) are $\frac{n}{2}$ disjoint pairs containing all vertices except $x_n = 1$. We equip the structure of a Farey symbol on the *g.F.S.* given by (12.1.1) by pairing the intervals $\{\infty, 0\}$ with $\{1, \infty\}$ and $\{x_i, x_{i+1}\}$, $0 \le i \le n - 1$ with the corresponding $\{x_{i^*}, x_{i^*+1}\}$. We shall also call this Farey symbol *balanced for N*. The significance of this notion is the following result.

(12.2) THEOREM. *Let $N \ge 4$ and*

$$n = \frac{N^2}{6} \prod_{p|N} \left(1 - \frac{1}{p^2}\right). \tag{12.2.1}$$

Then there exists a g.F.S. in the form (12.1.1) *which is balanced for N and the corresponding Farey symbol is a Farey symbol for* $\Gamma^1(N)$. *Moreover consider the g.F.S.*

$$y_{ij} = x_i + j, \qquad 0 \le i \le n, \qquad 0 \le j \le N - 1. \tag{12.2.2}$$

Then this g.F.S. can be equipped with the structure of a Farey symbol for $\Gamma(N)$.

The proof extends over (12.3)–(12.7). First we prove three lemmas on $\Gamma^1(N)$.

(12.3) LEMMA. *If $N \geq 4$ then $\Gamma^1(N)$ is torsionfree, and any special polygon for $\Gamma^1(N)$ has only free sides.*

Proof. The only elements of finite order in $\Gamma - \{e\}$ are of order 2 and 3, and they are characterized by the fact $|\text{trace}| = 0$ and 1 respectively. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\Gamma^1(N)$ then we have $a = 1 + a'N$ and $d = 1 + d'N$ for suitable integers a', d'. So

$$|\text{trace } A| = |a + d| = |2 + (a' + d')N|.$$

This equals 0 or 1 only if $N \leq 3$. The claim about a special polygon follows from (3.2). *q.e.d.*

(*Note*: $\Gamma^1(N)$ is not torsionfree for $N = 2$, or 3. Indeed for these values

$$\begin{pmatrix} 1 & -1 \\ N & 1 - N \end{pmatrix}^N = e.)$$

(12.4) LEMMA. *There is a special polygon P for $\Gamma^1(N)$ such that ∂P contains the sides $x = 0$, $y > 0$ and $x = 1$, $y > 0$ paired by the transformation $z \mapsto z + 1$.*

Proof. Since the stabilizer of the cusp ∞ in $\Gamma^1(N)$ is generated by $z \mapsto z + 1$ we see that ∞ has cuspwidth 1. So there is a unique even edge running into its image cusp on $\Gamma^1(N)\backslash\mathbf{H}$. While constructing a special polygon for $\Gamma^1(N)$, cf. the proof of Theorem (3.3), one has to cut along this even edge (and possibly also along an adjoining even edge if there exists one). In other words a special polygon for $\Gamma^1(N)$ containing ∞ necessarily has two sides of the form $x = k$, $y > 0$ and $x = k + 1$, $y > 0$ where k is an integer, and these two sides are paired by the transformation $z \mapsto z + 1$. By a further translation $z \mapsto z - k$, which is contained in $\Gamma^1(N)$, we can ensure the claim. *q.e.d.*

(12.5) LEMMA. *Let $N \geq 4$ and Φ a subgroup of finite index in Γ. Then Φ is a subgroup of $\Gamma^1(N)$ if and only if a Farey symbol for Φ satisfies* (12.1.2), (12.1.3). (*Clearly if one Farey symbol satisfies* (12.1.2), (12.1.3) *then all do.*)

Proof. First suppose that Φ is a subgroup of $\Gamma^1(N)$. Since $N \geq 4$, by (12.3) Φ is torsionfree. So all the intervals in its any Farey symbol are free. If a free interval with end-points $x_i = \frac{a_i}{b_i}$, $x_{i+1} = \frac{a_{i+1}}{b_{i+1}}$ is paired with $x_{i^*} = \frac{a_{i^*}}{b_{i^*}}$, $x_{i^*+1} = \frac{a_{i^*+1}}{b_{i^*+1}}$ then the condition that the pairing transformation belongs to $\Gamma^1(N)$ amounts to

$$\begin{pmatrix} a_i & -a_{i+1} \\ b_i & -b_{i+1} \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{i^*+1} & a_{i^*} \\ b_{i^*+1} & b_{i^*} \end{pmatrix} (N),$$

where it is to be recalled that all these matrices are determined upto a sign. This condition implies (12.1.2) or (12.1.3).

Conversely suppose that the conditions (12.1.2), (12.1.3) hold. We need to show that a pairing transformation (using the above notations)

$$A = \begin{pmatrix} a_i & -a_{i+1} \\ b_i & -b_{i+1} \end{pmatrix} \begin{pmatrix} a_{i^*+1} \\ b_{i^*+1} \end{pmatrix}^{-1} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (N).$$

By (12.1.2), (12.1.3) the A_{21}-entry is clearly $\equiv 0(N)$. For definiteness assuming (12.1.2) the A_{11}-entry is

$$a_i b_{i^*} + a_{i+1} b_{i^*+1} \equiv -a_i b_{i+1} + a_{i+1} b_i \equiv 1(N).$$

A similar calculation shows that the A_{22}-entry is $\equiv 1(N)$ also. A similar conclusion follows assuming (12.1.3). *q.e.d.*

(12.6). It is well-known that the index of $\Gamma(N)$ in Γ is $\frac{N^3}{2} \Pi_{p|N} (1 - \frac{1}{p^2})$ for $N \geq 3$, cf. [S], p. 76. It is easy to see that the index of $\Gamma(N)$ in $\Gamma^1(N)$ is N and so the index of $\Gamma^1(N)$ in Γ is $\frac{N^2}{2} \Pi_{p|N} (1 - \frac{1}{p^2})$. On the other hand since for $N \geq 4$, $\Gamma^1(N)$ is torsionfree its index by (7.2) is $3n$ where n is as in (12.1.1). This value of n agrees with the one in (12.2.1). If we now choose a special polygon for $\Gamma^1(N)$ as given by (12.4) then by (12.5) the corresponding Farey symbol is balanced for N. Conversely any Farey symbol balanced for N, and with the underlying *g.F.S.* of the form (12.1.1) and n as in (12.2.1) defines a subgroup of $\Gamma^1(N)$ and has the same index in Γ as $\Gamma^1(N)$. So this subgroup must equal $\Gamma^1(N)$. This proves the first part of the theorem.

(12.7). Now consider a new $g.F.S.$ consisting of ∞ and y_{ij} as in (12.2.2). From (7.2) we see that the index of any subgroup of Γ based on this $g.F.S.$ is $3nN$ which is the index of $\Gamma(N)$. So if we can show that this can be given a structure of a Farey symbol such that the side-pairing transformations lie in $\Gamma(N)$ we are done. Indeed let $i \leftrightarrow i^*$ be the correspondence given as in (12.1.2) or (12.1.3). For definiteness suppose that for a pair (i, i^*) the relation (12.1.2) holds. Given $0 \leq j < N$ we look for $0 \leq j^* < N$ such that the transformation pairing the side $\{y_{ij} y_{i+1j}\}$ to $\{y_{i^*j^*} y_{i^*+1j^*}\}$ lies in $\Gamma(N)$. Note first that since $i \neq i^*$ these sides are distinct. The corresponding side-pairing transformation lies in $\Gamma(N)$ iff

$$\begin{pmatrix} a_i + jb_i & -a_{i+1} - jb_{i+1} \\ b_i & -b_{i+1} \end{pmatrix} \equiv \begin{pmatrix} a_{i^*+1} + j^*b_{i^*+1} & a_{i^*} + j^*b_{i^*} \\ b_{i^*+1} & b_{i^*} \end{pmatrix} (N).$$

In view of (12.1.2) the bottom rows of the two matrices are already congruent mod N. To solve for j^* so that the top rows are also congruent mod N we need

$$\begin{pmatrix} a_{i^*+1} & b_{i^*+1} \\ a_{i^*} & b_{i^*} \end{pmatrix} \begin{pmatrix} 1 \\ j^* \end{pmatrix} \equiv \begin{pmatrix} a_i & b_i \\ -a_{i+1} & -b_{i+1} \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} (N).$$

Or

$$\begin{pmatrix} 1 \\ j^* \end{pmatrix} \equiv \begin{pmatrix} b_{i^*} & -b_{i^*+1} \\ -a_{i^*} & a_{i^*+1} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ -a_{i+1} & -b_{i+1} \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} (N).$$

It is easy to see that in view of the condition (12.1.2) this is possible for the unique value of j^* given by

$$0 \leq j^* < N, \qquad \text{and}$$

$$j^* \equiv -(a_{i^*}b_i + a_{i^*+1}b_{i+1})j - (a_{i^*}a_i + a_{i^*+1}a_{i+1})(N).$$

If instead of the condition (12.1.2) the condition (12.1.3) is satisfied the proof is similar. This completes the proof of the Theorem (12.2).

q.e.d.

(12.8). The above result excludes the cases $N = 2$, and 3. But it is easily verified that the Farey symbols

$$\{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{2} 1, \underbrace{\ \ }_{2} 2, \underbrace{\ \ }_{1} \infty\},$$

$$\{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{2} 1, \underbrace{\ \ }_{2} \tfrac{3}{2}, \underbrace{\ \ }_{3} 2, \underbrace{\ \ }_{3} 3, \underbrace{\ \ }_{1} \infty\},$$

lead to the subgroups $\Gamma(2)$, and $\Gamma(3)$ respectively and the Farey symbols

$$\{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{\circ} 1, \underbrace{\ \ }_{1} \infty\}, \qquad \{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{\bullet} 1, \underbrace{\ \ }_{1} \infty\}$$

lead to the subgroups $\Gamma^1(2)$, and $\Gamma^1(3)$ respectively.

(12.9) *Remark.* The Theorem (12.2) will be partially improved in (14.12).

