ΜΑΘΗΜΑ 20

ΤΕΧΝΙΚΕΣ ΟΛΟΚΛΗΡΩΣΗΣ

(Α) ΟΛΟΚΛΗΡΩΜΑ ΠΑΡΑΓΩΓΟΥ (Β΄ΘΘ)

$$\boxed{\int f' = f + c}$$

$$\int x^a dx = \frac{x^{a+1}}{a+1} + c, \quad a \neq -1.$$

$$\int x^{-1} dx = \int \frac{dx}{x} = \ln|x| + c$$

$$\int e^x dx = e^x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + c$$

$$\int \frac{1}{\sin^2 x} dx = -\cot x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

$$\int \frac{dx}{1+x^2} = \arctan x + c$$

(B) ΟΛΟΚΛΗΡΩΣΗ ΜΕ ΑΝΤΙΚΑΤΑΣΤΑΣΗ

B1 $\int f(\phi(x))\cdot\phi'(x)dx = \int f(\phi(x))d\phi(x) = \int f(u)du$.

Θέτουμε $u := \phi(x) \Rightarrow du = d\phi(x) = \phi'(x)dx$

Παραδείγματα:

(α) $\int \frac{\arctan x}{1+x^2}dx = \circledast$

$\left[u := \arctan x \Rightarrow du = \frac{dx}{1+x^2}\right]$

$\circledast = \int u\,du = \frac{u^2}{2} + c = \frac{(\arctan x)^2}{2} + c$.

(β) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \circledast$

$\left[u := \cos x \Rightarrow du = -\sin x\,dx\right]$

$\circledast = -\int \frac{-\sin x\,dx}{\cos x} = -\int \frac{du}{u} = -\ln|u| + c =$

$= -\ln|\cos x| + c$.

(γ) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \circledast$

$\left[u := \sqrt{x} = x^{1/2} \Rightarrow du = \frac{1}{2}x^{-1/2}dx = \frac{dx}{2\sqrt{x}}\right]$

$\circledast = \int \frac{2\cos\sqrt{x}}{2\sqrt{x}}dx = \int 2\cos u\,du = 2\int \cos u\,du =$

$= 2\sin u + c = 2\sin\sqrt{x} + c$

B2 Τριγωνομετρικά ολοκληρώματα: απλοποιούνται βάσει των ταυτοτήτων:

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

$$\boxed{1 + \tan^2 x = \frac{1}{\cos^2 x}}$$

$$1 + \cot^2 x = \frac{1}{\sin^2 x}$$

(1)

$$\boxed{\cos^2 x = \frac{1 + \cos 2x}{2}}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

(2)

$$\boxed{\sin 2x = 2 \sin x \cos x}$$

(3)

$$\sin ax \cdot \sin bx = \frac{\cos(a-b)x - \cos(a+b)x}{2}$$

$$\sin ax \cdot \cos bx = \frac{\sin(a+b)x + \sin(a-b)x}{2}$$

$$\boxed{\cos ax \cdot \cos bx = \frac{\cos(a+b)x + \cos(a-b)x}{2}}$$

(4)

Παραδείγματα:

(α) Υπολογισμός $\int \sin^k x \cos^\lambda x \, dx$ όταν ο ένας εκθέτης είναι περιττός (και ο άλλος άρτιος).

$$\int \cos^3 x \sin^4 x \, dx = \int (1 - \sin^2 x) \sin^4 x \, \underline{\cos x \, dx} = (*)$$

$$(*) = \int (1-u^2)u^4\,du = \int u^4\,du - \int u^6\,du =$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + c = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c.$$

(β) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \frac{1}{2}\,dx + \frac{1}{2}\int \cos 2x\,dx =$$

$$\left[\begin{array}{l} u = 2x \Rightarrow \\ du = 2dx \end{array}\right]$$

$$= \frac{x}{2} + \frac{1}{2}\int \cos u\,\frac{du}{2} =$$

$$= \frac{x}{2} + \frac{1}{4}\int \cos u\,du = \frac{x}{2} + \frac{\sin u}{4} + c =$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + c$$

