(1i) $\int \frac{2x}{x^2+2x+2}\,dx = \int \frac{2x+2-2}{x^2+2x+2}\,dx =$

$= \int \frac{2x+2}{x^2+2x+2}\,dx - 2\int \frac{dx}{x^2+2x+2} =$

$= \int \frac{d(x^2+2x+2)}{x^2+2x+2} - 2\underbrace{\int \frac{dx}{x^2+2x+2}}_{I_2 =} =$

$= \ln(x^2+2x+2) + c - 2I_2.$

$x^2+2x+2 = (x^2+2x+1)+1 = \underbrace{(x+1)}_{y}{}^2+1 \quad [y = x+1].$

$I_2 = \int \frac{dy}{y^2+1} = \arctan y + c =$

$= \arctan(x+1) + c.$

(1ii) $\int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx.$

$\frac{a}{x+3} + \frac{\beta}{x-1} + \frac{\gamma}{(x-1)^2} = \frac{a(x-1)^2 + \beta(x-1)(x+3) + \gamma(x+3)}{(x+3)(x-1)^2} =$

$= \frac{ax^2 - 2ax + a + \beta x^2 + 2\beta x - 3\beta + \gamma x + 3\gamma}{(x+3)(x-1)^2} =$

$= \frac{(a+\beta)x^2 + (2\beta+\gamma-2a)x + 3\gamma + a - 3\beta}{(x+3)(x-1)^2} = \frac{2x^2+x+1}{(x+3)(x-1)^2} \Rightarrow$

$\cancel{\begin{matrix} a+\beta=2 \\ 2\beta+\gamma-2a=1 \\ 3\gamma+a-3\beta=1 \end{matrix}} \Rightarrow \cancel{\begin{cases} a = 2-\beta \\ 2\beta+\gamma-4+2\beta=2 \Rightarrow \gamma=-4\beta+6 \\ -12\beta+18+2-\beta-3\beta=1 \rightarrow \end{cases}}$

$\cancel{\Rightarrow +2 =}$

$$\frac{a}{x+3}+\frac{b}{x-1}+\frac{\gamma}{(x-1)^2}=\frac{a(x-1)^2+b(x+3)(x-1)+\gamma(x+3)}{(\ )(\ )(\ )}\Rightarrow$$

$$\Rightarrow a(x^2-2x+1)+b(x^2+3x-x-3)+\gamma x+3\gamma=2x^2+x+1$$

$$\Rightarrow (a+b)x^2+(\gamma+2b-2a)x+(a+3\gamma-3b)=2x^2+x+1$$

$$\Rightarrow \begin{cases} a+b=2 \Rightarrow a=2-b. \\ \gamma+2b-2a=1 \Rightarrow \gamma+2b-4+2b=1 \Rightarrow \gamma+4b=5 \Rightarrow \\ a+3\gamma-3b=1 \qquad\qquad \Rightarrow \gamma=5-4b. \end{cases}$$

$$\Downarrow$$

$$2-b+15-12b-3b=1 \Rightarrow 16=16b \Rightarrow b=1$$

$$\Rightarrow (a,b,\gamma)=(1,1,1).$$

$$I=\int\frac{dx}{x+3}+\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2} \qquad [x-1=y].$$

$$=\ln|x+3|+\ln|x-1|+c+\int\frac{dy}{y^2}=$$

$$=\ln|x+3|+\ln|x-1|-\frac{1}{y}+c=$$

$$=\ln|x+3|+\ln|x-1|-\frac{1}{x-1}+c.$$

(iii) $\displaystyle\int\frac{3x^2+3x+1}{x^3+2x^2+2x+1}dx=I$

$$x^3+2x^2+2x+1=(x^3+x^2)+(x^2+x)+(x+1)=$$
$$=(x+1)x^2+(x+1)x+(x+1)=$$
$$=(x+1)(x^2+x+1).$$

$$\frac{a}{x+1}+\frac{bx+\gamma}{x^2+x+1}=\frac{ax^2+ax+a+bx^2+bx+\gamma x+\gamma}{(x+1)(x^2+x+1)}\Rightarrow$$