(12.10) *Example.* The case $N = 6$ already fairly illustrates the above process. Here $n = 4$. There is not much choice for the corresponding *g.F.S.*'s. Yet it is not unique. The *g.F.S.*'s

(12.10.1) $$\{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{2} \tfrac{1}{3}, \underbrace{\ \ }_{3} \tfrac{1}{2}, \underbrace{\ \ }_{3} \tfrac{2}{3}, \underbrace{\ \ }_{2} 1, \underbrace{\ \ }_{1} \infty\}$$

(12.10.2) $$\{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{2} \tfrac{1}{4}, \underbrace{\ \ }_{3} \tfrac{1}{3}, \underbrace{\ \ }_{3} \tfrac{1}{2}, \underbrace{\ \ }_{2} 1, \underbrace{\ \ }_{1} \infty\}$$

(12.10.3) $$\{\infty, \underbrace{\ \ }_{1} 0, \underbrace{\ \ }_{2} \tfrac{1}{2}, \underbrace{\ \ }_{3} \tfrac{2}{3}, \underbrace{\ \ }_{3} \tfrac{3}{4}, \underbrace{\ \ }_{2} 1, \underbrace{\ \ }_{1} \infty\}$$

with the indicated pairing are all balanced for 6. Each of them can serve as a Farey symbol for $\Gamma^1(6)$. Let us make the first choice. The side-pairing transformations are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 6 & -5 \end{pmatrix}, \begin{pmatrix} 5 & -3 \\ 12 & -7 \end{pmatrix}.$$

These are then the independent generators of $\Gamma^1(6)$. For example the second generator is obtained as

$$\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1},$$

cf. (6.1.3). Easy calculations using Section 7 show that $\Gamma^1(6)$ has genus 0, and four cusps ∞, 0, $\frac{1}{3}$, and $\frac{1}{2}$ with widths 1, 6, 3, and 2 respectively.

Applying the construction in the theorem to this Farey symbol for $\Gamma^1(6)$ gives the Farey symbol for $\Gamma(6)$.

$$(12.10.4)\quad \begin{array}{ccccccccccccccccccccccccccc} \{\infty, & & 0, & & \frac{1}{3}, & & \frac{1}{2}, & & \frac{2}{3}, & & 1, & & \frac{4}{3}, & & \frac{3}{2}, & & \frac{5}{3}, & & 2, & & \frac{7}{3}, & & \frac{5}{2}, & & \frac{8}{3}, & & 3, \\ & 1 & & 2 & & 3 & & 4 & & 5 & & 5 & & 6 & & 7 & & 8 & & 8 & & 9 & & 10 & & 11 & \end{array}$$

$$\begin{array}{cccccccccccccccccccccccccc} & \frac{10}{3}, & & \frac{7}{2}, & & \frac{11}{3}, & & 4, & & \frac{13}{3}, & & \frac{9}{2}, & & \frac{14}{3}, & & 5, & & \frac{16}{3}, & & \frac{11}{2}, & & \frac{17}{3}, & & 6, & & \infty\}. \\ 11 & & 4 & & 3 & & 12 & & 12 & & 7 & & 6 & & 13 & & 13 & & 10 & & 9 & & 2 & & 1 & \end{array}$$

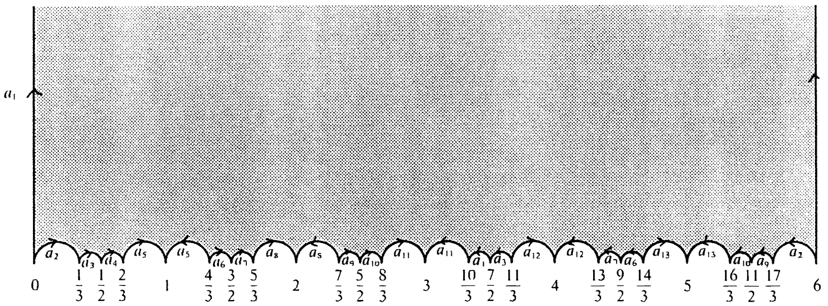
Since $\Gamma(6)$ is normal all cusps have the same widths. Since its index is 72, and ∞ has width 6, it follows that $\Gamma(6)$ has 12 cusps each of width 6, and has genus 1. A set of independent generators for $\Gamma(6)$ given by the above Farey symbol is

$$(12.10.5)\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -6 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} -5 & -18 \\ -12 & 43 \end{pmatrix}, \begin{pmatrix} 7 & -24 \\ 12 & -41 \end{pmatrix},$$

$$\begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix}, \begin{pmatrix} -17 & 78 \\ -12 & 55 \end{pmatrix}, \begin{pmatrix} 19 & -84 \\ 12 & 53 \end{pmatrix}, \begin{pmatrix} -11 & 24 \\ -6 & 13 \end{pmatrix}, \begin{pmatrix} -29 & 162 \\ -12 & 67 \end{pmatrix},$$

$$\begin{pmatrix} 31 & -168 \\ 6 & -65 \end{pmatrix}, \begin{pmatrix} -17 & 54 \\ -6 & 19 \end{pmatrix}, \begin{pmatrix} -23 & 96 \\ -6 & 25 \end{pmatrix}, \begin{pmatrix} -29 & 150 \\ -6 & 31 \end{pmatrix}.$$

The corresponding fundamental domain is



13. Semi-balanced Farey symbols (13.1). Let N be ≥ 2 and d, e_2, e_3, r, t, g be the invariants of $\Gamma_0(N)$ as described in Section 7. These invariants as functions of N are given as follows, cf. [S], Chapter 4, Section 8. First of all the index is

$$d = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Here p runs over all primes dividing N. Secondly if N is divisible by 4 or by a prime $\equiv -1(4)$, then $e_2 = 0$. If N is 2 then $e_2 = 1$. Otherwise N is of the form $2^{\epsilon}m$ where $\epsilon = 0$, *or* 1 and m is a product of say $u \geq 1$ primes $\equiv 1(4)$. Then $e_2 = 2^u$. Thirdly if N is divisible by 9 or by a prime $\equiv -1(3)$, then $e_3 = 0$. If N is 3 then $e_3 = 1$. Otherwise N is of the form $3^{\epsilon}m$ where $\epsilon = 0$, *or* 1 and m is a product of say $v \geq 1$ primes $\equiv 1(3)$. Then $e_3 = 2^v$. Now (7.1.2) determines r. Lastly

$$t = \Sigma_{a|N}\, \varphi\left(g.c.d.\left(a, \frac{N}{a}\right)\right),$$

where φ denotes the Euler's totient function. Now (7.1.1) determines g.

Let

(13.1.1) $$n = e_2 + e_3 + 2r - 2.$$

A $g.F.S.$ of the form

$$\{\infty, 0 = x_0, x_1, \ldots, x_n = 1, \infty\}, \tag{13.1.2}$$

where $x_i = \frac{a_i}{b_i}$ (reduced fractions with a_i, b_i nonnegative, and with usual conventions about ∞ as in (5.1)) is said to be *semi-balanced for N if*

i) *there are* e_2 *values* i, $0 \le i \le n - 1$, *s.t.*

$$b_i^2 + b_{i+1}^2 \equiv 0(N),$$

ii) *there are* e_3 *values* i, $0 \le i \le n - 1$, *s.t.*

$$b_i^2 + b_i b_{i+1} + b_{i+1}^2 \equiv 0(N),$$

iii) *the remaining* $2r - 2$ *values of* i, $0 \le i \le n - 1$, *are paired* $i \leftrightarrow i^*$ *s.t.*

$$b_i b_{i^*} + b_{i+1} b_{i^*+1} \equiv 0(N).$$

The significance of this notion is the following result.

(13.2) THEOREM. *Let* $N \ge 2$ *and* n *as in* (13.1.1). *Then there exists a g.F.S. in the form* (13.1.2) *which is semi-balanced for* N. *Moreover there exists a canonical structure of a Farey symbol on this g.F.S. such that the corresponding special polygon is an admissible fundamental polygon for* $\Gamma_0(N)$.

Proof. The proof parallels the proof of Theorem (12.2), so we shall be brief. First of all a semi-balanced $g.F.S.$ is made into a Farey symbol by declaring the e_2 (resp. e_3) intervals $\{x_i \quad x_{i+1}\}$ as in the case i) (resp. ii)) as described above to be even intervals (resp. odd intervals). The remaining intervals are to be considered as free intervals. The interval $\{\infty \quad 0\}$ is paired to the interval $\{1 \quad \infty\}$, and for the remaining $2r - 2$ values of i, $0 \le i \le n - 1$, paired $i \leftrightarrow i^*$ as in iii) above, the interval $\{x_i \quad x_{i+1}\}$ is paired to $\{x_{i^*} \quad x_{i^*+1}\}$. This pairing—without any reference to semi-balancing—leads to a subgroup of the same index as $\Gamma_0(N)$. So if we can ensure that the side-pairing transformations belong to $\Gamma_0(N)$ then the subgroup defined by this Farey symbol is $\Gamma_0(N)$. The conditions i), ii), iii) in (13.1) for semi-balancing precisely ensure this.

To show the existence of a semi-balanced *g.F.S.* for a given $N \geq 2$, observe again that the width of the cusp of $\Gamma_0(N)$ defined by ∞ is 1. As in the case of $\Gamma^1(N)$ in the proof of (12.2), since the interval $\{\infty \quad 0\}$ is paired to the interval $\{1 \quad \infty\}$, and since the transformation $z \mapsto z + 1$ is contained in $\Gamma_0(N)$, we see that there is a special polygon which has the sides $x = 0$, $y > 0$ and $x = 1$, $y > 0$ paired by the transformation $z \mapsto z + 1$. Now the condition that the side-pairing transformations belong to $\Gamma_0(N)$ ensures in view of (6.1) that the *g.F.S.* defined by this special polygon is necessarily semi-balanced. *q.e.d.*

(13.3). The subgroup $\Gamma_0(N)$ is normalized by the conjugation J_1, where as in the proof of (9.6), J_1 denotes $z \mapsto 1 - \bar{z}$. Let

$$\Gamma_0^*(N) = \langle \Gamma_0(N), J_1 \rangle. \tag{11.3.1}$$

Suppose now that N is a prime which we naturally denote by p. By the discussion in (11.1), in this case $d = p + 1$, and $\Gamma_0(p)$ has two cusps. Let P be a special polygon for $\Gamma_0(p)$ constructed in the proof of (13.2). Since ∞ has width 1 it follows that *all the other vertices of P represent a single cusp of* $\Gamma_0(p)$ *and this cusp has width* p. Now $\Gamma_0(p)\backslash\mathbf{H}$ has a canonical conjugation θ induced by J_1. It is not hard to check that $n = 1$ if $p = 2$ or 3, but $n > 1$ if $p \geq 5$.

