ΜΑΘΗΜΑ 21

Α) ΟΛΟΚΛΗΡΩΣΗ ΡΗΤΩΝ ΣΥΝΑΡΤΗΣΕΩΝ.

$f(x) = \frac{p(x)}{q(x)}$, p, q πολυώνυμα. $\int f(x)dx = ?$

1º Βήμα διαιρούμε: $f(x) = \pi(x) + \frac{v(x)}{q(x)}$ με $\deg v < \deg q$.

$$\int f(x)dx = \int \underbrace{\pi(x)}_{\text{πολ/μο}}dx + \underbrace{\int \frac{v(x)}{q(x)}dx}_{=;}$$

Πχ: $\int \frac{x^3+2x+1}{x^2-1}dx = I = ;$

$$\frac{x^3+2x+1}{x^2-1} = \frac{x^3-x+3x+1}{x^2-1} = \frac{x(x^2-1)}{x^2-1} + \frac{3x+1}{x^2-1} =$$

$$= x + \frac{3x+1}{x^2-1}$$

$\pi(x)$ $\quad v(x) = 3x+1$. $\quad \deg v = 1 < \deg(x^2-1) = 2$.

$$\Downarrow$$

$$I = \int x dx + \int \frac{3x+1}{x^2-1}dx = \left(\frac{x^2}{2} + c\right) + \underbrace{\int \frac{3x+1}{x^2-1}dx}_{=;}$$

Ζητάω $\int \frac{v(x)}{q(x)}dx$ με $\deg v < \deg q$

2º Βήμα Παραγοντοποιώ παρονομαστή $q(x)$ σε γινόμενο παραγόντων
$(1^{ου}\text{ βαθμού})^k \cdot (2^{ου}\text{ βαθμού})^\lambda$.

3ο Βήμα Διασπώ το $\frac{\nu(x)}{q(x)}$ σε απλά κλάσματα.

→ Κάθε $(x-a)^k$ του $q(x)$ δίνει

$$\frac{a_1}{x-a}+\frac{a_2}{(x-a)^2}+\dots+\frac{a_k}{(x-a)^k}$$

πχ:

$$\frac{3x-1}{(x-1)^2}=\frac{a}{x-1}+\frac{\beta}{(x-1)^2}=\frac{ax-a}{(x-1)^2}+\frac{\beta}{(x-1)^2}=$$

$$=\frac{ax+(\beta-a)}{(x-1)^2} \Rightarrow a=3 \wedge \beta=2. \text{ Άρα}$$

$$\frac{3x-1}{(x-1)^2}=\frac{3}{x-1}+\frac{2}{(x-1)^2}.$$

Τότε:

$$\int\frac{dx}{(x-a)^k}=\begin{cases}-\frac{1}{k-1}\cdot\frac{1}{(x-a)^{k-1}}+c & k\geq 2.\\ \ln|x-a|+c & k=1.\end{cases}$$

→ Κάθε $(x^2+\beta x+\gamma)^\lambda$ δίνει όρους της μορφής (με $\Delta<0$)

$$\frac{Bx+\Gamma}{(x^2+\beta x+\gamma)^k} \quad 1\leq k\leq\lambda.$$

$$\int\frac{Bx+\Gamma}{(x^2+\beta x+\gamma)^k}\,dx=\,?$$

4ο Βήμα Στα $\int\frac{Bx+\Gamma}{(x^2+\beta x+\gamma)^k}\,dx$

προσθαφαιρώντας εμφανίζω στον αριθμητή την παράγωγο του ~~παρονομαστή~~ $x^2+\beta x+\gamma$: $2x+\beta$

$$Bx+\Gamma = \frac{B}{2}\cdot 2x+\Gamma = B/2\,(2x+\beta) - \frac{B\beta}{2}+\Gamma$$
$$= B/2\,(2x+\beta) + \left(\Gamma - \frac{B\beta}{2}\right)$$
$$\Downarrow$$
$$\int \frac{Bx+\Gamma}{(x^2+\beta x+\gamma)^k}\,dx = \frac{B}{2}\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx + \int \frac{\Gamma-\frac{B\beta}{2}}{(x^2+\beta x+\gamma)^k}\,dx$$

όπου:

