A HISTORY OF THE DEFINITE INTEGRAL

by

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ABSTRACT

The definite integral has an interesting history. In this thesis we trace its development from the time of ancient Greece (500-200 B. C.) until the modern period. We place special emphasis on the work done in the nineteenth century and on the work of Lebesgue (1902).

The thesis is divided into four parts arranged roughly chronologically. The first part traces the developments in the period from the fifth century B. C. until the eighteenth century A. D. Secondary sources were used in writing this history. The second part recounts the contributions of the nineteenth century. The original works of Cauchy, Dirichlet, Riemann, Darboux, and Stieltjes are examined. The third part is concerned with the development of measures in the latter part of the nineteenth century. This work leads to the Lebesgue integral. The final part is a brief survey of modern ideas.

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CHAPTER ONE A Brief History of Early Contributions

The idea of the definite integral arose from the problems of calculating lengths, areas, and volumes of curvilinear geometric figures. These problems were first solved with some success by the mathematicians of ancient Greece.

Probably the earliest attempt at a solution was one devised for calculating areas of curvilinear figures. It can be traced back to two Greek geometers, Antiphon (430 B. C.) and Bryson (450 B. C.) . They attempted to find the area of a circle by inscribing regular polygons, and then successively doubling the number of sides. By this procedure they hoped to "exhaust" the area of the circle, believing that the polygon would eventually coincide with the circle. This implied that the circumference of the circle was not infinitely divisible, but must be made up of "indivisibles" or "infinitesimals". These ideas were vague and led to difficulties. In fact, the ideas of infinitesimals and the infinite caused so much difficulty that they were excluded from Greek geometry.

Eudoxus of Cnidus (408-355 B. C.) is generally credited with devising a method of finding areas and volumes which avoided these problems. This method, which later became known as the Method of Exhaustion, was the Greek equivalent of integration. It used the basic idea of approximating curvilinear figures by rectilinear figures but used only a finite number of these figures. It avoided the problems of the infinitesimal and the infinite by the judicious use of a double reductio ad absurdum argument.

The Method of Exhaustion was based on the following axiom, commonly called the lemma, or postulate, of Archimedes.

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. ([21], P. 14)

Using this principle, for example, one can conclude that a regular inscribed polygon can approximate a circle so that the difference in the areas can be made as small as one wishes. This is accomplished by successively doubling the number of sides, thereby decreasing the difference in area by more than half each time.

The following example from Euclid ([21], pp. 374-375) illustrates the procedure used in the Method of Exhaustion. (This is a condensed version of the actual procedure.)

> Suppose one wished to prove for two circles that $A_1 : A_2 = d_1^2 : d_2^2$. where A_1 , A_2 are areas of the circle and d_1 , d_2 , are their diameters. (The Greeks did not have numbers for geometrical quantities because

of the problem of the incommensurable but used proportions involving four geometrical quantities). One then used the double reductio ad absurdum argument. Suppose first that $A_1 : A_2 > d_1^2 : d_2^2$. Then by the lemma there exists a polygon P_1 included in A_1 and such that $P_1 : A_2 > d_1^2 : d_2^2$. Construct a similar polygon P_2 in A_2. From previous results one knows that $P_1 : P_2 = d_1^2 : d_2^2$. Now $P_1 : A_2 > P_1 : P_2$ which implies that $A_2 < P_2$. But this is impossible since the polygon P_2 is included in A_2. By a similar argument $A_1 : A_2 < d_1^2 : d_2^2$ leads to a contradiction. Hence the result $A_1 : A_2 = d_1^2 : d_2^2$ is proved.

This Method of Exhaustion was used extensively by Eudoxus and his successors until the seventeenth century. The procedure had the advantages of being logically correct and intuitively clear but had the disadvantages of being cumbersome to apply and difficult to deduce new results from.

Archimedes (287-212 B. C.), who is generally considered to be the greatest mathematician of antiquity, greatly extended the work of finding area and volumes of geometric figures. He supplemented the Method of Exhaustion and devised an ingenious heuristic method for finding results before proving them formally. He was then able to anticipate many of the results of integral calculus.

The heuristic method which Archimedes devised to get initial results was based on the mechanical law of the lever.

The geometrical figures in question were visualized as being "made up" of lines or planes. The lines or planes were then pictured as being hung from one end of a lever which was then balanced by a figure of known content and centre of gravity. From this procedure the content of the unknown figure could be calculated. The method is illustrated by the following example given by Archimedes ([20] P. 15-17).

The problem was to show that in the following diagram the parabolic segment ABC has area equal to $4/3 \Delta$ ABC .

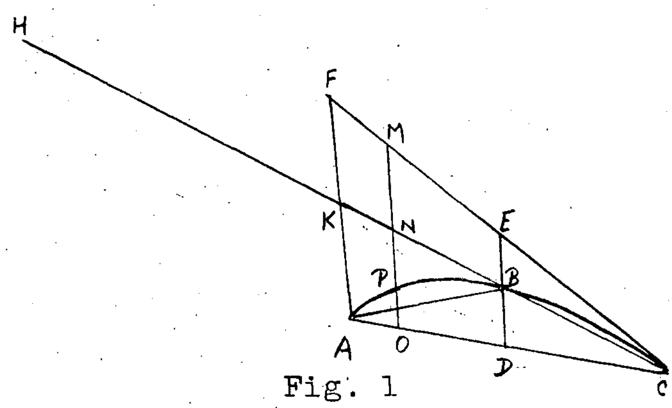

Fig. 1

In the diagram D is the midpoint of chord AC ,
DBE and AKF are drawn parallel to the axis of the parabola,
CF is a tangent,
CK = KH,
CH is visualized as the lever balanced at K ,
MO is any line in Δ AFC parallel to AKF and DBE .

Archimedes proceeded as follows: From the properties of the parabola and the constructions he showed that CK is the median of Δ AFC and that $\frac{MO}{OP} = \frac{CA}{AO} = \frac{CK}{KN} = \frac{HK}{KN}$. He considered the first and last term $\frac{MO}{OP} = \frac{HK}{KN}$ and interpreted this mechanically as meaning that line segment OP at H will balance MO at N with K being the fulcrum. This result is true for any position of MO in Δ AFC . Since the geometric figures are "made up" of lines he concluded that parabolic segment ABC at H will balance Δ AFC at its center of gravity. Since the center of gravity of a triangle is 1/3 the distance along its median he concluded that parabolic segment ABC = 1/3 Δ AFC . By a previous result Archimedes knew that Δ AFC = 4 Δ ABC . Hence parabolic segment ABC = 4/3 Δ ABC .