(13.4) PROPOSITION. *Let* $p \geq 5$. *Then the projection of the lines which join* ∞ *to* 0 *and* ∞ *to* $\frac{1}{2}$ *in* $\Gamma_0(p)\backslash\mathbf{H}$ *is the full fixed point set of* θ.

Proof. First of all J_1 fixes the line joining ∞ to $\frac{1}{2}$ and both J_1 and the transformation $z \mapsto z + 1$ map the line joining ∞ to 0 onto the line joining ∞ to 1. So the projection of the lines which join ∞ to 0 and ∞ to $\frac{1}{2}$ is contained in the fixed point set of θ.

Next notice that each of these lines is mapped injectively into $\Gamma_0(p)\backslash\mathbf{H}$. Indeed choose a special polygon P as in (13.2). The line joining ∞ to 0 is mapped by a pairing transformation of P onto the line joining ∞ to 1. So the line joining ∞ to 0 is mapped injectively into $\Gamma_0(p)\backslash\mathbf{H}$. (This step is valid for all primes p.) Now if $p \geq 5$ then the *g.F.S.* corresponding to P contains $\frac{1}{2}$ as one of the vertices, cf. (8.11). So the line joining ∞ to $\frac{1}{2}$ is contained in the interior of P. So this line is also mapped injectively into $\Gamma_0(p)\backslash\mathbf{H}$.

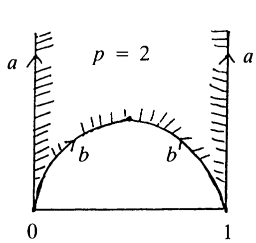
Now consider the compact Riemann surface X which is obtained by filling the punctures of $\Gamma_0(p)\backslash\mathbf{H}$, i.e. by adding the points say x and y. Then θ extends to a conjugation $\bar{\theta}$ on X, and the fixed point set of $\bar{\theta}$

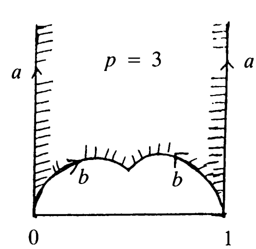
consists of one or more disjoint circles, cf. (9.1) and (9.5). For $p \geq 5$ the projections of the lines joining ∞ to 0 and $\frac{1}{2}$ together with x and y clearly form one such circle, say C. On the other hand the fixed point set of θ on $\Gamma_0(p)\backslash\mathbf{H}$ itself contains no circles, cf. (10.2). So each component of the fixed point set of θ has to end in one or two cusps, i.e. to say that each component of the fixed point set of $\bar{\theta}$ has to pass through x or y. Since the fixed point set of the conjugation on X consists of *mutually disjoint* circles it follows that C is the full fixed point set of $\bar{\theta}$. This also proves the proposition. *q.e.d.*

(13.5) THEOREM. *There exists a special polygon P for $\Gamma_0(p)$ with a reflection which lifts θ. Moreover we may choose P so that P has sides $x = 0$, $y > 0$ and $x = 1$, $y > 0$ paired by the transformation $z \mapsto z + 1$ and the reflection on P is induced by J_1. In other words there exists a Farey symbol for $\Gamma_0(p)$ which is semi-balanced for p and whose denominators have a symmetry around $\frac{1}{2}$.*

Proof. First let $p \geq 5$. The above proposition shows that $\Gamma_0^*(p)\backslash\mathbf{H}$ has exactly two boundary components one of which is an even line and the other is an odd line. So by (11.3) we get a special polygon P_1 for $\Gamma_0(p)$ with a reflection which lifts θ. Consider the tessellation of $\mathbf{H}$ corresponding to P_1 and consider a tile P of this tessellation which contains ∞ as a vertex. Since ∞ is a cusp of width 1, as in (13.2) by a further translation if necessary we may assume that P has $x = 0$, $y > 0$ and $x = 1$, $y > 0$ as its sides which are paired by the transformation $z \mapsto z + 1$. By construction P_1 and hence P have a reflection which lifts θ, and the fixed line of this reflection is an odd line passing through the cusp of width 1. It follows that the fixed line has to be the line which joins ∞ to $\frac{1}{2}$ and so the reflection on P is induced by J_1.

For $p = 2$ and 3 we simply exhibit the appropriate special polygons.





q.e.d.

(13.6) *Example.* Consider the cases $\Gamma_0(p)$ where $p = 11$ and 13. By the formulas in (13.1) in the first (resp. second) case we have $e_2 = e_3 = 0$ (resp. 2), and $r = 3$ (resp. 1). In both cases the value of n in (13.1.1) is 4. The only *g.F.S.* with $n = 4$ and whose denominators are symmetric around $\frac{1}{2}$ is

$$\{\infty, 0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1, \infty\}.$$

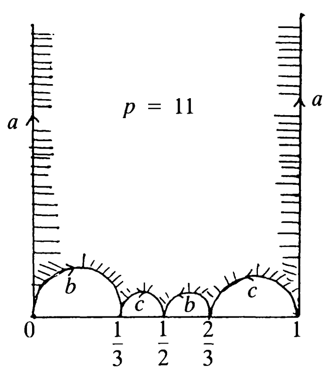
The corresponding Farey symbol for $p = 11$ is

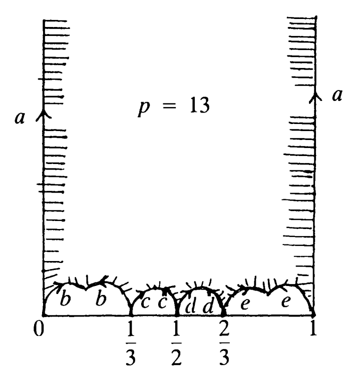
$$\begin{array}{ccccccccccccc} \{\infty, & & 0, & & \frac{1}{3}, & & \frac{1}{2}, & & \frac{2}{3}, & & 1, & & \infty\}, \\ & 1 & & 2 & & 3 & & 2 & & 3 & & 1 & \end{array}$$

whereas for $p = 13$ it is

$$\begin{array}{ccccccccccccc} \{\infty, & & 0, & & \frac{1}{3}, & & \frac{1}{2}, & & \frac{2}{3}, & & 1, & & \infty\}. \\ & 1 & & \bullet & & \circ & & \circ & & \bullet & & 1 & \end{array}$$

The corresponding fundamental domains are





Notice that the fundamental domain for $\Gamma_0(11)$ shows the handle! (As is well-known $\Gamma_0(11)$ has genus 1, and $N = 11$ is the least integer for which $\Gamma_0(N)$ has genus greater than 0.)

A set of independent generators for $\Gamma_0(11)$ is

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}3 & -2\\ 11 & -7\end{pmatrix}, \begin{pmatrix}4 & -3\\ 11 & -8\end{pmatrix},$$

and that for $\Gamma_0(13)$ is

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}4 & -1\\ 13 & -3\end{pmatrix}, \begin{pmatrix}5 & -2\\ 13 & -5\end{pmatrix}, \begin{pmatrix}8 & -5\\ 13 & -8\end{pmatrix}, \begin{pmatrix}9 & -7\\ 13 & -10\end{pmatrix}.$$

14. Fundamental domains for subgroups of Γ^* (14.1). In this section we shall consider arbitrary subgroups of finite index in the *extended* modular group Γ^* and briefly point out the modifications needed to extend the previously outlined procedures for obtaining the fundamental domains in this case. Geometrically speaking some modifications are necessary due to the fact that the odd reflections fix odd lines, and the boundaries of the corresponding surfaces may contain corners. If Φ is a subgroup of Γ whose normalizer contains a conjugation then the following modifications would usually lead to better-looking fundamental domains for Φ which are not necessarily special polygons.

(14.2). Let P be a hyperbolic polygon and e and f two arcs contained in its boundary. The boundary of P and so any arc contained in it inherits a canonical orientation. An element γ in Γ^* is said to *pair e* and f if $\gamma(e) = f$ or $\gamma(f) = e$ and $\gamma(\text{int } P) \cap \text{int } P = \phi$. Note that if γ is in Γ then the pairing is orientation-reversing, and if γ is a conjugation then the pairing is orientation-preserving. Notice also that given two even (resp. odd) lines e and f in ∂P there is a unique element in Γ which pairs them in an orientation-reversing way, and a unique conjugation in Γ^* which pairs them in an orientation-preserving way, cf. the discussion in (2.2), (2.3), and (10.3).

(14.3). We now consider the following two modifications of the notion of a special polygon. Consider a convex hyperbolic polygon P of finite area containing 0 and ∞ as vertices and so that each component of ∂P is of one of the forms i), ii), iii), iv) noted in (10.6) (and thus is a union of a certain number of even, odd, or f-edges) or is a union of two odd edges which meet at an odd vertex making an internal angle $\frac{2\pi}{3}$ as in S_2) of (2.4).

A *side-pairing* is of 4 types:

i) An even line is divided into two even edges and they are paired by a unique element of Γ. These edges are among the *sides* of P.

ii) Two odd edges making an internal angle $\frac{2\pi}{3}$ are paired to each other again by a unique element of Γ. These edges are also among the *sides* of P.

iii) An even (resp. odd) line is paired to another such line by an element of Γ^*. This element is determined uniquely *once it is specified that the pairing preserves or reverses the orientation.* Notice that in contrast to the special polygons, it is now possible to pair a line with itself *of course necessarily by a reflection.* These lines are also considered as the *sides* of P.

iv) An even edge, or an odd edge, or an f-edge which is a part of a component containing a corner of the form i)–iv) of (10.6) is paired to itself by an appropriate reflection in Γ^*. The parts of such components which connect a vertex in $\mathbf{R} \cup \{\infty\}$ to a vertex incident to a corner with angle $\frac{\pi}{2}$ or $\frac{\pi}{3}$ are also among the *sides* of P.