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{d(x^2+\beta x+\gamma)}{(x^2+\beta x+\gamma)^k} = \begin{cases} -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} & k\geq \\ \ln(x^2+\beta x+\gamma) & k=1 \end{cases}$$

$$\int \frac{dx}{(x^2+\beta x+\gamma)^k} = ?$$

5ο βήμα Γράφω το $x^2+\beta x+\gamma$ σαν y^2+1.

$$x^2+\beta x+\gamma = x^2+2\cdot x\cdot \beta/2 + \beta^2/4 + \gamma - \beta^2/4 =$$
$$= \left(x+\beta/2\right)^2 + \frac{4\gamma-\beta^2}{4} =$$
$$= \frac{4\gamma-\beta^2}{4}\left[\frac{4}{4\gamma-\beta^2}\left(x+\beta/2\right)^2+1\right]$$
$$= \frac{4\gamma-\beta^2}{4}\left[\left(\underbrace{\frac{2}{\sqrt{4\gamma-\beta^2}}\left(x+\beta/2\right)}_{y}\right)^2+1\right]$$

Παρατ: $4\gamma-\beta^2 = -\Delta > 0$.

$y = \delta(x+\beta/2) \Rightarrow dy = \delta dx$ και $dx = \frac{dy}{\delta}$

Οπότε:

$$\int \frac{dx}{(x^2+\beta x+\gamma)^k} = \int \frac{1/\delta \, dy}{(y^2+1)^k} = \frac{1}{\delta}\int \frac{dy}{(y^2+1)^k} = ?$$

6ο βήμα Υπολογισμός του $I_k = \int \frac{dy}{(y^2+1)^k}$:

$$I_1 = \int \frac{dy}{y^2+1} = \arctan y + c$$

$$I_k = \int \frac{dy}{(y^2+1)^k} = \int y' \frac{1}{(y^2+1)^k} dy =$$

$$= \frac{y}{(y^2+1)^k} - \int y(-k)(y^2+1)^{-k-1} \cdot 2y \, dy$$

$$= \frac{y}{(y^2+1)^k} + 2k \int \frac{y^2 dy}{(y^2+1)^{k+1}} =$$

$$= \frac{y}{(y^2+1)^k} + 2k \int \frac{(y^2+1) - 1}{(y^2+1)^{k+1}} dy =$$

$$= \frac{y}{(y^2+1)^k} + 2k \int \frac{dy}{(y^2+1)^k} - 2k \int \frac{dy}{(y^2+1)^{k+1}} =$$

$$= \frac{y}{(y^2+1)^k} + 2k I_k - 2k \cdot I_{k+1} \Rightarrow$$

$$\Rightarrow 2k I_{k+1} = \frac{y}{(y^2+1)^k} + (2k-1) I_k \Rightarrow$$

$$\Rightarrow I_{k+1} = \frac{1}{2k}\left[\frac{y}{(y^2+1)^k} + (2k-1) I_k\right].$$

Αναδρομική σχέση!!

Παραδ.

$$\int \frac{2x-1}{x^2+x+1}dx = \int \frac{2x+1-2}{x^2+x+1}dx =$$

$(x^2+x+1)' = 2x+1$

$$= \int \frac{2x+1}{x^2+x+1}dx - 2\int \frac{dx}{x^2+x+1} =$$

$$= \int \frac{d(x^2+x+1)}{x^2+x+1} - 2\int \frac{dx}{x^2+x+1} =$$

$$= \left[\ln(x^2+x+1)+c\right] - \underbrace{2\int \frac{dx}{x^2+x+1}}_{=?}$$

$$x^2+x+1 = x^2+2\cdot x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} =$$

$$= \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right] =$$

$$= \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right] = \frac{3}{4}(y^2+1).$$

$$y = \frac{2}{\sqrt{3}}(x+1/2) \Rightarrow dy = \frac{2}{\sqrt{3}}dx \Rightarrow dx = \frac{\sqrt{3}}{2}dy.$$

$$\Downarrow$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{\sqrt{3}/2\, dy}{3/4\,(y^2+1)} = \frac{4}{3}\,\frac{\sqrt{3}}{2}\int \frac{dy}{y^2+1} =$$

$$= \frac{2}{\sqrt{3}}\int \frac{dy}{y^2+1} = \frac{2}{\sqrt{3}}\arctan y + c =$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(x+1/2)\right)+c.$$