Archimedes then rigorously proved, by the Method of Exhaustion, every result suggested by the heuristic procedure because he did not consider it to be a valid mathematical demonstration. Many of his applications of the Method of Exhaustion were quite ingenious. In some problems, for example in finding the volume of a paraboloid, he approximated the figure both from the inside and from the outside with elementary figures. In other problems his procedure was very similar to that which we now use in integral calculus. For example, in his official proof that the area of the parabolic segment ABC is equal to 4/3 the area of Δ ABC he proceeded as follows ([5], pp. 51-52):

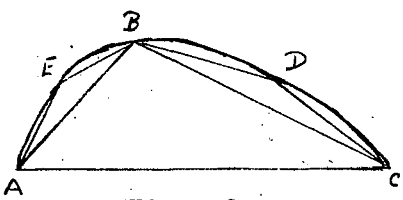

Fig. 2

He approximated the area of the parabolic segment ABC by successively forming triangles such as Δ AEB and Δ BDC . He showed that the area after the n^{th} step was $\Delta ABC \left(1 + \frac{1}{4} + \frac{1}{4}^2 + \frac{1}{4}^3 + \ldots \frac{1}{4}^{n-1} \right)$. Rather than considering a limit and showing that the limit is equal to $4/3\ \Delta ABC$ he completed the last step by the double reductio ad absurdum argument.

Using these methods Archimedes was able to find areas, volumes, and centers of gravity of numerous geometric figures. His results were a great incentive toward the further development of the subject, especially in the seventeenth century.

During the two thousand year period from Archimedes until the sixteenth century it appears that nothing significant was done in devising new methods and techniques for finding area and volumes. However, two new ideas, useful in the further development of integration, were advanced during this period. One was the study of variation. People began to study ideas such as velocity, acceleration, density, and thermal content as physical quantities rather than as qualities. This was the first step in the development of the idea of a function. The second idea, due to Nicole Oresme, (1323-1382) was the realization of a connection between certain geometrical pictures and physical situations. Oresme devised the equivalent of a Cartesian coordinate system and represented velocities by lines on the coordinate system. He even interpreted the area under the velocity curve as representing the distance that the body travelled. These ideas were probably incentives for the further development of integration.

The sixteenth century saw a revival of interest in the problems of quadratures, cubatures, and centers of gravity. This renewed interest was caused mainly by the translation of Archimedes' work into Latin in 1544. People first copied his formal method (The Method of Exhaustion) but soon they began to seek improvements and then to devise new methods for solving the problems.

The first suggested reform came from the Flemish engineer, Simon Stevins (1586) , and the Italian mathematician, Luca Valerio (1606) . They both attempted to avoid the double reductio ad absurdum argument by a direct passage to the limit. However, they still thought in geometrical terms and did not have the arithmetic ideas necessary to give precise definitions.

The unwieldiness of the Method of Exhaustion caused the mathematicians of the seventeenth century to drop the procedure completely and to adopt the less rigorous ideas of indivisibles or infinitesimals. In fact, the period in the seventeenth century until the time of Newton and Leibniz (1670) has been called the Period of the Indivisibles ([12], P. 341) . Integration became associated with the idea of summing these indivisibles.

The first to make extensive use of infinitesimals was Johann Kepler (1571-1650). He became interested in length, area, and volume problems while studying the laws of planetary motion. He was faced with the problems of finding the area of an elliptic segment and the length of an elliptic arc. Kepler was also interested in gauging the contents of wine casks. To solve these problems and others, Kepler visualized that geometric solids were made up of infinitesimals. For example, a circle was made up of an infinite number of triangles with a common vertex and an infinitely small base, and a sphere was made up of an infinite number of infinitely small pyramids. To find the content one merely added up the contents of the components. For example, the area of a circle is equal to the sum of the areas of the triangles and this is equal to one-half times the total sum of the bases (i.e. the circumference) times the radius.

Using procedures such as this, Kepler was able to find the contents of more than eighty new geometrical figures.

It was undoubtedly Kepler's work that led Bonaventura Cavalíeri (1598-1647), an Italian Jesuit mathematician, to develop his method of indivisibles. His work was probably the most influential one of this period.

Cavalieri was never too precise as to what he meant by an indivisible, but it seems he visualized points as being indivisibles of lines, lines as being indivisibles of surfaces, and planes as being indivisibles of volumes. To find lengths, areas, or volumes, he added up the indivisibles. To avoid the problem of the infinite he always considered two geometric figures and formed a correspondence between them. This approach is illustrated, by the so-called Cavalieri's Theorem.

> If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in that ratio.

Cavalieri's use of indivisibles to prove propositions can be illustrated by the following simple example ([5], p. 118). He was interested in proving that parallelogram ACDF has area equal to double the area of Δ CAF or Δ CDF and proceeded as follows:

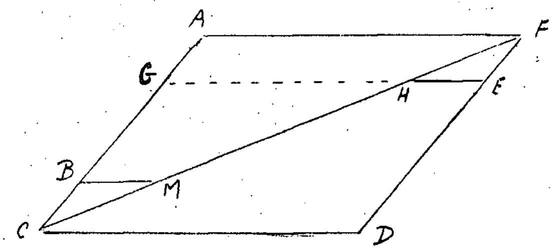

Fig. 3

If EF = CB and HE and BM are parallel to CD then the lines BM and HE are equal. Therefore, all the lines of Δ CAF are equal to all the lines of Δ CDF and the two triangles are therefore equal. Also the area of the parallelogram ACDF is equal to twice the area of either triangle.

By a similar but more involved procedure, Cavalieri was able to obtain results which have been interpreted ([5], p.120) as being equivalent to the formula $\int_0^a x^m dx = \frac{a^{m+1}}{m+1}$, although he thought of his work as pertaining only to geometrical considerations. His work was a generalization of Kepler's as it went beyond the specific geometric problems.