A *special *-polygon* is a polygon with a side-pairing of the type described above. A convenient geometric way to think about it is that it can be obtained as a convex hull of its vertices lying in $\mathbf{R} \cup \{\infty\}$ and attaching externally across even lines certain triangles each with two angles 0 and the third angle $\frac{2\pi}{3}$, or $\frac{\pi}{2}$, or $\frac{\pi}{3}$.

(14.4). A further weakening of the notion of a special polygon is a *weak *-polygon.* Its geometric shape is the same as a special *-polygon, but the side-pairing rule iv) is weakened to the following

iv)$'$ Let e and f be two sides of P making a corner as defined in iv). A side-pairing transformation γ is either a reflection in e or else it maps e onto another such side e' of a corner so that $\gamma(\text{int } P) \cap (\text{int } P) = \phi$. We require that in the first case there must also be a side-pairing transformation which is a reflection in f, whereas in the second case there must also be a side-pairing transformation which maps f onto f' so that e' and f' form a corner of P.

(14.5). It easily follows from surface topology that a torsionfree subgroup of Γ^* is free, but in general the subgroups of Γ^* containing elements of finite order are not free products of cyclic groups. So now the Rademacher's notion of *independent generators* does not make sense.

The following notion appears to be a reasonable substitute. A system of generators $\langle x_i \rangle$, $i \in I$ for a group is said to be *quasi-independent* if the only relations are of the following types

(14.5.1) a) x_i^2, x_j^3, b) $x_k^2, x_l^2, (x_k x_l)^2$, c) $x_u^2, x_v^2, (x_u x_v)^3$,

where all subscripts $i, j, \ldots$ are distinct. Let $\mathbf{Z}_2^2$ resp. S_3 denote $\mathbf{Z}_2 \times \mathbf{Z}_2$ resp. the symmetric group on three letters.

PROPOSITION. *A group admits a quasi-independent system of generators iff it is isomorphic to a free product of groups isomorphic to* $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}$, $\mathbf{Z}_2^2$ *and* S_3.

Proof. The relations of type b) resp. c) clearly define subgroups isomorphic to Z_2^2 resp. S_3. *q.e.d.*

(14.6) THEOREM. *Let P^* be a special *-polygon, and $\Phi_{P^*}^*$ the subgroup of Γ^* generated by the side-pairing transformations. Then $\Phi_{P^*}^*$ is a subgroup of finite index in Γ^*, P^* is a fundamental domain for $\Phi_{P^*}^*$, and the side-pairing transformations form a quasi-independent system of generators for $\Phi_{P^*}^*$. If ∂P^* does not contain a corner with angle $\frac{\pi}{2}$ or $\frac{\pi}{3}$ then the system of generators is independent.*

Proof. The argument is exactly as in (3.2). It is only a corner in ∂P^* with angle $\frac{\pi}{2}$ or $\frac{\pi}{3}$ which gives rise to a relation of the type b) or c) respectively. So the last assertion is clear. *q.e.d.*

(14.7) THEOREM. *Let Φ^* be a subgroup of finite index in Γ^* but $\neq \Gamma^*$ or Φ_0^*, cf.* (10.4). *Then it admits a fundamental domain which is a special *-polygon. There are only finitely many choices of such fundamental domains. In particular Φ^* is an internal free product of subgroups isomorphic to* $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}$, $\mathbf{Z}_2^2$ *and* S_3.

Proof. The argument is as in (3.3) or rather as in (11.3). We use the notation in (10.1). Let $\mathscr{E}_f^*$ denote the union of f-edges in S_Φ^* and let T^* be a maximal tree in $\mathscr{E}_f^*$ which contains the f-edges in ∂S_{Φ^*}. Now as in (3.3) or (11.3) cut S_Φ^* open into a space which is isometric to a connected simply connected convex hyperbolic polygon P^* and then obtain a special *-polygon which is a fundamental domain for Φ^*. Since we require that 0 and ∞ are among the vertices of a special *-polygon it is clear that there are only finitely many choices for such polygons. The last assertion follows from (14.5). *q.e.d.*

(*A note concerning* Γ^* *and* Φ_0^*: Since by definition a special *-polygon has 0 and ∞ as vertices it is not difficult to see that Γ^* and Φ_0^* do not admit a special *-polygon as a fundamental domain. Their presentations as given by their fundamental domains described at the beginning of the proof of (11.3) are

$$\Gamma^* = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^3 \rangle,$$

$$\Phi_0^* = \langle x, y, z \mid x^2, y^2, z^2, (xy)^3, (yz)^3 \rangle.$$

It may be noted that Γ^* (resp. Φ_0^*) is isomorphic to a free product of $\mathbf{Z}_2^2$ and S_3 (resp. two copies of S_3) each amalgamated over $\mathbf{Z}_2$. From this fact, and from a general fact in combinatorial group theory that a finite subgroup of a free product of groups is conjugate to a subgroup of one of the free factors, one may see that Γ^* and Φ_0^* in fact do not admit a quasi-independent system of generators.)

(14.8). There is an analogue of the arithmetic part of Section 5 in the present case. This is perhaps the most succinct way of describing a subgroup. A *-*g.F.S.* is exactly a *g.F.S.* as in (5.1) except that we allow x_0 and x_n to be integers or half-integers, and in (5.1.2) we allow the possibility that $|a_i b_{i+1} - b_i a_{i+1}|$ equals 1 *or* 2. A *-Farey symbol based on a *-*g.F.S.* is as before corresponding to the sidepairing rules i) or ii) of (14.3) with the following additional stipulation for iii) and iv). As for iii) of (14.3) if the pairing is by a conjugation it will be indicated by a symbol

$$\underset{a}{\underbrace{x_i \quad x_{i+1}}} \qquad\qquad \underset{a}{\underbrace{x_{i'} \quad x_{i'+1}}}$$

in case two distinct intervals are paired, and by

$$\underbrace{x_i \quad x_{i+1}}$$

in case the interval is paired to itself by a reflection. For iv) we use one of the following symbols.

$$\underset{c_2}{\overset{x_i \quad x_{i+1},}{\curvearrowright}} \qquad \underset{c_2}{\overset{x_i \quad x_{i+1},}{\curvearrowleft}} \qquad \underset{c_3}{\overset{x_i \quad x_{i+1},}{\curvearrowright}} \qquad \underset{c_3}{\overset{x_i \quad x_{i+1}.}{\curvearrowleft}}$$

Here the markings c_2 and c_3 indicate a corner with an angle $\frac{\pi}{2}$ and $\frac{\pi}{3}$ respectively, and the direction indicates that the "smaller" side, i.e. the one not containing one or two f-edges, is incident to the initial end-vertex of the arrow.

The special $*$-polygons are in an obvious natural one-to-one correspondence with $*$-Farey symbols which in turn classify the subgroups of finite index in Γ^*.

(14.9). There is a more or less obvious modification of Theorem (6.1) to get a set of quasi-independent generators for a subgroup given by its $*$-Farey symbol. For the convenience of the reader we merely record some computations beyond (6.1) needed for this modification. Notice first that if $\frac{b}{d}$, $\frac{a}{c}$ are two reduced fractions which are endpoints of an even line so that $ad - bc = 1$, then

$$z \mapsto \frac{az + b}{cz + d}, \qquad z \mapsto \frac{a\bar{z} - b}{c\bar{z} - d},$$

are the unique orientation-preserving and orientation-reversing elements of Γ^* mapping ∞ to $\frac{a}{c}$ and 0 to $\frac{b}{d}$. From these we easily get the unique orientation-preserving and orientation-reversing elements of Γ^* carrying one even line onto the other. On the other hand if $\frac{a}{c}$ and $\frac{b}{d}$ (reduced fractions) are the endpoints of an odd line so that $ad - bc = 2$, then $a \pm b$, $c \pm d$ are necessarily even integers, cf. (2.3). So the unique element of Γ carrying ∞ to $\frac{a}{c}$ and $\frac{1}{2}$ to $\frac{b}{d}$ is

$$z \mapsto \frac{az + \dfrac{b - a}{2}}{cz + \dfrac{d - c}{2}}$$

and the unique conjugation in Γ^* carrying ∞ to $\frac{a}{c}$ and $\frac{1}{2}$ to $\frac{b}{d}$ is

$$z \mapsto \frac{a\bar{z} - \dfrac{b+a}{2}}{c\bar{z} - \dfrac{d+c}{2}}.$$

We now easily get the unique orientation-preserving and orientation-reversing elements of Γ^* carrying one odd line onto the other, and so eventually a set of quasi-independent generators for the subgroup given by its *-Farey symbol.

(14.10). The following theorem partially extends and further clarifies the Theorem (11.3). We use the notation of (10.1). Our concern is whether we can lift the reflection θ on S_Φ to a reflection on some "nice" fundamental domain for Φ.