$$\left.\begin{array}{l} a+b=3 \\ a+b+\gamma=3 \end{array}\right\}\Rightarrow \gamma=0$$

$$I = \int \frac{dx}{x+1} + \int \frac{2x\,dx}{x^2+x+1} = \int \frac{d(x+1)}{x+1} + \int \frac{2x+1-1}{x^2+x+1}\,dx$$

$$= \ln|x+1| + c + \int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{dx}{x^2+x+1} =$$

$$= \ln|x+1| + \ln(x^2+x+1) + c - \underbrace{\int \frac{dx}{x^2+x+1}}_{I_1}$$

$$x^2+x+1 = x^2 + 2\cdot x \cdot \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} =$$

$$= \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right] = \frac{3}{4}\left[\Big(\underbrace{\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)}_{y}\Big)^2 + 1\right] \Rightarrow$$

$$\Rightarrow \left[y = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \Rightarrow dy = \frac{2}{\sqrt{3}}\,dx \Rightarrow dx = \frac{\sqrt{3}}{2}\,dy\right]$$

και

$$I_1 = \int \frac{\frac{\sqrt{3}}{2}\,dy}{\frac{3}{4}(y^2+1)} = \frac{2}{\sqrt{3}} \int \frac{dy}{y^2+1} = \frac{2}{\sqrt{3}} \arctan y + c =$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right) + c.$$

(2i) $\displaystyle\int \frac{dx}{x^4+1} = I$

$$x^4+1 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}\,x)^2 =$$
$$= (x^2+\sqrt{2}\,x+1)(x^2-\sqrt{2}\,x+1).$$

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{\gamma x+\delta}{x^2-\sqrt{2}x+1} \Rightarrow$$

$$\Rightarrow (ax+b)(x^2-\sqrt{2}x+1) + (\gamma x+\delta)(x^2+\sqrt{2}x+1) = 1 \Rightarrow$$

$$\Rightarrow ax^3 - \sqrt{2}ax^2 + ax + bx^2 - \sqrt{2}bx + b +$$
$$\gamma x^3 + \sqrt{2}\gamma x^2 + \gamma x + \delta x^2 + \sqrt{2}\delta x + \delta = 1 \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} a+\gamma = 0 \\ b-\sqrt{2}a+\sqrt{2}\gamma+\delta = 0 \\ a-\sqrt{2}b+\gamma+\sqrt{2}\delta = 0 \\ b+\delta = 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \gamma = -a \\ b-\sqrt{2}a-\sqrt{2}a+1-b = 0 \\ a-\sqrt{2}b-a+\sqrt{2}(1-b) = 0 \\ \delta = 1-\beta \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} 1 = 2\sqrt{2}a \Rightarrow a = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \Rightarrow \gamma = -\frac{\sqrt{2}}{4} \\ -\sqrt{2}b - \sqrt{2}b + \sqrt{2} = 0 \Rightarrow -2b+1 = 0 \Rightarrow b = 1/2 = \delta \end{array}\right.$$

$$I = \int \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2+\sqrt{2}x+1}\,dx - \int \frac{\frac{\sqrt{2}}{4}x - \frac{1}{2}}{x^2-\sqrt{2}x+1}\,dx$$
$$= \frac{\sqrt{2}}{4}\underbrace{\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx}_{I_1} - \frac{\sqrt{2}}{4}\underbrace{\int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx}_{I_2}.$$

$$I_1 = \frac{1}{2}\int \frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx = \frac{1}{2}\int\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}}{x^2+\sqrt{2}x+1}\right]dx =$$

$$= \frac{1}{2}\int \frac{d(x^2+\sqrt{2}x+1)}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}}{2}\underbrace{\int \frac{dx}{x^2+\sqrt{2}x+1}}_{I_3} =$$

$$= \frac{1}{2}\ln(x^2+\sqrt{2}x+1)+c-\frac{\sqrt{2}}{2}I_3.$$

Για το $I_3$:

$$x^2+\sqrt{2}x+1 = x^2+2\cdot x\cdot\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)^2+1-\left(\frac{\sqrt{2}}{2}\right)^2 =$$