Cavalieri's work on indivisibles stimulated more mathematicians to work on problems involving areas and volumes. Also some mathematicians, such as the Frenchman Roberval (1634), developed ideas of indivisibles independently. Thus there emerged, in the period from 1630 to 1660, a myriad of individual methods for solving these problems. As Struik ([42], p. 138) points out, however, there evolved two distinct trends in the work. Cavalieri, Toricelli, and Barrow, (Newton's teacher) concentrated on a geometrical approach while Fermat, Pascal, Descartes, and Wallis used more of the new algebra and also more of the new analytic geometry which had been developed in this period. Both groups were concerned with the same basic problem:

> Practically all authors in the period from 1630 to 1660 confined themselves to questions dealing with algebraic curves, especially those with the equations $a^m y^n = b^n x^m$ and they found each in his own way, formulas equivalent to $\int_0^a x^m dx = \frac{a^{m+1}}{m+1}$ first for positive integers m, later for m negative integer and fractional. ([42], p. 138)

We will consider in detail two of the methods devised in this period, first that of Pierre Fermat, and then that of John Wallis. These methods were the most advanced of the period in that the techniques used most closely resemble the modern approach to the integral.

Pierre Fermat devised a precedure for calculating area under the curve for special curves. His ingenious procedure used a geometric series and the new idea of a limit.

Fermat devised this procedure for finding the area under the curve of $y = x^{\frac{p}{q}}$ from 0 to b ([5], pp. 160-161; [43] pp. 53-54).

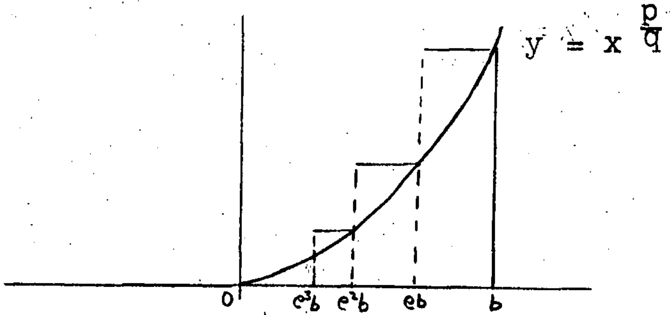

Fig. 4

He first subdivided the interval from 0 to b , not into a finite number of subintervals, but into an infinite number of intervals of unequal length. He selected $e < 1$ and then partitioned the interval by the points $b, eb, e^2b, e^3b, \ldots\ldots$. He formed the approximating sum and found it formed an infinite geometric progression. The formula for the sum was known at the time.

$$S = b^{\frac{p}{q}}(b - eb) + (eb)^{\frac{p}{q}}(eb - e^2b) + (e^2b)^{\frac{p}{q}}(e^2b - e^3b) + \ldots.$$
$$= b^{\frac{p}{q}}(b - eb)\left[1 + e^{\frac{p}{q}+1} + e^{\frac{2p}{q}+2} + \ldots \quad\right]$$
$$= b^{\frac{p}{q}}(b - eb)\left[\frac{1}{1-e^{\frac{p}{q}+1}}\right] = \frac{b^{\frac{p+q}{q}}(1-e)}{1-e^{\frac{p+q}{q}}}$$

Substituting $e = E^q$ he found:

$$S = b^{\frac{p+q}{q}}\left(\frac{1-E^q}{1-E^{p+q}}\right) = \frac{b^{\frac{p+q}{q}}(1-E)(1 + E + E^2 \quad + E^{q-1})}{(1-E)(1 + E + E^2 \quad + E^{p+q-1})}$$

To make the size of the rectangle "infinitely small" he let $e = 1$ (insinuating a limit as e approaches one). The widths of the rectangles approach zero and E approaches one. He substituted $E = 1$ into the sum and found it to be equal to $\frac{q}{p+q} \; b^{\frac{p+q}{q}}$. Hence $\int_0^b x^{\frac{p}{q}} dx = \frac{q}{p+q} \; b^{\frac{p+q}{q}}$.

As Boyer points out ([5], p. 161), Fermat's demonstration possesses many of the important characteristics of the definite integral. There is an equation of a curve, a partition of the x-axis, a sum formed from the areas of approximating rectangles, and some idea of a limit of the sum as the widths of the rectangles approach zero. Fermat, however, did not realize the significance of the operation. He regarded the procedure as a method of solving a particular geometrical problem and had no thought of a generalized procedure.

John Wallis (1616-1703) was an English mathematician. He devised an integration procedure which introduced arithmetic into the geometrical procedure and introduced the idea of a limit.

Wallis' procedure is illustrated by the following example taken from Hooper ([29], pp. 256-258) . In this example, Wallis was interested in comparing the area under the curve $y = x^2$ between O and B with the area in the rectangle OBAC .

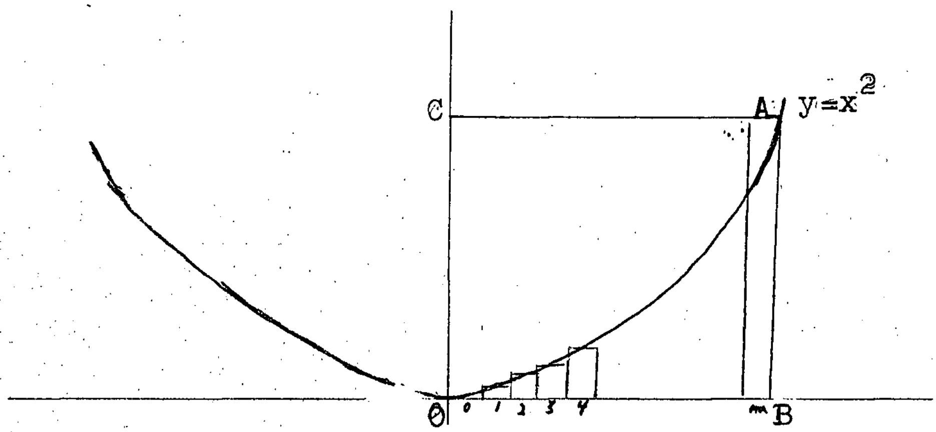

Fig. 5

He began by subdividing the interval OB into $m + 1$ equal parts and formed approximating rectangles with the heights selected so that the total area would be proportional to $0^2 + 1^2 + 2^2 \ldots + m^2$. The area of rectangle OBAC is proportional to $(m + 1)m^2$. Hence the ratio of the areas is $\frac{0^2 + 1^2 + 2^2 + \ldots m^2}{m^2(m + 1)}$. Substituting values for m he found:

(1) $m = 1$ $\qquad \frac{0 + 1}{1 + 1} = 1/3 + 1/6$

(2) $m = 2$ $\qquad \frac{0 + 1 + 4}{4 + 4 + 4} = 1/3 + 1/12$

(3) $m = 3$ $\qquad \frac{0 + 1 + 4 + 9}{9 + 9 + 9 + 9} = 1/3 + 1/18$

He noted that the greater the number of terms, the closer the ratio approximates 1/3 . If this is continued to infinity the difference "will be about to vanish completely " ([5], p. 172). "Consequently the ratio for an infinite number of terms is 1/3 " ([5], p. 172). This result is equivalent to the formula $\int_0^a x^2 dx = \frac{a^3}{3}$. Wallis was able by a similar procedure to derive the formula $\int_0^a x^m dx = \frac{a^{m+1}}{m+1}$ for higher powers of integers and then he apparently affirmed the rule for all powers, rational and irrational except $n = -1$. He was able to apply these results to problems of quadratures and cubatures.

Wallis and Fermat came very close to our present idea of the definite integral. In fact, according to Boyer ([5],p 173)

> ...the basis for the concept of the definite integral may be considered fairly well established in the work of Fermat and Wallis.

But, as he points out

> ...it was to become confused later by the introduction of the conceptions of fluxions and differentials.

These two contributions came from Newton and Leibniz. Newton and Leibniz are generally considered to be the inventors of calculus, as they devised algorithms for differentiation and integration, but their work marks a change in the concept of the integral.

Isaac Newton (1642-1727) was primarily interested in the idea of the derivative, which was also being studied at the time. He showed that the area under the curve could be calculated, not by a summation process as his predecessors had done, but by a process which depended on the idea of differentiation. For example, ([5], p. 191) he considered a curve with abscissa x and ordinate y, with area under the curve being given by $z = \left(\frac{n}{m+n}\right) ax^{\frac{m+n}{n}}$. If o represents the infinitesimal increase in the abscissa then the augmented area will be $z + o\cdot y = (\frac{n}{m+n}) a(x + o)^{\frac{m+n}{n}}$. If, in this equation, one uses the binomial theorem, divides through by o, and then neglects the terms involving o (Newton was uncertain of the justification for this procedure but was thinking in terms of a limit concept), the result will be $y = ax^{\frac{m}{n}}$. Hence, if the area is $z = (\frac{n}{m+n}) ax^{\frac{m+n}{n}}$ the curve will be $y = ax^{\frac{m}{n}}$. Conversely, if the curve is $y = ax^{\frac{m}{n}}$ then the area will be $z = (\frac{n}{m+n}) ax^{\frac{m+n}{n}}$. Thus to find the area one could work backwards from the derivative. Newton, consequently defined the integral, or fluent, as he called it, as the inverse of the fluxion or derivative and concentrated on the methods for finding derivatives.

Leibniz, (1646-1716) working at the same time as Newton, was interested in developing operational rules for sums and differences of infinitesimals. He introduced the notation $\int x$ and later $\int x\,dx$ to represent the sum of all the values of the magnitudes x - or the integral of x, a name which was suggested by the Bernoulli brothers. However, in devising rules for the sum of the infinitesimals, Leibniz relied upon the fact that sums and differences are inverse operations and he used the rules for finding differences. For example, he derived the rule that the difference (or derivative) of x^n was nx^{n-1}. Hence, the sum or integral of x^n must be $\frac{x^{n+1}}{n+1}$.

With the work of Newton and Leibniz, the idea of the integral had changed . It was no longer associated with the idea of a sum, but was now viewed as a secondary operation.

CHAPTER TWO Developments in Integration During the Nineteenth Century

From the time of Newton and Leibniz until the beginning of the nineteenth century, integration was viewed as the inverse operation to differentiation. As we have noted, Newton had defined the integral as the inverse of the fluxion or derivative, while Leibniz in practice used the idea of an antiderivative. In the further development of the subject, Johann Bernoulli and Euler also stressed the integral as the inverse of the differential. Euler, in fact, in the publication of his Institutiones calculi integralis of 1768, defined integral calculus as the method of finding from a given relation of differentials, the quantities themselves ([32], p. 664). He used the sum concept only as a means of approximating the value of the integral.

The concept of a function in use at this time was rather restricted. It usually meant a quantity y related to a variable x by an equation involving certain constants, together with symbols to represent arithmetic, trigonometric, exponential or logarithmic operations. For example, $y = 3x^2$, $y = \sin x + 4x$, $y = a^x$ ... would be classified as functions. Functions could also be defined and represented geometrically, but it appears as if the graph must be a smooth continuous curve before it represented a true function. Also it was assumed that somehow these true geometrical functions could be represented by a single analytic expression, while arbitrary curves could not be ([35], p. 3) .

The work of J. B. Fourier, published in his famous book "The Analytic Theory of Heat" (1807-1822) forced a reexamination of these fundamental ideas.

Fourier first showed that some discontinuous functions could be represented by a single analytic expression, namely a trigonometric series. For example, a function equal to 1 from 0 to α , and 0 from α to π has a trigonometric expansion. Thus the requirement of having an analytic expression did not distinguish between a true function and some arbitrary functions. Moreover it seemed no longer necessary to associate the existence of a single analytic expression with the definition of a function because such expressions could apparently be determined afterwards. This work suggested a more general concept of a function.

It also forced a re-examination of the notion of integral. In the development of the trigonometric or Fourier series of a discontinuous function, the coefficients are defined in terms of the integral of discontinuous functions. For example, in expanding the function f in a trigonometric series, the coeffients a_i are given by $\frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \sin ix \, dx$ or $\frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \cos ix \, dx$. These definite integrals could not be defined as the inverse of a derivative but they seemed to have some interpretation in terms of area ([15], p. 196). Therefore they added impetus to the development of the integral in terms of approximating sums.

A. L. Cauchy (1823) was the person who clarified these concepts. He suggested a more general definition of a function and he restored integration to a primary idea rather than a secondary operation.

He first considered the concept of a function. He began by defining an independent variable ([8], p. 17):

> When variable quantities are related in such a manner that given one of them one can conclude the value of all the others, the first quantity is called an independent variable.