THEOREM. *The reflection θ on S_Φ can be lifted to a reflection on a special *-polygon P which is a fundamental domain for Φ iff ∂S_Φ^* has at most one component with a corner. In this case in fact the side-pairing transformations of P are independent. In any case the reflection θ always lifts to a weak *-polygon which is a fundamental domain for Φ.*

Proof. If Φ^* is Γ^* or Φ_0^*, cf. (10.4), then the assertions are clear. So we assume that this is not the case. In particular a component of ∂S_Φ^* contains at most one corner. Suppose a special *-polygon P exists and σ is the reflection on P which lifts θ. Notice that no side-pairing transformation of P can be a reflection since it is a fundamental domain for a subgroup of Γ. So ∂P itself has no corners with angles $\frac{\pi}{2}$ or $\frac{\pi}{3}$ and the side-pairing transformations of P are independent. Since no two even lines intersect, an even line can intersect exactly one odd line, and an odd line intersects exactly two other odd lines it is clear that the fixed line of σ can pass through at most one even vertex or at most one odd vertex on ∂P. If it does, there will be a corner on the corresponding component of ∂S_Φ^* and this is the only way a corner on ∂S_Φ^* can occur. So ∂S_Φ^* has at most one component with a corner. Conversely as in (11.3) we take two copies P_1^*, P_2^* of a special *-polygon P^* for Φ^*, and glue them along the boundary-components which correspond to the same component of ∂S_Φ^*. We then obtain a fundamental domain P for Φ. In general P is a weak *-polygon. If ∂S_Φ^* has only one component with a corner we glue along the larger sides of the corresponding components of ∂P_1^*, ∂P_2^* and actually obtain a special *-polygon. *q.e.d.*

(14.11). We can now extend the Theorem (13.5) to the case of $\Gamma_0(N)$ where N is not necessarily a prime. A *-*g.F.S.* is said to be *semi-balanced for N* if it satisfies the conditions i), ii), for a semi-balanced *g.F.S.*, cf. (13.1), where in i) or ii) the reduced fractions $\frac{a_i}{b_i}$, $\frac{a_{i+1}}{b_{i+1}}$ must satisfy $|a_i b_{i+1} - b_i a_{i+1}| = 1$, whereas iii) is replaced by

iii)$'$ *the remaining* $2r - 2$ *values of* i, $0 \le i \le n - 1$, *are paired* $i \leftrightarrow i^*$ *s.t.*

$$b_i b_{i^*} + b_{i+1} b_{i^*+1} \equiv 0(N) \text{ (resp. } 2N)$$

according as $|a_i b_{i+1} - b_i a_{i+1}| = 1$, (*resp.* 2). We call the corresponding *-Farey symbol also as being semi-balanced for N. (The difference in iii) and iii)$'$ arises from the differences for the expressions for the pairing of even and odd edges as explained in (14.9). These conditions are necessary and sufficient for the pairing transformations to be contained in $\Gamma_0(N)$.)

THEOREM. *There exists a special* *-*polygon P for* $\Gamma_0(N)$ *with a reflection which lifts the canonical reflection on* $\Gamma_0(N)\backslash\mathbf{H}$. *Moreover we may choose P so that P has sides* $x = 0$, $y > 0$ *and* $x = 1$, $y > 0$ *paired by the transformation* $z \mapsto z + 1$ *and the reflection on P is induced by* $J_1 : z \mapsto 1 - \bar{z}$. *In other words there exists a* *-*Farey symbol for* $\Gamma_0(N)$ *which is semi-balanced for N and whose denominators have a symmetry around* $\frac{1}{2}$.

Proof. Let $Y(N)$ resp. $Y^*(N)$ denote the surfaces $\Gamma_0(N)\backslash\mathbf{H}$ resp. $\Gamma_0^*(N)\backslash\mathbf{H}$. In view of (14.10) to prove the first part of the theorem we need only prove that $Y^*(N)$ has at most one boundary component with a corner. A more precise statement in this direction is the following

Assertion. i) $\partial Y^*(2)$ *contains only one corner and its angle is* $\frac{\pi}{2}$. ii) $\partial Y^*(3)$ *contains only one corner and its angle is* $\frac{\pi}{3}$. iii) *For* $N \ge 4$, $\partial Y^*(N)$ *has no corner.*

Proof. Consider the fundamental domains for $\Gamma_0(2)$ and $\Gamma_0(3)$ which were exhibited in (13.5). The parts of these lying within the strip bounded by $x = 0$ and $x = \frac{1}{2}$ can serve as fundamental domains for

$\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ respectively. These are special *-polygons in which each side is paired to itself. The one for $\Gamma_0^*(2)$ resp. $\Gamma_0^*(3)$ contains a corner of angle $\frac{\pi}{2}$ resp. $\frac{\pi}{3}$ in its boundary. This proves the first two parts. Now assume $N \geq 4$. By (10.3) iii) amounts to the fact that for $N \geq 4$, $\Gamma_0^*(N)$ does not contain two reflections fixing an even or odd vertex. There are two cases to consider.

Case 1: *$\Gamma_0^*(N)$ contains two reflections fixing an even vertex.* Then these reflections are conjugate by an element of Γ with the reflections $z \mapsto -\bar{z}$ and $z \mapsto \frac{1}{\bar{z}}$ which fix $i = \sqrt{-1}$. In terms of matrices the conjugates of these latter reflections by an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} * & * \\ -2cd & * \end{pmatrix}, \tag{14.11.1}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} * & * \\ d^2 - c^2 & * \end{pmatrix}. \tag{14.11.2}$$

If these lie in $\Gamma_0^*(N)$ then we have

$$2cd \equiv d^2 - c^2 \equiv 0(N), \qquad g.c.d.(c, d) = 1. \tag{14.11.3}$$

If p is an odd prime dividing N then by (14.11.3) either p divides c or d, and p divides $c + d$ or $c - d$. These equations are incompatible. So N must be a power of 2. If $N \geq 4$ we again see that 2 divides c or d, and 2 divides $c + d$ or $c - d$. Again these equations are incompatible. So the Case 1 does not occur.

Case 2: *$\Gamma_0^*(N)$ contains two reflections fixing an odd vertex.* Then these reflections are conjugate by an element of Γ with the reflections $z \mapsto 1 - \bar{z}$ and $z \mapsto \frac{1}{\bar{z}}$ which fix $\rho = e^{i\pi/3}$. Their conjugates by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are given by (14.11.2) and

$$(14.11.4) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} * & * \\ -2cd - c^2 & * \end{pmatrix}.$$

So

$$(14.11.5) \quad d^2 - c^2 \equiv 2cd + c^2 \equiv 0(N), \qquad g.c.d.(c, d) = 1.$$

In turn this implies $2cd + d^2 \equiv 0(N)$. It is easy to see that N cannot be even. If p is an odd prime dividing N then either p divides c or $c + 2d$, and p divides d or $2c + d$. These equations are incompatible unless p divides $c + 2d$ and $2c + d$. So p must divide $3(c + d)$. Now if $p \geq 5$ then p divides $c + d$. Now again it is easy to see that these equations are incompatible. So N must be a power of 3. If $N > 3$ then as above 3 divides $c + d$, and we again see that these equations are incompatible. So the Case 2 does not occur. *q.e.d. of the assertion.*

Now to prove the latter half of the theorem we may omit the simple cases $N = 2$ and 3, and first observe that since J_1 normalizes $\Gamma_0(N)$ it induces the canonical reflection θ on $Y(N)$, and by the above assertion its fixed line projects *onto* a boundary component say C of $Y^*(N)$. Let P_0^* be the space isometric to a special *-polygon obtained by cutting $Y^*(N)$, and P_0 the space obtained by doubling P_0^* along C. We can now isometrically develop P_0 onto a J_1-invariant special *-polygon P so that C is mapped onto the fixed line of J_1 taking care that the "cusp" on C corresponding to ∞ is indeed mapped onto ∞. By this construction the canonical projection $\mathbf{H} \to Y(N)$ maps the interior of P homeomorphically onto an open dense set of $Y(N)$. So P is indeed a fundamental domain for $\Gamma_0(N)$. Since the width of ∞ is 1 for $\Gamma_0(N)$ the other assertions in the theorem are now clear. *q.e.d.*

(14.12). We can now also partially refine the Theorem (12.2). As observed there a *g.F.S.* which is balanced for N naturally leads to a Farey symbol for $\Gamma^1(N)$. Now $\Gamma^1(N)$ equals $\Gamma_0(N)$ for $N = 2$ and 3 and in any case it is a subgroup of $\Gamma_0(N)$. It is easily checked that it is normalized by J_0 and J_1 and its cusp-width at ∞ is 1. So the *Assertion* in the theorem in (14.11) concerning the corners in $\partial\Gamma_0^*(N)$ and the rest of the proof is valid also for $\partial\Gamma^{1*}(N)$, where naturally $\Gamma^{1*}(N) = \langle\Gamma^1(N), z \mapsto -\bar{z}\rangle$. A purely arithmetic notion describing a *-Farey symbol for $\Gamma^1(N)$ is the following modification of (12.1): a **-g.F.S.* is said to be *balanced* for N if the conditions (12.1.1) and (12.1.2) are replaced by:

(14.12.1) $$b_i \equiv b_{i^*+1}(N), \qquad \text{and} \quad b_{i+1} \equiv -b_{i^*}(N),$$

or

$$b_i \equiv -b_{i^*+1}(N), \qquad \text{and} \quad b_{i+1} \equiv b_{i^*}(N)$$

if $|a_i b_{i+1} - b_i a_{i+1}| = 1$, whereas

(14.12.2) $$b_i \equiv b_{i^*+1}(N), \qquad b_{i+1} \equiv -b_{i^*}(N), \qquad \text{and}$$

$$b_i \equiv b_{i^*+1}(2N) \quad \text{iff} \quad b_{i+1} \equiv -b_{i^*}(2N),$$

or

$$b_i \equiv -b_{i^*+1}(N), \qquad b_{i+1} \equiv b_{i^*}(N) \quad \text{and}$$

$$b_i \equiv -b_{i^*+1}(2N), \qquad \text{iff} \quad b_{i+1} \equiv b_{i^*}(2N)$$

if $|a_i b_{i+1} - b_i a_{i+1}| = 2$. An elementary but a bit tedious calculation using (14.9) along the lines of (12.5) shows that these are indeed the necessary and sufficient conditions for the side-pairing transformations to be in $\Gamma^1(N)$. To summarize: the Theorem (12.2) partially improves to the following.

THEOREM. *Let N, n be as in* (12.2). *Then there exists a *-g.F.S. in the form* (12.1.1) *which is balanced for N, and whose denominators have a symmetry around $\frac{1}{2}$. It leads to a special *-polygon P for $\Gamma^1(N)$ with a reflection which lifts the canonical reflection on $\Gamma^1(N)\backslash\mathbf{H}$ and we may choose P so that P has sides $x = 0$, $y > 0$ and $x = 1$, $y > 0$ paired by the transformation $z \mapsto z + 1$ and the reflection on P is induced by J_1 : $z \mapsto 1 - \bar{z}$. Moreover let y_{ij} be defined as in* (12.2.2). *Then there exists a unique structure of a *-Farey symbol on a *-g.F.S. containing y_{ij} and*

∞ *such that the corresponding special *-polygon is an admissible fundamental polygon for* $\Gamma(N)$.