$$=\left(x+\sqrt{2}/2\right)^2+1/2 = \frac{1}{2}\left[2\left(x+\frac{\sqrt{2}}{2}\right)^2+1\right]=$$

$$=\frac{1}{2}\left[\left(\sqrt{2}\cdot\left(x+\frac{\sqrt{2}}{2}\right)\right)^2+1\right]=$$

$$=\frac{1}{2}\left[(\sqrt{2}x+1)^2+1\right]$$

Θέτω $y=\sqrt{2}x+1 \Rightarrow dy=\sqrt{2}\,dx \Rightarrow dx=\frac{\sqrt{2}}{2}dy$, και

$$I_3=\int\frac{dx}{x^2+\sqrt{2}x+1}=\int\frac{\frac{\sqrt{2}}{2}dy}{\frac{1}{2}(y^2+1)}=\sqrt{2}\int\frac{dy}{y^2+1}=$$

$$=\sqrt{2}\arctan y+c=\sqrt{2}\arctan(\sqrt{2}x+1)+c.$$

Παρόμοια το $I_2$.

(2ii) $\displaystyle\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=I=?$

Θέτω $y=\sqrt[6]{x} \Rightarrow x=y^6$ και $dx=6y^5dy$.

$\sqrt{x}=y^3$

$\sqrt[3]{x}=y^2$

$$I=\int\frac{6y^5dy}{y^3+y^2}=6\int\frac{y^3dy}{y+1}$$

$$\int \frac{6y^3}{y+1}\,dy = 6\int \frac{y^3+y^2-y^2-y+y+1-1}{y+1}\,dy$$

$$= 6\int \frac{(y+1)(y^2-y+1)-1}{y+1}\,dy =$$

$$= 6\int (y^2-y+1)\,dy - 6\int \frac{dy}{y}$$

$$= 6\left(\frac{y^3}{3} - \frac{y^2}{2} + y\right) - 6\ln|y| + c =$$

$$= 2y^3 - 3y^2 + y - 6\ln|y| + c =$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + \sqrt[6]{x} \underbrace{- 6\ln\sqrt[6]{x}}_{-\ln x} + c$$

(2iv) $$\int \frac{dx}{\sqrt{1+e^x}} = I = ?$$

$$u := \sqrt{1+e^x} \Rightarrow u^2 = 1+e^x \Rightarrow x = \ln(u^2-1) \Rightarrow$$

$$\Rightarrow dx = \frac{2u\,du}{u^2-1}$$

$$I = \int \frac{2u\,du}{u(u^2-1)} = \int \frac{2\,du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} =$$

$$= \ln|u-1| - \ln|u+1| + c =$$

$$= \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right) + c.$$

(3i) $I = \int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx =$

$= \int (1-\sin^2 x)(\sin x)'dx \qquad [u = \sin x \Rightarrow du = (\sin x)'dx]$

$= \int (1-u^2)du = \int du - \int u^2 du = u - \frac{u^3}{3} + c =$

$= \sin x - \frac{\sin^3 x}{3} + c.$

(3iv) $I = \int \frac{dx}{\cos^4 x}$

$\frac{1}{\cos^2 x} = 1 + \tan^2 x = (\tan x)' \Rightarrow$

$I = \int (1+\tan^2 x)(\tan x)'dx =$

$[u = \tan x \Rightarrow du = (\tan x)'dx].$

$= \int (1+u^2)du = u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c.$

(3v) $I = \int \sqrt{\tan x}\,dx$

$[u = \sqrt{\tan x} \Rightarrow u^2 = \tan x \Rightarrow x = \arctan u^2 \Rightarrow$
$\Rightarrow dx = \frac{2u\,du}{u^4+1}]$

$I = \int u \frac{2u\,du}{u^4+1} = 2\int \frac{u^2 du}{u^4+1}$ $\qquad u \lambda \pi.$

(5i) $I = \int \frac{x^2 dx}{(x^2-4)(x^2-1)}$

$(x^2-4)(x^2-1) = (x+1)(x-1)(x+2)(x-2)$.