The definition of function followed directly:

> ... and the other quantities, expressible by means of the independent variable, are called functions of this variable.

Similarly, functions of more than one variable were defined. Cauchy did not, however, think in terms of the modern notion of function because his later work suggested that he thought of the variables being related, not by any arbitrary rule, but by an equation.

Cauchy next considered a special type of function, which he named continuous and which he defined as follows ([8], pp. 19-20) :

> When the function $f(x)$ has unique and finite values for all x between two given limits, and the difference $f(x+i) - f(x)$ is an infin-

itely small quantity, one says that the function $f(x)$ is a continuous function of x between the given limits.

The stage was now set for Cauchy's definition of the integral (1823). He arbitrarily restricted himself by defining the integral only for continuous functions, probably because continuous functions or those with a finite number of discontinuities were the only functions which, at the time, were considered important. An outline of his procedure ([8], pp. 122-125) is as follows:

> Let $f(x)$ be a continuous function of x defined between the two finite limits $x=x_o$ and $x=X$. Let $x_o, x_1, x_2, \ldots x_n = X$ be a partition of $[x_o, X]$ and form the sum $S = (x_1 - x_o) f(x_o) + (x_2 - x_1) f(x_1) \ldots + (X - x_{n-1}) f(x_{n-1})$. Then the sum S approaches a definite limit as the differences $(x_i - x_{i-1})$ become infinitely small. This limit which depends only on the function $f(x)$ and the values x_o and X is called the definite integral of $f(x)$ and is represented by the notation $\int_{x_o}^{X} f(x)\, dx$. (The notation is due to Fourier.)

It is interesting to note that Cauchy's proof of the existence of the integral is incomplete as he assumed uniform continuity of the function.

Cauchy then proved the standard algebraic properties of the integral. He also apparently ([5], p. 280) gave the first rigorous demonstration of the fundamental theorem of calculus i.e. if f is a continuous function and $F(x) = \int_{x_0}^{X} f(x)dx$ then $F'(x) = f(x)$.

Cauchy next extended integration to a certain class of unbounded function ([8], p. 143) . The following is an outline of the procedure:

If the function $f(x)$ becomes infinite between $x=x_0$ and $x=X$ at the points (finite in number) $x_1, x_2 \ldots x_m$ then the integral $\int_{x_0}^{X} f(x)dx$ is defined as:

$$\int_{x_0}^{X} f(x)dx = \lim_{\epsilon \to 0} \left[\int_{x_0}^{x_1 - \epsilon\mu_1} f(x)dx + \int_{x_1+\epsilon\gamma_1}^{x_2-\epsilon\mu_2} f(x)dx \ldots + \int_{x_m+\epsilon\gamma_m}^{X} f(x)dx \right]$$

provided the limit exists, where $\mu_1 \gamma_1, \mu_2 \cdots \mu_m, \gamma_m$ and ϵ are arbitrary positive constants.
If the limits of integration are infinite then the integral $\int_{-\infty}^{\infty} f(x)dx$ is defined as :

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\epsilon \to 0} \left[\int_{-1/\epsilon u}^{x_1-\epsilon u_1} f(x)dx + \int_{x_1+\epsilon\gamma_1}^{x_2-\epsilon u_2} f(x)dx \ldots\ldots\ldots + \int_{x_m+\epsilon\gamma_m}^{1/\epsilon\gamma} f(x)dx \right]$$

provided the limit exists where u and γ are arbitrary positive constants.

If in the previous definitions all of the arbitrary constants are reduced to unity one gets Cauchy's definition of the principal value.

Thus Cauchy's work gave integration its modern character. Later developments in the field were based on the foundation which he had provided.

The work of Lejeune Dirichlet, a contemporary of Cauchy, on Fourier series motivated a further development of the integral. Dirichlet, in 1829, devised sufficient conditions under which a function could be represented by a convergent Fourier series. These conditions were ([11], p. 18) :

(1) The function has only a finite number of maxima and minima.

(2) The function has only a finite number of discontinuities.

The second condition was included because it was only under this condition that the integrals defining the coefficients were considered.

The next step in the development seemed to be to alter this second condition by extending the idea of the integral. First Dirichlet himself attempted to do this by extending the integral to functions whose set e of discontinuities has a finite number of accumulation points. An example of this type of function is $\frac{1}{\sin\frac{1}{x}}$, for the only accumulation point is 0. The integral was defined as follows ([35], p. 10) :

> The accumulation points of e will divide the interval $[a, b,]$ into a finite number of partial intervals. Let $[\alpha, \beta]$ be one of them. The interval $[\alpha + h, \beta - k]$ will contain only a finite number of points of e and one can consider the Cauchy integral $\int_{\alpha+h}^{\beta-k} f(x)dx$ provided it exists. Then
>
> $\int_{\alpha}^{\beta} f(x)dx = \lim_{\substack{h\to 0 \\ k\to 0}} \int_{\alpha+h}^{\beta-k} f(x)dx$ provided the

limit exists. The integral over $[a,b]$ is then just the sum of the integrals over the intervals.

This integral apparently was not extensively used, partly because the original paper was never published,(1) but mainly because it was superceded by the integral of Riemann.

G. B. Riemann (1854) was also interested in extending the conditions of Dirichlet. In fact, he was interested in finding not only sufficient but necessary conditions under which the representation can occur. This led him quite naturally to an investigation of the meaning of the symbol $\int_a^b f(x)dx$. The result of the investigation was the famous Riemann integral.

Riemann began by considering what $\int_a^b f(x)dx$ meant first for bounded functions. Unlike Cauchy he made no other assumptions about the functions. An outline of his procedure ([46], p. 239) is as follows:

Let $x_1, x_2, \ldots x_{n-1}$ be an increasing sequence of values in (a,b) and let $\delta_1 = x_1 - a$, $\delta_2 = x_2 - x_1, \ldots, \delta_n = b - x_{n-1}$. Form the sum $S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + \ldots + \delta_n f(x_{n-1} + \epsilon_n\delta_n)$ where the ϵ_i are positive proper fractions. The value of the sum S depends upon the choice of the intervals δ_i and the numbers ϵ_i. If this sum has the property that it approaches a finite number A as the δ_i approach zero, no matter how

(1) It is mentioned in [35] p. 10

δ_i and ϵ_i are chosen, the value A is the definite integral $\int_a^b f(x)dx$. If the sum S does not have this property then $\int_a^b f(x)dx$ has no meaning.