(14.13) *Remark.* Instead of having a special *-polygon bounded by $x = 0, y > 0$ and $x = 1, y > 0$ to be symmetric around $\frac{1}{2}$ it may be convenient to replace it by a special *-polygon bounded by $x = -\frac{1}{2}$, $y > 0$ and $x = \frac{1}{2}, y > 0$ and symmetric around 0. This luxury was not available with special polygons. The reader may think of some other variations of this theme.

(14.14) PROPOSITION.

i) $\Gamma_0^*(2) = \Gamma^{1*}(2) \approx \mathbf{Z}_2 * \mathbf{Z}_2^2$.
ii) $\Gamma_0^*(3) = \Gamma^{1*}(3) \approx \mathbf{Z}_2 * S_3$.
iii) *For* $N \geq 4$ *both* $\Gamma_0^*(N)$ *and* $\Gamma^{1*}(N)$ *admit an independent system of generators.*

Proof. It is easy to see that $\Gamma^{1*}(N) = \Gamma_0^*(N)$ for $N = 2$ or 3. So the assertion is immediate from (14.6) and the statements concerning the fundamental domains for $\Gamma_0^*(N)$ and $\Gamma^{1*}(N)$ in the proofs of (14.11) and (14.12). *q.e.d.*

(14.15) *Example.* Let us consider the case $N = 8$. A Farey symbol for $\Gamma_0(8)$ is

$$\{\underbrace{\infty, 0}_{1}, \underbrace{\tfrac{1}{4}}_{2}, \underbrace{\tfrac{1}{3}}_{2}, \underbrace{\tfrac{1}{2}}_{3}, \underbrace{1}_{3}, \underbrace{\infty}_{1}\}.$$

$$\{\infty,\ 0,\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ 1,\ \infty\}.$$
$$1\quad 2\quad 2\quad 3\quad 3\quad 1$$

It is easy to check that there is no Farey symbol in this case in which the denominators are symmetric around $\frac{1}{2}$. On the other hand there is a *-Farey symbol which has such symmetry. Namely it is

$$\{\infty,\ 0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4},\ 1,\ \infty\}.$$
$$1\quad 2\quad 3\quad 3\quad 2\quad 1$$

Similarly a Farey symbol for $\Gamma^1(8)$ is

$$\{\infty,\ 0,\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{3}{5},\ \tfrac{5}{8},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ 1,\ \infty\}.$$
$$1\quad 2\quad 3\quad 3\quad 4\quad 4\quad 5\quad 5\quad 2\quad 1$$

Again it is easily checked that there is no Farey symbol in this case in which the denominators are symmetric around $\frac{1}{2}$. On the other hand a *-Farey symbol with this symmetry is

$$\{\infty,\ 0,\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{1}{2},\ \tfrac{5}{8},\ \tfrac{2}{3},\ \tfrac{3}{4},\ 1,\ \infty\}.$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1$$

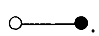
Even when N is a prime p, in contrast with $\Gamma_0(p)$, in general $\Gamma^1(p)$ does not admit a special polygon with a reflection symmetry. This happens already for $p = 5$.

In Appendix 4 we have described the *-Farey symbols for $\Gamma_0(N)$, for the values of $N \leq 25$ which are not primes, and in Appendix 5 the *-Farey symbols for $\Gamma^1(N)$, for the values $4 \leq N \leq 12$.

Appendix 1 subgroups of index $\leq$ 6 (A1.1). A special polygon for Γ is the triangle with vertices ∞, 0, and $\rho = e^{\pi i/3}$. Its Farey symbol is

$$\{\infty,\ 0,\ \infty\}.$$
$$\circ \quad \bullet$$

The tree diagram is



(A1.2). The unique subgroup of index 2 in Γ has a Farey symbol

$$\{\infty,\ 0,\ \infty\}.$$
$$\bullet \quad \bullet$$

The tree diagram is

(A1.3). The possible tree diagrams for index 3 subgroups are

The Farey symbol

$$\begin{array}{c}\{\infty,\quad 0,\quad 1,\quad \infty\}\\ \smile\smile\smile\\ \circ\quad\ \circ\quad\ \circ\end{array}$$

corresponds to the first possibility. It is a normal subgroup. Its independent generators are

$$\left\langle z \mapsto -\frac{1}{z},\ z \mapsto \frac{z-1}{2z-1},\ z \mapsto \frac{z-2}{z-1}\right\rangle.$$

The Farey symbols

$$\begin{array}{c}\{\infty,\quad 0,\quad 1,\quad \infty\},\\ \smile\smile\smile\\ \circ\quad\ 1\quad\ 1\end{array}\qquad \begin{array}{c}\{\infty,\quad 0,\quad 1,\quad \infty\},\\ \smile\smile\smile\\ 1\quad\ 1\quad\ \circ\end{array}\qquad \begin{array}{c}\{\infty,\quad 0,\quad 1,\quad \infty\}\\ \smile\smile\smile\\ 1\quad\ \circ\quad\ 1\end{array}$$

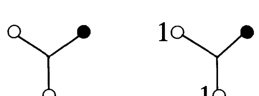
correspond to three subgroups in the same conjugacy class. These subgroups are respectively

$$\Gamma_\theta = \left\langle z \mapsto -\frac{1}{z},\ z \mapsto \frac{2z-1}{z}\right\rangle,$$

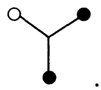
$$\Gamma^0(2) = \left\langle z \mapsto \frac{-z}{z-1},\ z \mapsto \frac{z-2}{z-1}\right\rangle,$$

$$\Gamma_0(2) = \left\langle z \mapsto z+1,\ z \mapsto \frac{z-1}{2z-1}\right\rangle.$$

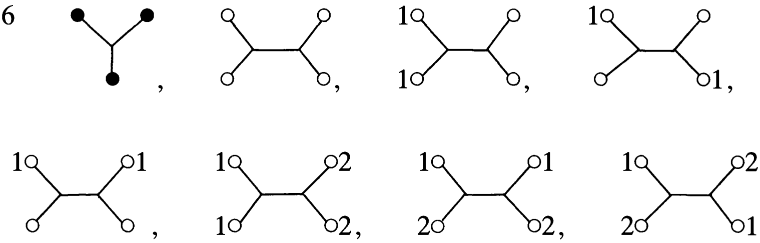
(A1.4). We list the tree diagrams for the subgroups of indices 4 through 6. The corresponding Farey symbols for the subgroups in the corresponding conjugacy classes and their independent generators may be found by the procedures in Section 4 and Section 6.

$d = 4$ 

$d = 5$



$d = 6$



Appendix 2. Balanced Farey symbols for $4 \leq N \leq 12$. In these symbols we label the intervals by the pairing which is a part of the definition of a balanced Farey sequence. We agree not to list the initial and final ∞. As it stands it is a Farey symbol for $\Gamma^1(N)$. To obtain the Farey symbol or the corresponding special polygon for $\Gamma(N)$ we shall need to follow the procedure of Theorem (12.2). The following constructions were made by using the remarks in Section 5 and Section 8, especially (5.7), (8.12), to generate candidates and then checking them by some trial and error.

$N = 4$ $\quad \{0,\ \tfrac{1}{2},\ 1\}.$

$\quad\quad 1 \quad 1$

$N = 5$ $\quad \{0,\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2},\ 1\}.$

$\quad\quad 1 \quad 2 \quad 2 \quad 1$

$N = 6$ $\quad \{0,\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ 1\}.$

$\quad\quad 1 \quad 2 \quad 2 \quad 1$

$N = 7$ $\quad \{0,\ \tfrac{1}{2},\ \tfrac{4}{7},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{5}{7},\ \tfrac{3}{4},\ \tfrac{4}{5},\ 1\}.$

$\quad\quad 1 \quad 2 \quad 2 \quad 3 \quad 4 \quad 4 \quad 3 \quad 1$

$$N = 8 \qquad \{0, \underbrace{\ }_{1} \tfrac{1}{3}, \underbrace{\ }_{2} \tfrac{1}{2}, \underbrace{\ }_{2} \tfrac{3}{5}, \underbrace{\ }_{3} \tfrac{5}{8}, \underbrace{\ }_{3} \tfrac{2}{3}, \underbrace{\ }_{4} \tfrac{3}{4}, \underbrace{\ }_{4} \tfrac{4}{5}, \underbrace{\ }_{1} 1\}.$$

$$N = 9 \qquad \{0, \underbrace{\ }_{1} \tfrac{1}{6}, \underbrace{\ }_{2} \tfrac{1}{5}, \underbrace{\ }_{3} \tfrac{2}{9}, \underbrace{\ }_{3} \tfrac{1}{4}, \underbrace{\ }_{4} \tfrac{1}{3}, \underbrace{\ }_{4} \tfrac{2}{5}, \underbrace{\ }_{5} \tfrac{1}{2}, \underbrace{\ }_{6} \tfrac{5}{9}, \underbrace{\ }_{6} \tfrac{4}{7}, \underbrace{\ }_{5} \tfrac{3}{5}, \underbrace{\ }_{2} \tfrac{2}{3}, \underbrace{\ }_{1} 1\}.$$

$$N = 10 \qquad \{0, \underbrace{\ }_{1} \tfrac{1}{7}, \underbrace{\ }_{2} \tfrac{1}{6}, \underbrace{\ }_{3} \tfrac{1}{5}, \underbrace{\ }_{3} \tfrac{1}{4}, \underbrace{\ }_{2} \tfrac{2}{7}, \underbrace{\ }_{4} \tfrac{3}{10}, \underbrace{\ }_{4} \tfrac{1}{3}, \underbrace{\ }_{5} \tfrac{1}{2}, \underbrace{\ }_{6} \tfrac{3}{5}, \underbrace{\ }_{6} \tfrac{5}{8}, \underbrace{\ }_{5} \tfrac{2}{3}, \underbrace{\ }_{1} 1\}.$$

