Απειροστικός Λογισμός ΙΙ
Μάθημα 30ο (02-07-2014)

III Ολοκλήρωση με αντικατάσταση (α')

Ολοκλήρωμα της μορφής $\int f(\varphi(x))\varphi'(x)dx$

- Θέτουμε $y=\varphi(x)$ (οπότε $dy=\varphi'(x)dx$)
- Υπολογίζουμε το $\int f(y)dy = \dots = G(y)$
- Το ολοκλήρωμα που ζητάμε είναι $G(\varphi(x))+c$.

Παραδείγματα

(a) $\int \frac{\arctan x}{x^2+1}dx$

$$\begin{cases} \varphi(x)=\arctan x \\ \varphi'(x)=\frac{1}{x^2+1} \\ f(y)=y \rightsquigarrow f(\varphi(x))=\varphi(x)=\arctan x \end{cases}$$

$y=\arctan x$

$dy=\frac{1}{x^2+1}dx$

Υπολογίζω το $\int y\,dy=\frac{y^2}{2}+c=\frac{(\arctan x)^2}{2}+c$.

(b) $\int \tan x\,dx$

$\int \tan x\,dx=\int\frac{\sin x}{\cos x}dx$ $\quad y=\cos x,\ dy=-\sin x\,dx$

Υπολογίζω το $-\int\frac{dy}{y}=-\ln|y|+c=-\ln|\cos x|+c$.

(c) $\int\frac{\cos(\sqrt{x})}{\sqrt{x}}dx$

$y=\sqrt{x},\quad dy=\frac{1}{2\sqrt{x}}dx \rightsquigarrow \frac{1}{\sqrt{x}}dx=2dy$

Υπολογίζω το: $2\int\cos y\,dy=2\sin y+c=2\sin(\sqrt{x})+c$

IV Βασικά τριγωνομετρικά ολοκληρώματα

Τριγωνομετρία

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos(ax)\cdot\cos(bx) = \frac{\cos(a-b)x + \cos(a+b)x}{2}$$

$$\sin(ax)\cdot\sin(bx) = \frac{\cos(a-b)x - \cos(a+b)x}{2}$$

$$\sin(ax)\cdot\cos(bx) = \frac{\sin(a+b)x + \sin(a-b)x}{2}$$

(Ισχύουν με πιθανότητα 60%)

Παραδείγματα

(a) $\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + c.$

(b) $\int \sin^5 x\, dx = \int \sin^4 x \cdot \sin x\, dx = \int \underbrace{(1-\cos^2 x)^2}_{(1-y^2)^2} \cdot \underbrace{\sin x\, dx}_{-dy}$

$y = \cos x$
$dy = -\sin x\, dx$

Υπολογίζω το: $-\int \underbrace{(1-y^2)^2}_{(1-2y^2+y^4)} dy = -y + \frac{2}{3}y^3 - \frac{y^5}{5} + c =$

$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c.$

(c) $\int \tan^2 x\, dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \tan x - x + c$ (διότι $(\tan x)' = \frac{1}{\cos^2 x}$)

(d) $\int \cot^2 x\, dx = \int \left(\frac{1}{\sin^2 x} - 1\right) dx = -\cot x - x + c.$

V Ολοκλήρωση κατά μέρη

$$\int f'(x) g(x)\, dx = f(x) g(x) - \int f(x) g'(x)\, dx.$$

Παραδείγματα

(a) $\int x \log x\, dx$ / $\int x \log x\, dx = \int \left(\frac{x^2}{2}\right)' \log x\, dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2} (\log x)'\, dx =$

$= \frac{x^2 \log x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx = \frac{x^2 \log x}{2} - \frac{x^2}{4} + c.$

(b) $\int x\cos x\, dx = \int x(\sin x)' dx = x\sin x - \int \cancel{(x)'}^{1} \sin x\, dx = x\sin x + \cos x$.

(c) $\int e^x \sin x\, dx = \int (e^x)' \sin x\, dx = e^x \sin x - \int e^x (\sin x)' dx =$

$= e^x \sin x - \int (e^x)' \cos x\, dx =$

$= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx \Rightarrow$

$\Rightarrow 2\int e^x \sin x\, dx = e^x(\sin x - \cos x) \Rightarrow$

$\Rightarrow \int e^x \sin d x = \frac{e^x}{2}(\sin x - \cos x)$.

(d) $\int \ln(x+\sqrt{x})dx = \int (x)' \ln(x+\sqrt{x})dx = x\ln(x+\sqrt{x}) - \int x \frac{1+\frac{1}{2\sqrt{x}}}{x+\sqrt{x}} dx =$

$= x\ln(x+\sqrt{x}) - \int \frac{\sqrt{\cancel{x}}(2\sqrt{x}+1)}{2\sqrt{\cancel{x}}(1+\sqrt{x})} dx =$

$= x\ln(x+\sqrt{x}) - \int \frac{2\sqrt{x}+1}{2(1+\sqrt{x})} dx$

$y = \sqrt{x}$
$dy = \frac{dx}{2\sqrt{x}}$

Υπολογίζουμε το: $\int \frac{2y+1}{2(y+1)} 2y\, dx = \int \frac{2y^2+y}{y+1} dy$.