Riemann also defined $\int_a^b f(x)dx$ for functions f which have a singularity at a point c , $a \leq c \leq b$ ([46], p.240).

Form the integrals $S = \int_a^{c-a_1} f(x)dx + \int_{c+a_2}^b f(x)dx$ for a_1, a_2, arbitrary positive constants. Now let a_1 and a_2 approach zero independently. If S approaches a limit then this limit is defined as $\int_a^b f(x)dx$.

Under what conditions imposed on the function f will the integral exist? This is the next question that Riemann answered. He showed that a necessary and sufficient condition for the integral of a function to exist is that given $\sigma > o$ then the sum of the lengths of intervals where the oscillation of the function is greater than that σ can be made as small as one would like.

This statement suggested the idea of a measure of a set and may have been a stimulus in the development of that concept ([4], p. 249).

Riemann's work thus introduced the property of integrability of a function and widened the class of integrable functions to include many discontinuous ones.

G. M. Darboux [10] (1875) while attempting to make Riemann's work on integration more precise, suggested a different approach to the integral through the use of upper and lower sums.

Darboux began by defining precisely the supremum M_i and infimum m_i of a bounded function f on an interval $[a,b]$. He then subdivided the interval (a,b) by the points $x_1, x_2, \ldots x_{n-1}$ and formed the new sums:

$$M = M_1\delta_1 + M_2\delta_2 + \ldots M_n\delta_n$$
$$m = m_1\delta_1 + m_2\delta_2 \qquad + m_n\delta_n$$

where $\delta_i = x_i - x_{i-1} \qquad \delta_o = x_1 - a \ , \ \delta_n = b - x_{n-1}$

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \qquad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$$

Then he proved what became known as the Darboux Theorem ([10], p. 65):

> If $\delta_i \leq \delta$ for all i, then there exists finite numbers M_{ab}, m_{ab} such that $\lim_{\delta \to 0} M = M_{ab}$, $\lim_{\delta \to 0} m = m_{ab}$.

He did not call these limits the upper and lower integrals nor did he use the notation $\overline{\int} f(x)dx$ and $\underline{\int} f(x)dx$. These contributions apparently came from Jordan in 1892 ([28], p. 464).

In considering the integral $\int_a^b f(x)dx$ Darboux started with the characteristic sum.

$$\Sigma = \delta_1 f(a+\theta_1\delta_1) + \delta_2 f(x_1+\theta_2\delta_2) \ldots\ldots\ldots + \delta_n f(x_{n-1}+\theta_n\delta_n)$$

He then formed his upper and lower sums, M and m, discussed above, and showed that a necessary and sufficient condition that the sum Σ has a limit as $\delta \to o$ ($\delta_i \leq \delta$) is that the limits M_{ab} and m_{ab} of M and m are equal.

A similar procedure to this is used in many modern texts to define the Riemann integral.

Darboux was also able to give a completely valid proof that a continuous function was integrable. It is difficult to say for certain but this seems to be one of the earliest proofs of this result. (Heine [23] (1872) had considered the idea of uniform continuity and had shown that a continuous function on [a,b] is uniformly continuous. This could have led to earlier proofs of this result).

During the latter part of the nineteenth century apparently many integrals were devised for unbounded functions ([44], p. 236). These integrals were extensions of the Riemann integral. They did not achieve lasting importance but have some historical interest. We will consider one example to illustrate the type of procedure.

A. Harnack (1884) [16] devised an integral for unbounded functions whose set of singularities can be enclosed in a finite number of intervals with total length as small as one would wish. (By modern terminology the set of singularities has content zero.) An outline of his procedure ([16], p. 220) is as follows:

> Let f be the function and enclose the singularity points in a set E consisting of a finite number of intervals of total length ϵ. Let f_1 be equal to 0 in E and to f everywhere else and suppose $\int_a^b f_1(x)dx$ exists. If this integral approaches a finite limit as ϵ approaches 0 this limit is said to be the integral of f from a to b.

In 1894 T.J. Stieltjes [40] introduced a completely new idea into the history of integration, an idea which was unrelated to other developments in the field. While working on questions involving the distribution of mass along a line, Stieltjes suggested the idea of an integral involving two functions.

He began by considering a monotone increasing function φ defined on the positive x-axis with $\varphi(o) = o$. The function could be visualized as representing a distribution of mass with the points of discontinuity representing the points of condensation of mass. With this interpretation an increasing function represents a physical example of a measure.

Stieltjes then considered the moment about the origin of such a distribution in [a,b] and proceeded as follows ([40], p. J71):

> Let $a=x_o$, $b=x_n$, and place between x_o and x_n the n-1 values $x_o < x_1 < x_2 \ldots < x_n$. Next pick n numbers $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ such

that $x_{k-1} \leq \epsilon_k \leq x_k$. Then form the sum

$$\epsilon_1[\varphi(x_1)-\varphi(x_o)]+\epsilon_2[\varphi(x_2)-\varphi(x_1)]\ldots+\epsilon_n[\varphi(x_n)-\varphi(x_{n-1})].$$

The limit of the sum (as max $(x_i - x_{i-1})$ approaches 0) is by definition the moment of the distribution about the origin.

Stieltjes then generalized this procedure by considering the sum $f(\epsilon_1)[\varphi(x_1)-\varphi(x_o)]+f(\epsilon_2)[\varphi(x_2)-\varphi(x_1)]+\ldots f(\epsilon_n)[\varphi(x_n)-\varphi(x_{n-1})]$ where f is any continuous function. This sum will have a limit as max $(x_i - x_{i-1})$ approaches 0. This limit is designated by $\int_a^b f(x)d\varphi(x)$ and is now called the Stieltjes integral of f with respect to φ .

Stieltjes did not extend this integral beyond the case where f is continuous and φ is monotone. The only property he proved was the following:

$$\int_a^b f(x)d\varphi(x)=f(b)\varphi(b)-f(a)\varphi(a)-\int_a^b \varphi(x)df(x) .$$

CHAPTER THREE The Development of Measures - The Lebesgue Integral

As we have seen, the concept of the integral is closely related to the concept of area. Area, along with length and volume, were historically amongst the first examples of the general idea of measure. These examples all have the characteristic property of being non-negative and additive.