$$N = 11 \qquad \{0, \underbrace{\ }_{1} \tfrac{1}{7}, \underbrace{\ }_{2} \tfrac{1}{6}, \underbrace{\ }_{3} \tfrac{2}{11}, \underbrace{\ }_{3} \tfrac{1}{5}, \underbrace{\ }_{4} \tfrac{1}{4}, \underbrace{\ }_{5} \tfrac{1}{3}, \underbrace{\ }_{6} \tfrac{2}{5}, \underbrace{\ }_{2} \tfrac{3}{7}, \underbrace{\ }_{7} \tfrac{1}{2}, \underbrace{\ }_{8} \tfrac{6}{11}, \underbrace{\ }_{8} \tfrac{5}{9}, \underbrace{\ }_{7} \tfrac{4}{7}, \underbrace{\ }_{4} \tfrac{3}{5}, \underbrace{\ }_{6} \tfrac{5}{8}, \underbrace{\ }_{9} \tfrac{7}{11}, \underbrace{\ }_{9} \tfrac{2}{3}, \underbrace{\ }_{5} \tfrac{5}{7}, \underbrace{\ }_{10} \tfrac{8}{11}, \underbrace{\ }_{10} \tfrac{3}{4}, \underbrace{\ }_{1} 1\}.$$

$$N = 12 \qquad \{0, \underbrace{\ }_{1} \tfrac{1}{8}, \underbrace{\ }_{2} \tfrac{1}{7}, \underbrace{\ }_{3} \tfrac{1}{6}, \underbrace{\ }_{3} \tfrac{1}{5}, \underbrace{\ }_{4} \tfrac{2}{9}, \underbrace{\ }_{5} \tfrac{1}{4}, \underbrace{\ }_{5} \tfrac{1}{3}, \underbrace{\ }_{4} \tfrac{2}{5}, \underbrace{\ }_{6} \tfrac{1}{2}, \underbrace{\ }_{6} \tfrac{4}{7}, \underbrace{\ }_{7} \tfrac{7}{12}, \underbrace{\ }_{7} \tfrac{3}{5}, \underbrace{\ }_{8} \tfrac{2}{3}, \underbrace{\ }_{8} \tfrac{5}{7}, \underbrace{\ }_{2} \tfrac{3}{4}, \underbrace{\ }_{1} 1\}.$$

Appendix 3. Semibalanced Farey symbols (A3.1). For simplicity we shall omit the initial and final ∞ in a semi-balanced Farey symbol. We have already considered $\Gamma_0(2)$ in (A1.3). Moreover $\Gamma_0(3)$ has index 4 and has a Farey symbol

$$\{\underbrace{0,\quad 1}_{\bullet}\}. \qquad (\textit{Note}: 1^2 + 1\cdot 1 + 1^2 \equiv 0(3).)$$

(A3.2). $\Gamma_0(9)$ has index 12 and has $r = 3$, $e_2 = e_3 = 0$. It has a Farey symbol

$$\begin{array}{ccccccccc} \{0, & & \tfrac{1}{3}, & & \tfrac{1}{2}, & & \tfrac{2}{3}, & & 1\}. \\ & 1 & & 1 & & 1' & & 1' & \end{array}$$

Correspondingly its independent generators are

$$z \mapsto z + 1,$$

$$z \mapsto \frac{2z - 1}{9z - 4}, \qquad (\textit{Note}: 1\cdot 3 + 3\cdot 2 \equiv 0(9).)$$

$$z \mapsto \frac{5z - 4}{9z - 7}, \qquad (\textit{Note}: 2\cdot 3 + 3\cdot 1 \equiv 0(9).)$$

(A3.3). $\Gamma_0(12)$ has index 24 and has $r = 5$, $e_2 = e_3 = 0$. It has a Farey symbol

$$\begin{array}{ccccccccccccccccc} \{0, & & \tfrac{1}{6}, & & \tfrac{1}{5}, & & \tfrac{1}{4}, & & \tfrac{1}{3}, & & \tfrac{1}{2}, & & \tfrac{2}{3}, & & \tfrac{3}{4}, & & 1\}. \\ & 1 & & 1 & & 2 & & 3 & & 4 & & 4 & & 3 & & 2 & \end{array}$$

Correspondingly its independent generators are

$$z \mapsto z + 1,$$

$$z \mapsto \frac{5z - 1}{36z - 7}, \qquad (\textit{Note}: 1\cdot 6 + 6\cdot 5 \equiv 0(12).)$$

$$z \mapsto \frac{5z - 4}{24z - 19}, \qquad (\textit{Note}\colon 5 \cdot 4 + 4 \cdot 1 \equiv 0(12).)$$

$$z \mapsto \frac{7z - 5}{24z - 17}, \qquad (\textit{Note}\colon 4 \cdot 3 + 3 \cdot 4 \equiv 0(12).)$$

$$z \mapsto \frac{5z - 3}{12z - 7}, \qquad (\textit{Note}\colon 3 \cdot 2 + 2 \cdot 3 \equiv 0(12).)$$

(A3.4). Now we shall concentrate on the case of $\Gamma_0(p)$, where p is a prime. Here is some basic data.

i) $p \equiv 1(3), p \equiv 1(4)$. *Then* $\Gamma_0(p)$ *has* $r = \frac{1}{6}(p - 7)$, $e_2 = 2$, $e_3 = 2$.

ii) $p \equiv 1(3)$, $p \equiv -1(4)$. *Then* $\Gamma_0(p)$ *has* $r = \frac{1}{6}(p - 1)$, $e_2 = 0$, $e_3 = 2$.

iii) $p \equiv -1(3)$, $p \equiv 1(4)$. *Then* $\Gamma_0(p)$ *has* $r = \frac{1}{6}(p + 1)$, $e_2 = 2$, $e_3 = 0$.

iv) $p \equiv -1(3)$, $p \equiv -1(4)$. *Then* $\Gamma_0(p)$ *has* $r = \frac{1}{6}(p + 7)$, $e_2 = 0$, $e_3 = 0$.

Also in cases i) and ii) a semi-balanced Farey sequence has $\frac{p+2}{3}$ terms, and in cases iii) and iv) it has $\frac{p+4}{3}$ terms.

(A3.5). We shall now list the Farey sequences which are semi-balanced for the primes ≤ 100. They exhibit certain remarkable features. One of these features, as observed in the theorem in (13.5), is that *the denominators in this Farey symbol have a symmetry around* $\frac{1}{2}$, *or what is the same the corresponding special polygon has a reflection symmetry in the vertical line* $x = \frac{1}{2}$, $y > 0$.

Due to this symmetry it will be convenient to reduce the Farey symbol to a symbol of the form

$$\{\underbrace{0,\ x_1}_{\circ}\underbrace{,\ x_2}_{1},\ \ldots\ldots\ldots \underbrace{.\,\tfrac{1}{2}}_{\bullet} \mid \text{reflect}\}.$$

Here the even intervals and the odd intervals are symmetrically placed around $\frac{1}{2}$. In fact an even interval (resp. an odd interval) occurs exactly once to the left of $\frac{1}{2}$ iff $p \equiv 1(4)$, (resp. $p \equiv 1(3)$); otherwise it does not occur. On the other hand if there is a free interval with a label a to the left of $\frac{1}{2}$ then its reflection in $\frac{1}{2}$ has necessarily a *different* label a'—they are *not* paired. On the left of $\frac{1}{2}$ there will be another free interval with a label either a or a', and its reflected interval will have a label a' or a respectively.

Example. A Farey symbol for $\Gamma_0(41)$ is

$$\underbrace{\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{5}{7},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{5}{6},\ 1\}}_{1\quad 1'\quad \circ\quad 2\quad 3\quad 2'\quad 3\quad 3'\quad 2\quad 3'\quad 2'\quad \circ\quad 1\quad 1'}.$$

This will be shortened to

$$\underbrace{\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2}}_{1\quad 1'\quad \circ\quad 2\quad 3\quad 2'\quad 3}\ |\ \text{reflect}\}.$$

(A3.6). In the definition of a semi-balanced Farey sequence for N, cf. (12.1), we require the congruences $b_i^2 + b_{i+1}^2 \equiv 0(N)$, to be satisfied by the denominators. For N = a prime ≤ 100 it is possible to construct semi-balanced Farey sequences where *these congruences can be lifted to equalities in natural numbers*! The author has not been able to justify these on general grounds.

Example. A Farey symbol for $\Gamma_0(43)$ is

$$\underbrace{\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2}}_{\bullet\quad 1\quad 2\quad 3\quad 2'\quad 1\quad 3}\ |\ \text{reflect}\}.$$

Corresponding to the odd interval $\{0, \frac{1}{6}\}$ we have not only $1^2 + 1\cdot 6 + 6^2 \equiv 0(43)$ but actually $1^2 + 1\cdot 6 + 6^2 = 43$. Similarly corresponding to the paired free intervals $\{\frac{1}{6}, \frac{1}{5}\}$ and $\{\frac{1}{3}, \frac{2}{5}\}$ we have not only $6\cdot 3 + 5\cdot 5 \equiv 0(43)$ but actually $6\cdot 3 + 5\cdot 5 = 43$.

(A3.7). *The modified Farey symbols for* $\Gamma_0(p)$, *where* p *is a prime*, $5 \leq p \leq 100$: The following Farey symbols were constructed by hand. The following empirical rules were followed. i) The first nonzero term was chosen to be $\frac{1}{a}$ where $a = [\sqrt{p}]$. ii) Whenever a, b are the denominators in two consecutive terms we solved $ax + by = p$, in natural numbers x, y and we looked for terms with x, y in the denominators. The rest of the terms were more or less forced by the remarks in (8.12), (A3.5) and (A3.6).