Άσκηση

$$I=\int \frac{x^5+3x^3-x^2+3x-1}{x^3+2x-3}\,dx = ?$$

1 Διαιρώ:

$$\begin{array}{l|l}
x^5+3x^3-x^2+3x-1 & x^3+2x-3 \\ \hline
-x^5-2x^3+3x^2 & x^2+1 \\
x^3+2x^2+3x-1 & \\
-x^3-2x+3 & \\
2x^2+x+2 &
\end{array}$$

$$I=\underbrace{\int (x^2+1)\,dx}_{I_1} + \underbrace{\int \frac{2x^2+x+2}{x^3+2x-3}\,dx}_{I_2 = ;}$$

$$I_1 = \frac{x^3}{3}+x+c$$

2 παραγοντοποιώ παρονομαστή: $x=1$ ρίζα.

$$\begin{array}{l|l}
x^3+2x-3 & x-1 \\ \hline
-x^3+x^2 & x^2+x+3 \\
x^2+2x-3 & \\
-x^2+x & \\
3x-3 & \\
- &
\end{array}$$

$$x^3+2x-3=(x-1)(x^2+x+3)$$

$\Delta<0$.

3. Διασπώ σε απλά κλάσματα:

$$\frac{2x^2+x+2}{(x-1)(x^2+x+3)} = \frac{a}{x-1} + \frac{\beta x+\gamma}{x^2+x+3} =$$

$$= \frac{ax^2+ax+3a+\beta x^2+\gamma x-\beta x-\gamma}{(\ \ )(\ \ )} \Rightarrow$$

$$\Rightarrow \begin{cases} a+\beta=2 \\ a+\gamma-\beta=1 \\ 3a-\gamma=2 \end{cases} \Rightarrow \begin{cases} \beta=2-a \\ a+\gamma+a-2=1 \\ 3a-\gamma=2 \end{cases} \Rightarrow \begin{cases} 2a+\gamma=3 \\ 3a-\gamma=2 \end{cases}$$

$$\Rightarrow \begin{cases} a=1 \\ \beta=1 \\ \gamma=1. \end{cases}$$

$$I_2 = \int \frac{2x^2+x+2}{x^3+2x-3}dx = \underbrace{\int \frac{dx}{x-1}}_{I_3} + \underbrace{\int \frac{x+1}{x^2+x+3}dx}_{I_4}$$

$$I_3 = \int \frac{d(x-1)}{x-1} = \ln|x-1| + c.$$

$$I_4 = ?$$

4. Εμφανίζω στον αριθμ. την παράγωγο του παρονομαστή $2x+1$:

$$x+1 = \frac{1}{2}(2x+2) = \frac{1}{2}(2x+1+1) \Rightarrow$$

$$I_4 = \int \frac{x+1}{x^2+x+3}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+3}dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+3}dx =$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+3}dx + \frac{1}{2}\int \frac{dx}{x^2+x+3} =$$

$$= \underbrace{\frac{1}{2}\int \frac{d(x^2+x+3)}{x^2+x+3}}_{\overset{\shortparallel}{\ln(x^2+x+3)+c}} + \underbrace{\frac{1}{2}\int \frac{dx}{x^2+x+3}}_{I_5 = ;}$$

5. Γράφω το x^2+x+3 σαν y^2+1:

$$x^2+x+3 = x^2+2\cdot x\cdot \frac{1}{2}+\frac{1}{4}-\frac{1}{4}+3 =$$

$$= \left(x+\frac{1}{2}\right)^2+\frac{11}{4} = \frac{11}{4}\left[\frac{4}{11}\left(x+\frac{1}{2}\right)^2+1\right] =$$

$$= \frac{11}{4}\left[\Big(\underbrace{\frac{2}{\sqrt{11}}\left(x+\frac{1}{2}\right)}_{y}\Big)^2+1\right].$$

Άρα: $y = \frac{2}{\sqrt{11}}\left(x+\frac{1}{2}\right) \Rightarrow dy = \frac{2}{\sqrt{11}}dx \Rightarrow dx = \frac{\sqrt{11}}{2}dy.$

$$\Downarrow$$

$$I_5 = \int \frac{dx}{x^2+x+3} = \int \frac{\sqrt{11}/2\; dy}{11/4\,(y^2+1)} = \frac{2}{\sqrt{11}}\int \frac{dy}{y^2+1} =$$

$$= \frac{2}{\sqrt{11}}\arctan y + c =$$

$$= \frac{2}{\sqrt{11}}\arctan\left(\frac{2}{\sqrt{11}}\left(x+\frac{1}{2}\right)\right) + c.$$