From antiquity until the nineteenth century, these measures were calculated only for very regular geometric sets such as the set of points under a continuous curve. The procedure, as we have noted in the case of area, was to approximate the sets from the inside and/or the outside by a finite number of simple figures. For example, Archimedes, in calculating the volume of a paraboloid used approximations by rectilinear solids, both from the inside and from the outside.

The advances in analysis in the nineteenth century seemed to motivate a more intensive study of measures. As we have seen Riemann's condition for the integrability of a function (that the sum of the lengths of the intervals on which the oscillation $> \sigma$ can be made as small as we like) suggests the idea of a measure for certain new subsets ([4], p. 249). With the development of set theory, many more sets were considered. The problem then presented (according to some sources ([13], p 150)) was how to associate a measure not only with the regular sets but also with arbitrary subsets.

The first methods introduced by Stolz, ([41], p. 151), Harnack, ([17], p. 241), and Cantor, ([7], pp. 473-475), (1884-1885), all used the same basic idea. A set E in R_1, for example, was covered by a finite number of intervals. The measure, m(E), was defined as the limit of the sum of the lengths as the longest of the intervals approached zero. This measure, however, was unsatisfactory because it did not have the additive property even for commonly used sets. For example, if A is the set of rationals in [0,1], A' is the complement of A, m is the measure, then $m(A') = 1$, $m(A) = 1$, $m(A \cup A') = 1$. Hence $m(A \cup A') \neq m(A) + m(A')$.

Probably to overcome these difficulties, C. Jordan [30] (1894) suggested a more refined approach to the problem of measures. He first of all considered approximating a set not only from the outside but also from the inside, using in each case a finite number of elementary figures. He then calculated limits as the size of the figures approached zero. He illustrated his procedure by considering a set E in the plane.

> ...Decompose this plane by parallels to the coordinate axes, into squares of sides r.
>
> The set of those squares which are interior to E form a domain S interior to E; The set of those which are interior to E or which meet its boundary form a new domain S + S' to which E is interior. We can represent the areas of these domains by S and S + S'.

> Let us now vary our decomposition into squares in such a way that r tends to zero: the areas S and S+S' will tend to some fixed limits. ([30], p. 28)

These limits A and a are called respectively the interior area and exterior area of E . If these two numbers are equal the set E is called "quarrable" and has area or measure a=A . Jordan then restricted himself to these quarrable sets and showed that the measure has the additive property.

This was probably the first time that, in order to achieve this additive property, the measure was restricted to a family of subsets rather than being calculated for all subsets.

Jordan also mentioned that this procedure can be adopted for sets of any number of dimensions. The interior and exterior "content" (étendue) can be determined and if these two numbers are equal the set is called measurable. The measure is additive on these measurable sets.

Apparently G. Peano ([4], p. 249) , at approximately the same time, developed similar ideas of measure and measurability.

These ideas of Jordan and Peano, however, had limited applicability because too many commonly used sets were not measurable. For example, the set of irrationals in [0,1] has inner content equal to zero and outer content equal to one and is therefore not measurable.

All of these previously mentioned measures, using finite approximations from the outside, were very coarse. They would not, in fact, distinguish between a set and its closure.

E. Borel (1898), apparently ([12], p. 342) while studying series of functions, found the need for a measure with the property that the measure of countable sets was zero. To fulfill this need, he introduced ([3], pp. 46-50) a new property for a measure and a new method for calculating the measure of certain sets. The new property was countable additively, ie. the measure of the union of a countable number of disjoint sets is equal to the sum of their measures. The new method involved considering how certain sets were constructed and deducing what the measure should be. Restricting himself to subsets of the interval $[0,1]$ Borel began by considering an interval with or without end points. The measure should be its length. Since an open set G can be expressed as the union of a countable number of disjoint intervals E_i, $i = 1,2, \ldots$, the measure of $G, m(G)$, should be equal to the sum of the measures $m(E_i)$ $i=1,2, \ldots$ A closed set F is the complement of an open set G. Its measure $m(F)$ should therefore be $1-m(G)$.

Borel continued to calculate the measure of a set by this step by step procedure, using the following two properties:

(1) countable additivity.

(2) if $E \supset E'$ and $m(E) = S$, $m(E') = S'$, then $m(E - E') = S - S'$.

Those sets which had a measure defined by this procedure were called measurable sets. By a transfinite procedure, Borel constructed all the sets which belong to what we now call the Borel σ - ring. He then showed that the measure defined on these sets was a non-negative countably additive set function.

In 1902 Henri Lebesgue introduced, in his thesis [34], some powerful, new ideas on measure and integration. His work marks the beginning of a new era in these fields.

Lebesgue was interested in the problems of finding a function knowing its derivative. This led him quite naturally to a consideration of area under the curve, hence to area in the plane and to the more general problem of measures. He introduced a new approach to measures, and a more general class of measurable sets. These ideas led directly to a definition of the integral for a wider class of functions. His integral possessed some important new properties and was used in the solution of the original problem of finding a function knowing its derivative.

Lebesgue began his work on measures by stipulating the conditions Borel had apparently suggested a measure must satisfy ([34], p. 232), the measure being restricted to bounded sets:

> We propose to attach to each bounded set a number, positive or zero, which we will call its measure and which will satisfy the following conditions:
> (1) There exists some sets for which the measure is not zero.
> (2) Two equal sets have the same measure (sets are equal if they can be made to coincide by displacement.)
> (3) The measure of the sum of a finite or countable number of disjoint sets is the sum of the measures of the sets. ([34],p.236)

In order to achieve this goal Lebesgue used a much simpler and, as it turned out, more general approach than had Borel. He amended the procedure of Jordan by approximating sets with countable covers rather than just finite ones. This idea is Lebesgue's key contribution to measure theory.

He considered bounded sets E first on the real line and covered E with a countable number of intervals. These intervals formed a set E_1. He defined the measure of an interval as its length and defined $m(E_1)$ as the sum of the lengths of the component intervals. He then defined the outer measure of E, $m_e(E)$, as the inf of the numbers $m(E_1)$ taken over all possible countable covers by intervals. To get the inner measure of E he let I represent an interval containing E and defined the inner measure $m_i(E)$ by $m_i(E) = m(I) - m_e(I-E)$.