$p = 5.$
$$\begin{array}{ccc} \{0, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & \circ & \end{array}$$

$p = 7.$
$$\begin{array}{ccc} \{0, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & \bullet & \end{array}$$

$p = 11.$
$$\begin{array}{ccccc} \{0, & & \tfrac{1}{3}, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & 1 & & 1' & \end{array}$$

$p = 13.$
$$\begin{array}{ccccc} \{0, & & \tfrac{1}{3}, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & \bullet & & \circ & \end{array}$$

$p = 17.$
$$\begin{array}{ccccccc} \{0, & & \tfrac{1}{4}, & & \tfrac{1}{3}, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & \circ & & 1 & & 1' & \end{array}$$

$p = 19.$
$$\begin{array}{ccccccc} \{0, & & \tfrac{1}{4}, & & \tfrac{1}{3}, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & 1 & & 1' & & \bullet & \end{array}$$

$p = 23.$
$$\begin{array}{ccccccccc} \{0, & & \tfrac{1}{4}, & & \tfrac{1}{3}, & & \tfrac{2}{5}, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & 1 & & 2 & & 1' & & 2 & \end{array}$$

$p = 29.$
$$\begin{array}{ccccccccccc} \{0, & & \tfrac{1}{5}, & & \tfrac{1}{4}, & & \tfrac{1}{3}, & & \tfrac{2}{5}, & & \tfrac{1}{2} \mid \text{reflect}\}. \\ & 1 & & 1' & & 2 & & 2' & & \circ & \end{array}$$

$p = 31.$ $\{0,\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$\bullet \quad 1 \quad 1' \quad 2 \quad 2'$

$p = 37.$ $\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$\circ \quad 1 \quad 2 \quad \bullet \quad 2' \quad 1'$

$p = 41.$ $\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$1 \quad 1' \quad \circ \quad 2 \quad 3 \quad 2' \quad 3$

$p = 43.$ $\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$\bullet \quad 1 \quad 2 \quad 3 \quad 2' \quad 1 \quad 3$

$p = 47.$ $\{0,\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$1 \quad 2 \quad 3 \quad 4 \quad 3 \quad 4 \quad 1 \quad 2'$

$p = 53.$ $\{0,\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$1 \quad 2 \quad 3 \quad 4 \quad 1 \quad 3' \quad 2' \quad 4 \quad \circ$

$p = 59.$ $\{0,\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$1 \quad 2 \quad 3 \quad 2 \quad 3 \quad 4 \quad 1 \quad 4' \quad 5 \quad 5'$

$p = 61.$ $\{0,\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$1 \quad 2 \quad \circ \quad \bullet \quad 3 \quad 3' \quad 4 \quad 1' \quad 4' \quad 2$

$p = 67.$ $\{0,\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$

$1 \quad 2 \quad 3 \quad 4 \quad 2' \quad 5 \quad 3 \quad 1 \quad 5 \quad 4' \quad \bullet$

$$p = 71. \qquad \{0,\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{3}{11},\ \tfrac{2}{7},\ \tfrac{3}{10},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$$

$$1 \quad 1' \quad 2 \quad 2' \quad 3 \quad 3' \quad 4 \quad 5 \quad 6 \quad 5 \quad 6 \quad 4'$$

$$p = 73. \qquad \{0,\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{4}{9},\ \tfrac{1}{2} \mid \text{reflect}\}.$$

$$\bullet \quad 1 \quad 2 \quad 3 \quad 4 \quad 2' \quad 1' \quad \circ \quad 3 \quad 5 \quad 4' \quad 5'$$

$$p = 79. \qquad \{0,\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{2}{9},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{4}{9},\ \tfrac{1}{2} \mid \text{reflect}\}.$$

$$1 \quad 2 \quad 3 \quad 4 \quad 4' \quad 5 \quad 5' \quad \bullet \quad 6 \quad 6' \quad 3' \quad 1 \quad 2'$$

$$p = 83. \qquad \{0,\ \tfrac{1}{9},\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{3}{11},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{4}{9},\ \tfrac{1}{2} \mid \text{reflect}\}.$$

$$1 \quad 2 \quad 3 \quad 4 \quad 3 \quad 5 \quad 6 \quad 5 \quad 2' \quad 7 \quad 4' \quad 6' \quad 7' \quad 1'$$

$$p = 89. \qquad \{0,\ \tfrac{1}{9},\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{2}{9},\ \tfrac{3}{13},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{3}{10},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2} \mid \text{reflect}\}.$$

$$1 \quad 1' \quad 2 \quad 3 \quad 4 \quad 3 \quad 5 \quad 4' \quad 6 \quad 6' \quad 7 \quad 7' \quad \circ \quad 2 \quad 5$$

$$p = 97. \qquad \{0,\ \tfrac{1}{9},\ \tfrac{1}{8},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{2}{9},\ \tfrac{1}{4},\ \tfrac{3}{11},\ \tfrac{2}{7},\ \tfrac{3}{10},\ \tfrac{1}{3},\ \tfrac{3}{8},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{4}{9},\ \tfrac{1}{2} \mid \text{reflect}\}.$$

$$1 \quad 2 \quad 3 \quad 3' \quad 4 \quad 5 \quad \circ \quad 6 \quad 4' \quad 1 \quad 7 \quad \bullet \quad 5' \quad 6 \quad 7 \quad 2$$

Appendix 4. *-Farey symbols for $\Gamma_0(N)$, $N \leq 25$, $N \neq$ a prime. Again it will be convenient to omit the initial and final ∞ from these *-Farey symbols. Moreover as in (A3.5)—now using the theorem in (14.11)—we shall reduce the symbol to the form

$$\{\underbrace{0,\ x_1}_{1}\underset{\circ}{}\underbrace{,\ x_2}_{2},\ \ldots\ldots\underbrace{,\ \tfrac{1}{2}}_{3} \mid \text{reflect}\}.$$

In contrast to the observations in the case when N is a prime, cf. (A3.5), it is now possible that if there is a free interval with a label a to the left of $\frac{1}{2}$ then its reflection in $\frac{1}{2}$ may have a label a or a different label a'. If the free interval and its reflection have different labels a and a' then again on the left of $\frac{1}{2}$ there will be another free interval with a label either a or a', and its reflected interval will have a label a' or a respectively.

$N = 4$. $$\{\underbrace{0,\ \tfrac{1}{2}}_{1} \mid \text{reflect}\}.$$

$N = 6$. $$\{\underbrace{0,\ \tfrac{1}{3}}_{1}\underbrace{,\ \tfrac{1}{2}}_{2} \mid \text{reflect}\}.$$

$N = 8$. $$\{\underbrace{0,\ \tfrac{1}{4}}_{1}\underbrace{,\ \tfrac{1}{2}}_{2} \mid \text{reflect}\}.$$

$N = 9$. $$\{\underbrace{0,\ \tfrac{1}{3}}_{1}\underbrace{,\ \tfrac{1}{2}}_{1} \mid \text{reflect}\}.$$

$N = 10$. $$\{\underbrace{0,\ \tfrac{1}{3}}_{\circ}\underbrace{,\ \tfrac{2}{5}}_{1}\underbrace{,\ \tfrac{1}{2}}_{2} \mid \text{reflect}\}.$$

$N = 12.$ $\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 3 4

$N = 14.$ $\{0, \frac{1}{5}, \frac{1}{3}, \frac{3}{7}, \frac{1}{2} \mid \text{reflect}\}.$

1 1′ 2 3

$N = 15.$ $\{0, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 3 1

$N = 16.$ $\{0, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 3 4

$N = 18.$ $\{0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{9}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 2 3 4 1

$N = 20.$ $\{0, \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 3 4 5 3′

$N = 21.$ $\{0, \frac{1}{4}, \frac{4}{15}, \frac{2}{7}, \frac{1}{3}, \frac{1}{2} \mid \text{reflect}\}.$

● 1 2 3 1

$N = 22.$ $\{0, \frac{1}{5}, \frac{1}{4}, \frac{3}{11}, \frac{2}{7}, \frac{1}{3}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 3 4 1 2′

$N = 24.$ $\{0, \frac{1}{12}, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \mid \text{reflect}\}.$

1 2 3 4 5 6 7 3

$N = 25.$ $\{0, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2} \mid \text{reflect}\}.$

○ 1 1 2 2

Appendix 5. *-Farey symbols for $\Gamma^1(N)$, $4 \le N \le 12$. In writing these symbols we follow the pattern used in the previous appendix.

$N = 4$. $$\{0 \underbrace{\quad}_{1} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 5$. $$\{0 \underbrace{\quad}_{1} \tfrac{2}{5} \underbrace{\quad}_{2} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 6$. $$\{0 \underbrace{\quad}_{1} \tfrac{1}{3} \underbrace{\quad}_{2} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 7$. $$\{0 \underbrace{\quad}_{1} \tfrac{2}{7} \underbrace{\quad}_{2} \tfrac{1}{3} \underbrace{\quad}_{3} \tfrac{3}{7} \underbrace{\quad}_{4} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 8$. $$\{0 \underbrace{\quad}_{1} \tfrac{1}{4} \underbrace{\quad}_{2} \tfrac{1}{3} \underbrace{\quad}_{3} \tfrac{3}{8} \underbrace{\quad}_{4} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 9$. $$\{0 \underbrace{\quad}_{1} \tfrac{2}{9} \underbrace{\quad}_{2} \tfrac{1}{4} \underbrace{\quad}_{3} \tfrac{1}{3} \underbrace{\quad}_{3} \tfrac{2}{5} \underbrace{\quad}_{4} \tfrac{4}{9} \underbrace{\quad}_{5} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 10$. $$\{0 \underbrace{\quad}_{1} \tfrac{1}{5} \underbrace{\quad}_{2} \tfrac{1}{4} \underbrace{\quad}_{3} \tfrac{3}{10} \underbrace{\quad}_{4} \tfrac{1}{3} \underbrace{\quad}_{5} \tfrac{2}{5} \underbrace{\quad}_{6} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 11$. $$\{0 \underbrace{\quad}_{1} \tfrac{2}{11} \underbrace{\quad}_{2} \tfrac{1}{5} \underbrace{\quad}_{3} \tfrac{1}{4} \underbrace{\quad}_{4} \tfrac{3}{11} \underbrace{\quad}_{5} \tfrac{1}{3} \underbrace{\quad}_{6} \tfrac{4}{11} \underbrace{\quad}_{7} \tfrac{2}{5} \underbrace{\quad}_{3'} \tfrac{3}{7} \underbrace{\quad}_{8} \tfrac{5}{11} \underbrace{\quad}_{9} \tfrac{1}{2} \mid \text{reflect}\}.$$

$N = 12$. $$\{0 \underbrace{\quad}_{1} \tfrac{1}{6} \underbrace{\quad}_{2} \tfrac{1}{5} \underbrace{\quad}_{3} \tfrac{1}{4} \underbrace{\quad}_{4} \tfrac{1}{3} \underbrace{\quad}_{4} \tfrac{3}{8} \underbrace{\quad}_{3'} \tfrac{2}{5} \underbrace{\quad}_{5} \tfrac{5}{12} \underbrace{\quad}_{6} \tfrac{1}{2} \mid \text{reflect}\}.$$

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