$$\frac{x^2}{(x^2-4)(x^2-1)} = \frac{a}{x-1} + \frac{\beta}{x+1} + \frac{\gamma}{x-2} + \frac{\delta}{x+2} \Leftrightarrow$$

$$\Leftrightarrow x^2 = a(x+1)(x^2-4) + \beta(x-1)(x^2-4) + \gamma(x+2)(x^2-1) + \\ + \delta(x-2)(x^2-1)$$

$$\Leftrightarrow \begin{cases} a+\beta+\gamma+\delta = 0 \\ a-\beta+2\gamma-2\delta = 1 \\ -4a-4\beta-\gamma-\delta = 0 \\ -4a+4\beta-2\gamma+2\delta = 0 \end{cases} \Leftrightarrow \begin{cases} a = -1/6 \\ \beta = 1/6 \\ \gamma = 1/3 \\ \delta = -1/3 \end{cases}$$

$$I = -\frac{1}{6}\int \frac{dx}{x-1} + \frac{1}{6}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{x-2} - \frac{1}{3}\int \frac{dx}{x+2}$$

$$= -\frac{1}{6}\ln|x-1| + \frac{1}{6}\ln|x+1| + \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+2| + c.$$

5(vi) $I = \int x\sin^2 x\, dx$ $\quad \left[\sin^2 x = \frac{1-\cos 2x}{2}\right]$

$$= \int x\frac{1-\cos 2x}{2}dx = \int \frac{x\,dx}{2} - \frac{1}{2}\int x\cos 2x\, dx$$

$$= \frac{x^2}{4} + c - \frac{1}{2}\underbrace{\int x\cos 2x\, dx}_{I_1}$$

$[y = 2x \Rightarrow x = y/2$ και $dx = dy/2]$.

$$I_1 = \int \frac{y}{2}\cos y \frac{dy}{2} = \frac{1}{4}\int y\cos y\, dy = \frac{1}{4}\int y(\sin y)' dy$$

$$= \frac{1}{4}\left[y\sin y - \int \sin y\, dy\right] = \frac{1}{4}\left[y\sin y + \cos y\right] + c$$

$$= \frac{1}{4} 2x\sin 2x + \frac{1}{4}\cos 2x + c.$$

(5 vii) $I = \int \ln(x+\sqrt{x})\,dx = \int x' \ln(x+\sqrt{x})\,dx =$

$$= x\ln(x+\sqrt{x}) - \int x \cdot \frac{1}{x+\sqrt{x}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right) dx =$$

$$= x\ln(x+\sqrt{x}) - \underbrace{\int \frac{x}{x+\sqrt{x}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}} \cdot dx}_{I_1}$$

$[u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow dx = 2u\,du]$.

$$I_1 = \int \frac{u^2}{u^2+u} \cdot \frac{2u+1}{2u} \cdot 2u\,du =$$

$$= \int \frac{u(2u+1)}{u+1}\,du = \int \frac{2u^2+u}{u+1}\,du =$$

$$= \int \frac{2u^2+2u-u-1+1}{u+1}\,du =$$

$$= \int \left(2u - 1 + \frac{1}{u+1}\right) du = \frac{2u^3}{3} - \frac{u^2}{2} + \ln|u+1| + c.$$

(5x)

$$I = \int \frac{x}{1+\sin x}\,dx$$

θέτουμε $\boxed{y = \tan\frac{x}{2}} \Rightarrow \arctan y = \frac{x}{2} \Rightarrow \boxed{x = 2\arctan y}$

$\Rightarrow \boxed{dx = 2\frac{dy}{1+y^2}}$ και $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2y}{1+y^2}$

$$I = \int \frac{2\arctan y}{1+\frac{2y}{1+y^2}} \cdot 2\frac{dy}{1+y^2} =$$

$$= \int \frac{4\arctan y\,dy}{\frac{(1+y^2+2y)}{\cancel{1+y^2}}\cdot\cancel{(1+y^2)}} =$$

$$= 4\int \frac{\arctan y}{(1+y)^2}\,dy = 4\int \arctan y\left(-\frac{1}{1+y}\right)'dy$$

$$= -4(\arctan y)\frac{1}{1+y} + 4\underbrace{\int \frac{1}{1+y}\cdot\frac{1}{1+y^2}\,dy}_{}$$