(γ) Δύο χρήσιμα αποτελέσματα:

$$\int \tan^2 x\,dx = \int \left(\frac{1}{\cos^2 x} - 1\right)dx = \int \left((\tan x)' - 1\right)dx =$$

$$= \int (\tan x)'\,dx - \int 1\,dx = \tan x - x + c.$$

$$\int \cot^2 x\,dx = \int \left(\frac{1}{\sin^2 x} - 1\right)dx = \int \left((-\cot x)' - 1\right)dx =$$

$$= \int (-\cot x)'\,dx - \int 1\,dx = -\cot x - x + c.$$

ή:

$$\int \tan^2 x\,dx = \int (1+\tan^2 x - 1)\,dx = \int ((\tan x)' - 1)\,dx = \tan x - x + c,$$

$$\int \cot^2 x\,dx = \int (\cot^2 x + 1 - 1)\,dx = \int ((-\cot x)' - 1)\,dx = -\cot x - x + c.$$

$\boxed{B3}$ $\int f(x)dx = \int f(\phi(t))\,d\phi(t) = \int f(\phi(t))\phi'(t)\,dt.$

(Μέθοδος Β1, ανάποδα!)

Παραδείγματα

(α) $\sqrt{a^2-x^2} \rightsquigarrow x = a\sin t$ $\quad (t\in[-\pi/2,\pi/2],\ a>0.)$

$\Rightarrow \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 t} = a\cos t$ &

$dx = a\,d\sin t = a\cos t\,dt.$

πχ $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}} = (*)$ $\quad\left[\begin{array}{l} x:=3\sin t, \quad t\in[-\pi/2,\pi/2] \\ dx = 3\cos t\,dt \\ \sqrt{9-x^2} = \sqrt{9-9\sin^2 t} = 3\cos t. \end{array}\right]$

$\displaystyle (*) = \int \frac{3\cos t\,dt}{9\sin^2 t\, 3\cos t} = \frac{1}{9}\int\frac{dt}{\sin^2 t} = -\frac{1}{9}\cot t + c \quad (**).$

$\displaystyle \cot t = \frac{\cos t}{\sin t} = \frac{\sqrt{1-\sin^2 t}}{\sin t} = \frac{\sqrt{9-9\sin^2 t}}{3\sin t} = \frac{\sqrt{9-x^2}}{x}$

$\displaystyle (**) = -\frac{1}{9}\cdot\frac{\sqrt{9-x^2}}{x} + c$

(β) $\sqrt{x^2-a^2} \rightsquigarrow x = a/\cos t.$ $\quad (t\in[0,\pi/2)$ ή $t\in(\pi/2,\pi]$ $)$ $\quad a>0$

$\displaystyle \Rightarrow dx = \frac{a\sin t}{\cos^2 t}dt$ & $\sqrt{x^2-a^2} = \sqrt{a^2/\cos^2 t - a^2} = a\tan t$

πχ $\displaystyle\int \frac{\sqrt{x^2-4}}{x}dx = (*)$

$\displaystyle \Big[\ x := \frac{2}{\cos t} \Rightarrow dx = \frac{2\sin t}{\cos^2 t}dt$ και

$$ⓐ = \int \frac{2\tan t}{\frac{2}{\cos t}} \cdot \frac{2\sin t}{\cos^2 t} dt = 2\int \frac{\sin^2 t}{\cos^2 t} dt =$$

$$= 2\int \tan^2 t \, dt = 2(\tan t - t + c) \quad ⓐⓐ$$

$$2\tan t = \sqrt{x^2-4} \Rightarrow \tan t = \frac{\sqrt{x^2-4}}{2} \Rightarrow t = \arctan\left(\frac{\sqrt{x^2-4}}{2}\right)$$

$$ⓐⓐ = 2\left(\frac{\sqrt{x^2-4}}{2} - \arctan\frac{\sqrt{x^2-4}}{2}\right) + c.$$

