Άσκηση 9

Να υπολογιστούν τα ολοκληρώματα:

(α) $\int_0^{\pi/4} \frac{x}{\cos^2 x}\,dx$

Παρατηρούμε ότι:

$$\int \frac{x\,dx}{\cos^2 x} = \int x(\tan x)'\,dx = x\tan x - \int x'\tan x\,dx =$$

$$= x\tan x + \int \frac{-\sin x}{\cos x}\,dx =$$

$$= x\tan x + \int \frac{(\cos x)'\,dx}{\cos x} =$$

$$[u = \cos x, \quad du = d\cos x = (\cos x)'dx]$$

$$= x\tan x + \int \frac{du}{u} = x\tan x + \ln|u| + c =$$
$$= x\tan x + \ln|\cos x| + c.$$

Οπότε:

$$\int_0^{\pi/4} \frac{x\,dx}{\cos^2 x} = \left(x\tan x + \ln|\cos x|\right)\Big|_0^{\pi/4} =$$
$$= \pi/4\,\tan \pi/4 + \ln|\cos \pi/4| - 0 - \ln|\cos 0|$$
$$= \pi/4 \cdot 1 + \ln \sqrt{2}/2 - 0$$
$$= \pi/4 + \ln \sqrt{2}/2.$$

(β) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\tan^3 x}{\cos^3 x}\,dx$

Παρατηρούμε ότι:

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^6 x}\,dx = \int -\frac{\sin^2 x}{\cos^6 x}(\cos x)'\,dx =$$

$$[u = \cos x \Rightarrow du = (\cos x)'\,dx]$$

$$= \int \frac{u^2-1}{u^6}\,du = \int \frac{du}{u^4} - \int \frac{du}{u^6} =$$

$$= -\frac{1}{3u^3} + \frac{1}{5u^5} + c = -\frac{1}{3\cos^3 x} + \frac{1}{5\cos^5 x} + c.$$

Άρα:

$$\int_{-\pi/4}^{\pi/4} \frac{\tan^3 x\,dx}{\cos^3 x} = \left(-\frac{1}{3\cos^3 x} + \frac{1}{5\cos^5 x}\right)\Bigg|_{-\pi/4}^{\pi/4} =$$

$$= -\frac{1}{3\cos^3 \pi/4} + \frac{1}{5\cos^3 \pi/4} + \frac{1}{3\cos^5(-\pi/4)} - \frac{1}{5\cos^5(-\pi/4)} =$$

$$= 0 + 0 = 0$$

(γ) $\displaystyle\int_0^5 x \ln(\sqrt{1+x^2})\,dx.$

Θέτουμε $u = \sqrt{1+x^2} \Rightarrow u^2 = 1+x^2 \Rightarrow x^2 = u^2-1 \Rightarrow$

$$\Rightarrow \underbrace{2x\,dx} = 2u\,du.$$

$$\int x \ln(\sqrt{1+x^2})\,dx = \int u \ln u\,du = \frac{u^2}{2}\ln u - \int \frac{u^2}{2}\frac{du}{u}$$

$$= \frac{u^2}{2}\ln u - \frac{u^2}{4} + c = \frac{1+x^2}{2}\ln(\sqrt{1+x^2}) - \frac{1+x^2}{4} + c.$$

Άρα:

$$\int_0^5 x\ln(\sqrt{1+x^2})\,dx = \left(\frac{1+x^2}{2}\ln\sqrt{1+x^2} - \frac{1+x^2}{4}\right)\Bigg|_0^5 =$$

$$= \frac{1+25}{2}\ln\sqrt{26} - \frac{1+25}{4} - \frac{1}{2}\ln 1 + \frac{1}{4}$$

$$= 13\ln\sqrt{26} - \frac{13}{2} + \frac{1}{4} =$$

$$= 13\ln\sqrt{26} - 25/4.$$

(δ) $\int_0^{\pi/4} x\tan^2 x\,dx$

Έχουμε:

$$\int x\tan^2 x\,dx = \int x\,\frac{1-\cos^2 x}{\cos^2 x}\,dx =$$

$$= \int \frac{x\,dx}{\cos^2 x} - \int x\,dx \overset{(\alpha)}{=}$$

$$= x\tan x + \ln|\cos x| - \frac{x^2}{2} + c.$$

Άρα

$$\int_0^{\pi/4} x\tan^2 x\,dx = \left(x\tan x + \ln|\cos x| - \frac{x^2}{2}\right)\Bigg|_0^{\pi/4} =$$