$\int \frac{2y^2+y}{y+1} dy = \int \frac{2y(y+1)-(y+1)+1}{y+1} dy = \int (2y-1)dy + \int \frac{1}{y+1} dy =$

$= y^2 - y + \ln|y+1| + C =$

$= x - \sqrt{x} + \ln|\sqrt{x}+1| + C$

Τελικά: $x\ln(x+\sqrt{x}) + x - \sqrt{x} + \ln|\sqrt{x}+1| + C$.

(e) $\int x\sin^2 x\, dx = \int x \frac{1-\cos 2x}{2} dx = \int \frac{x}{2} dx - \frac{1}{2}\int x\cos 2x\, dx$.

VI Ολοκλήρωση με αντικατάσταση (β΄)

Έχουμε το $\int f(x)dx$.

- Θέτουμε $x=\varphi(y)$ (όπου φ αντιστρέψιμη $\rightsquigarrow y=\varphi^{-1}(x)$).
  $dx=\varphi'(y)dy$
- Υπολογίζουμε το $\int f(\varphi(y))\varphi'(y)dy = \dots = G(y)+c =$
  $= \boxed{G(\varphi^{-1}(x))+c}$

Παραδείγματα

(a) $\int \frac{dx}{x^2\sqrt{9-x^2}}$

$x=3\sin y \Rightarrow \frac{x}{3}=\sin y \Rightarrow y=\arcsin\frac{x}{3}$.

$x^2=9\sin^2 y$

$\sqrt{9-x^2}=\sqrt{9-9\sin^2 y}=3\sqrt{1-\sin^2 y}=3\cos y$.

$dx=3\cos y\, dy$

Υπολογίζουμε το: $\int \frac{\cancel{3\cos y}\,dy}{9\sin^2 y\,\cancel{3\cos y}} = \frac{1}{9}\int\frac{dy}{\sin^2 y} = -\frac{\cot y}{9}+c =$

$= -\frac{1}{9}\cot\left(\arcsin\frac{x}{3}\right)+c$.

(b) $\int \frac{\sqrt{x^2-4}}{x}dx$

$x=\frac{2}{\cos y} \Rightarrow \cos y=\frac{2}{x} \Rightarrow y=\arccos\frac{2}{x}$

$\sqrt{x^2-4}=\sqrt{\frac{4}{\cos^2 y}-4}=2\sqrt{\frac{1}{\cos^2 y}-1}=2\tan y$

$dx=\frac{2\sin y}{\cos^2 y}dy$.

Υπολογίζουμε το: $\int \frac{2\cdot\frac{\sin y}{\cancel{\cos y}}}{\frac{2}{\cancel{\cos y}}}\cdot\frac{2\sin y}{\cos^2 y}dy = 2\int\frac{\sin^2 y}{\cos^2 y}dy = 2\int\left(\frac{1}{\cos^2 y}-1\right)dy =$

$= 2\tan y-2y+c =$

$= 2\tan\left(\arccos\frac{2}{x}\right)-2\arccos\frac{2}{x}+c$.

Ασκήσεις

2) $\int \frac{dx}{x^4+1}$

$$x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-2x^2 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1).$$

$$\int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}$$

Βρίσκουμε A, B, Γ, Δ:

$$\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{\Gamma x+\Delta}{x^2-\sqrt{2}x+1} \quad \text{κλπ.}$$

$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

$$y=\sqrt[6]{x} \Rightarrow x=y^6 \Rightarrow dx=6y^5dy.$$

$$\sqrt{x} = (\sqrt[6]{x})^3 = y^3$$

$$\sqrt[3]{x} = (\sqrt[6]{x})^2 = y^2.$$

Υπολογίζουμε το: $\int \frac{6y^5\,dy}{y^3+y^2} dy = 6\int \frac{y^3}{y+1}dy = 6\int \underbrace{\frac{y^3+1}{y+1}}_{y^2-y+1} dy - 6\int \frac{1}{y+1}dy.$ κλπ.

$\int \frac{dx}{\sqrt{1+e^x}}$

$$y=\sqrt{1+e^x} \Rightarrow y^2 = 1+e^x \Rightarrow e^x = y^2-1$$

$$dy = \frac{e^x}{2\sqrt{1+e^x}}dx = \frac{y^2-1}{2y}dx \Rightarrow dx = \frac{2y}{y^2-1}dy. \qquad \frac{1}{y^2-1} = \frac{1}{2(y-1)} - \frac{1}{2(y+1)}$$

Υπολογίζουμε το: $\int \frac{2y}{y(y^2-1)}dy = \int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy = \ln\left|\frac{y-1}{y+1}\right| + c =$

$$= \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + c. \qquad \square$$

3 $\int \cos^2 x \cdot \sin^3 x \, dx$

$y = \cos x$

$dy = -\sin x \, dx$

$$\int \cos^2 x \cdot \sin^3 x \, dx = -\int y^2(1-y^2)\,dy = -\frac{y^3}{3} + \frac{y^5}{5} + c = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

$\int \frac{dx}{\cos^4 x}$

$$\int \frac{dx}{\cos^4 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^4 x}\,dx = \int \tan^2 x \cdot \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\cos^2 x}\,dx =$$

$$= \int \tan^2 x (\tan x)'\,dx + \tan x \quad (*)$$

$y = \tan x$

$dy = (\tan x)'\,dx$

Άρα $(*) \Rightarrow \int y^2\,dy + y = \frac{y^3}{3} + y + c = \frac{\tan^3 x}{3} + \tan x + c$. □