(γ) $\sqrt{x^2+a^2} \rightsquigarrow x = a\tan t \Rightarrow$

$\Rightarrow \sqrt{x^2+a^2} = a/\cos t$ & $dx = \frac{a}{\cos^2 t} dt.$

Πχ. $\int \frac{\sqrt{x^2+1}}{x^4} dx = ⓐ$

$$\left[\begin{array}{l} x = \tan t \\ dx = \frac{1}{\cos^2 t} dt \\ \sqrt{x^2+1} = \sqrt{\tan^2 t + 1} = \frac{1}{\cos t} \end{array}\right]$$

$$ⓐ = \int \frac{1}{\cos t} \cdot \frac{1}{\tan^4 t} \cdot \frac{1}{\cos^2 t} dt = \int \frac{\cos t}{\sin^4 t} dt =$$

$$\left[\begin{array}{l} u := \sin t \\ du = \cos t\, dt \end{array}\right]$$

$$= \int \frac{du}{u^4} = \int u^{-4} du = \frac{u^{-3}}{-3} = -\frac{1}{3} \frac{1}{\sin^3 t} + c = ⓐⓐ$$

$$x = \tan t \Rightarrow \sin t = x \cdot \cos t = x \frac{1}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

$$ⓐⓐ = -\frac{1}{3} \cdot \frac{\sqrt{x^2+1}^{\,3}}{x^3} + c.$$

$$\cos t = \frac{1}{\sqrt{1+\tan^2 t}}$$

Γ) ΟΛΟΚΛΗΡΩΣΗ ΚΑΤΑ ΜΕΡΗ / ΠΑΡΑΓΟΝΤΙΚΗ ΟΛΟΚΛΗΡΩΣΗ

$$\int fg' = fg - \int f'g.$$

Παραδείγματα

(α) $\int x\ln x\,dx = \int \left(\frac{x^2}{2}\right)' \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}(\ln x)'\,dx =$

$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx =$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

(β) $\int x\cos x\,dx = \int x(\sin x)'\,dx = x\sin x - \int \sin x\,dx =$

$= x\sin x + \int (\cos x)'\,dx = x\sin x + \cos x + c$

(γ) $\int e^x \sin x\,dx =: I =$

$= \int (e^x)'\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx =$

$= e^x\sin x - \int (e^x)'\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

$= e^x(\sin x - \cos x) - I \Rightarrow 2I = e^x(\sin x - \cos x) \Rightarrow$

$\Rightarrow I = \frac{e^x(\sin x - \cos x)}{2}$

(δ) $\int x\sin^2 x\,dx = \int x\,\frac{1-\cos 2x}{2}\,dx = \int \frac{x}{2}\,dx - \int \frac{x\cos 2x}{2}\,dx =$

$u := 2x$

$$= \frac{x^2}{4} - \int \frac{u}{4}\cdot\cos u\cdot\frac{du}{2} =$$

$$= \frac{x^2}{4} - \frac{1}{8}\int u\cos u\,du =$$

$$= \frac{x^2}{4} - \frac{1}{8}\left(u\sin u + \cos u\right) + C$$

$$= \frac{x^2}{4} - \frac{1}{8}\left(2x\sin 2x + \cos 2x\right) + C$$

(ε) $\int \ln(x+\sqrt{x})\,dx = \int x'\ln(x+\sqrt{x})\,dx =$

$$= x\ln(x+\sqrt{x}) - \int x\cdot\frac{1}{x+\sqrt{x}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)dx =$$

$$= x\ln(x+\sqrt{x}) - \int \frac{x}{x+\sqrt{x}}\cdot\frac{2\sqrt{x}+1}{2\sqrt{x}}\,dx = (*)$$

$$\left[\begin{array}{l} u := \sqrt{x} \\ u^2 = x \Rightarrow dx = 2u\,du \end{array}\right]$$

$(*) = x\ln(x+\sqrt{x}) - \int \frac{u^2}{u^2+u}\cdot\frac{2u+1}{2u}\,2u\,du =$

$= x\ln(x+\sqrt{x}) - \int \frac{2u^2+u}{u+1}\,du =$ (διαίρεση πολ/νύμων)

$= x\ln(x+\sqrt{x}) - \int\left(2u-1+\frac{1}{u+1}\right)du =$

$= x\ln(x+\sqrt{x}) - u^2 + u - \int \frac{1}{u+1}\,d(u+1) =$

$= x\ln(x+\sqrt{x}) - x + \sqrt{x} - \ln|u+1| + c =$