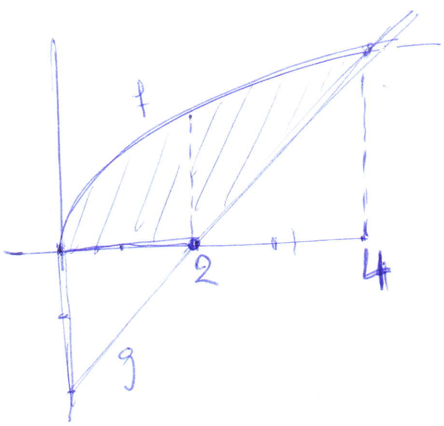
$$= \pi/4\cdot 1 + \ln\frac{\sqrt{2}}{2} - \frac{\pi^2}{32} - 0 - 0 + 0 =$$

$$= \frac{8\pi-\pi^2}{32} + \ln\frac{\sqrt{2}}{2}.$$

$\boxed{\text{Ασκ } 10}$

Υπολογίστε τα εμβαδά:

(1) Στο $1^{ο}$ τεταρτημόριο, μεταξύ των $f(x)=\sqrt{x}$, $g(x)=x-2$ & x-άξονα.



$$f(x)=g(x) \Rightarrow \sqrt{x}=x-2 \Rightarrow$$

$$\Rightarrow x = x^2-4x+4 \Rightarrow$$

$$\Rightarrow x^2-5x+4=0 \Rightarrow$$

$$\Rightarrow (x-4)(x-1)=0.$$

$x=1$ απορρ. $\Rightarrow \boxed{x=4}$

$$E=\int_0^4 f(x)\,dx - \int_2^4 g(x)\,dx =$$

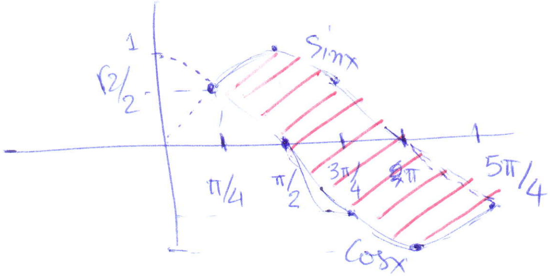
$$=\int_0^4 x^{1/2}\,dx - \int_2^4 x\,dx + \int_2^4 2\,dx =$$

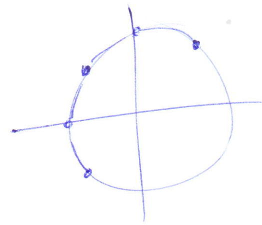
$$= \frac{2}{3}x^{3/2}\Big|_0^4 - \frac{x^2}{2}\Big|_2^4 + 2\cdot 2 = \frac{2}{3}\cdot 2^3 - \frac{16}{2} + \frac{4}{2} + 4 =$$

$$= \frac{16}{3} - 8 + 6 = 10/3.$$

(β) Του χωρίου μεταξύ $f(x)= \cos x$

$g(x)= \sin x$ στο $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.





Στο $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ $\sin x \geq \cos x \Rightarrow \sin x - \cos x \geq 0$

$$E = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = \int_{\pi/4}^{5\pi/4} (-\cos x)' - (\sin x)'\,dx =$$

$$= -\cos x \Big|_{\pi/4}^{5\pi/4} - \sin x \Big|_{\pi/4}^{5\pi/4} =$$

$$= -\cos\frac{5\pi}{4} + \cos\pi/4 - \sin 5\pi/4 + \sin\pi/4 =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}.$$