(σπάζει → ανάλ. σε απλά κλάσματα, κλπ

(6i) $\int \sin(\ln x)dx = I$

$[u = \ln x \Rightarrow e^u = x \Rightarrow dx = e^u du]$

$I = \int \sin u \cdot e^u du = \int \sin u . (e^u)' du =$

$= e^u \sin u - \int e^u \cos u du =$

$= e^u \sin u - e^u \cos u + \int e^u(-\sin u)du =$

$= e^u(\sin u - \cos u) - I \Rightarrow$

$\Rightarrow 2I = e^u(\sin u - \cos u) \Rightarrow$

$\Rightarrow I = \frac{1}{2} \cdot x \cdot (\sin(\ln x) - \cos(\ln x)) + c.$

(6ii) $\int \frac{1}{x\sqrt{x}} \ln(1-x)\,dx = I =$

$= \int (-2)\left(\frac{1}{\sqrt{x}}\right)' \ln(1-x)dx =$

$= -\frac{2\ln(1-x)}{\sqrt{x}} + 2\int \frac{1}{\sqrt{x}} \cdot \frac{(-1)}{1-x}dx =$

$= -\frac{2\ln(1-x)}{\sqrt{x}} - 2\underbrace{\int \frac{dx}{\sqrt{x}(1-x)}}_{I_1}$

$\left(\begin{array}{l} u = \sqrt{x}. \\ x = u^2 \\ dx = 2udu \end{array}\right)$

$I_1 = \int \frac{2udu}{u(1-u^2)} = \int \frac{2du}{1-u^2} = \int \left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$

$= \ln|1-u| + \ln|1+u| + c =$

$= \ln|1+\sqrt{x}| + \ln|1-\sqrt{x}| + c.$

(7i) $\displaystyle\int \frac{x \arctan x}{(1+x^2)^2}\,dx = \int \left(-\frac{1}{2}\,\frac{1}{1+x^2}\right)' \cdot \arctan x\,dx =$

$\displaystyle = -\frac{1}{2}\cdot\frac{1}{1+x^2}\cdot\arctan x - \int\left(-\frac{1}{2}\cdot\frac{1}{1+x^2}\right)\cdot\frac{1}{1+x^2}\,dx$

$\displaystyle = -\frac{\arctan x}{2(1+x^2)} + \frac{1}{2}\underbrace{\int\frac{dx}{(1+x^2)^2}}_{\text{υπολογίζεται αναδρομικά.}}$

(7ii) $\displaystyle\int \frac{xe^x}{(1+x)^2}\,dx = \int\left(-\frac{1}{1+x}\right)' \cdot xe^x\,dx =$

$\displaystyle = -\frac{xe^x}{1+x} - \int\left(-\frac{1}{1+x}\right)\cdot(e^x + xe^x)\,dx =$

$\displaystyle = -\frac{xe^x}{1+x} + \int\frac{(1+x)e^x}{1+x}\,dx =$

$\displaystyle = -\frac{xe^x}{1+x} + e^x + c.$

(8i) $\displaystyle\int\frac{e^x}{1+e^{2x}}\,dx = I$ $\qquad [u = e^x \Rightarrow du = de^x = e^x dx]$

$\displaystyle I = \int\frac{du}{1+u^2} = \arctan u + c =$

$= \arctan(e^x) + c.$

(8ii) $I = \int \frac{\ln(\tan x)}{\cos^2 x}dx \quad [u = \tan x \Rightarrow du = \frac{dx}{\cos^2 x}]$.

$= \int \ln u\, du = \int u' \ln u\, du =$

$= u\ln u - \int u(\ln u)' du = u\ln u - \int du =$

$= u\ln u - u + c =$

$= (\tan x)\cdot \ln(\tan x) - \tan x + c.$