The important sets considered were those for which the two measures were equal:

> We call sets measurable if the outer measure and the inner measure are equal. ([34],p. 238)

Lebesgue showed that this class of measurable sets was closed under countable unions and intersections and included the classes of Jordan and Borel measurable sets. He also showed that the measure restricted to these sets had the desired properties for a measure.

Lebesgue then stated that these considerations could easily be extended to bounded sets E of any dimension. He contented himself, however, with considering only dimension two and suggested a procedure which was completely analogous to the procedure for dimension one. It is interesting to note that he used triangles to cover the plane sets. They would be more useful in the extension to surface area.

Having settled the problem of measures, Lebesgue was led quite naturally to the following definition of the integral for bounded functions ([34], p. 250) : (This is a paraphrase of the actual definition):

> Let f be a bounded function defined on $[a,b]$, let m be the plane measure, and let
> $E_1 = \{(x,y) \mid a \leq x \leq b \quad o \leq y \leq f(x)\}$,
> $E_2 = \{(x,y) \mid a \leq x \leq b \quad f(x) \leq y < o\}$.
> If E_1 and E_2 are measurable sets then the integral of f is defined as the quantity $m(E_1) - m(E_2)$, and the function f is called summable.

The next step was to try to define the integral for unbounded functions. One procedure, which Lebesgue acknowledged but did not follow, was to extend the measure to unbounded sets. Instead he used a new procedure which involved subdividing the y-axis.

His inspiration for this procedure came from considering a continuous monotone increasing function f defined on $[\alpha,\beta]$ with range $[a,b]$ $(a<b)$. Corresponding to a subdivision $\alpha = x_o < x_1 < x_2 \ldots < x_n = \beta$ of $[\alpha,\beta]$ was a subdivision $a = a_o < a_1 < a_2 \ldots < a_n = b$ of $[a,b]$. Lebesgue noted that the classical integral of the function which was usually defined as the common limit of the two sums

$$\sum_1^n (x_i - x_{i-1})a_{i-1}, \quad \sum_1^n (x_i - x_{i-1})a_i$$

as $\max\ (x_i - x_{i-1})$ approached zero could also be defined as the common limit as $\max\ (a_i - a_{i-1})$ approached zero. Generalizing this idea he associated the following sums with an arbitrary bounded function f and any subdivision $a = a_o < a_1 < a_2 \ldots < a_n = b$ of an interval $[a,b]$ containing the range:

$$\sigma = \sum_o^n a_i m(e_i) + \sum_o^{n-1} a_i m(e_i{}')$$

$$\Sigma = \sum_o^n a_i m(e_i) + \sum_o^{n-1} a_{i+1} m(e_i{}')$$

where $e_i = \{x\colon f(x) = a_i\}$

$e_i{}' = \{x\colon a_i < f(x) < a_{i+1}\}$

m is the measure on the line.

These sums will be defined only if $m(e_i)$ and $m(e'_i)$ are defined. Consequently, Lebesgue considered the sums only for the functions for which, given any a and b, the set $\{x: a < f(x) < b\}$ is measurable. This condition, it turned out, is equivalent to the condition that the function is summable. Therefore, for these functions the sums σ and Σ are defined and, as Lebesgue showed, they have the same limit as $\max (a_i - a_{i-1})$ approaches zero. This limit is equal to the integral of the function.

This procedure thus suggested another definition of summable function and integral which is applicable to unbounded functions as well:

> A (bounded or unbounded) function f is called summable if for any a and b the set $\{x: a < f(x) < b\}$ is measurable. ([34], p. 258)

Thus Lebesgue's concept of a summable function is equivalent to our present concept of a measurable function.

To define the integral he considered a subdivision $\ldots m_{-2} < m_{-1} < m_o < m_1 < m_2 \ldots$ of the y-axis, varying between $-\infty$ and $+\infty$ and such that $m_i - m_{i-1}$ is bounded, and he let:

$$\sigma = \Sigma\, m_i m(e_i) + \Sigma\, m_i m(e_i')$$
$$\Sigma = \Sigma\, m_i m(e_i) + \Sigma\, m_{i+1} m(e_i')$$

He then showed that if one of these sums is finite then both will converge to the same finite limite as max $(m_i - m_{i-1})$ approaches zero. This limit, if it exists, is defined as the integral of the function. Lebesgue noted, however, that the integral does not necessarily exist for unbounded summable functions; hence the reason for our use of the term measurable instead.

This integral of Lebesgue has some interesting properties.

Although it is a generalization of the proper Riemann integral, it is not a generalization of the improper one. For example, the function $f(x) = \frac{(-1)^r}{r}$ for $r-1 \leq x < r \quad r = 1,2 \ldots$ has an improper Riemann integral but is not Lebesgue integrable.

However, unlike any other integral considered before, it possesses the following important property which is of paramount interest in analysis ([34], p. 259)

> If a sequence of summable functions $f_1, f_2, f_3 \ldots$, having integrals, has a limit f, and if $|f - f_n| < M, \forall n,$ where M is some fixed number, then f has an integral which is the limit of the integrals of functions f_n.

Moreover, the integral can be used to find primitives for a wider class of functions than those considered heretofore.

CHAPTER FOUR A Modern Glimpse

In this part of the thesis, we will give some indication of the developments in integration in the period after Lebesgue. The amount of material on this period is tremendous; we will confine ourselves to a very brief coverage.

The notions of measure and integral are intimately connected. Measure assigns numbers to sets while the integral assigns numbers to functions, that is, it is a functional. Given a measure, one can define an integral by a procedure like Lebesgue's or one devised by W. H. Young (1905) which uses Darboux sums. Similarly, given an integral, one can assign a measure to a set by considering the integral of its characteristic function if it is integrable. These points of view are reflected in developments along two broad lines, a set theoretic approach and a functional approach.

The work of Radon (1913), Fréchet (1915), and Carathéodory (1914, 1918) stressed the measure theory approach. Their work represents a natural generalization of the works of Lebesgue and Stieltjes.

Radon suggested replacing the n-dimensional Lebesgue measure by any completely additive set function defined on the Lebesgue measurable sets.

Fréchet generalized this idea by considering any completely additive set function defined on the subsets of any abstract space. He postulated the measurable subsets to be a σ-field.

The corresponding integrals in both these cases are defined in any of the usual ways using sums.

Carathéodory next devised a procedure for generating a measure rather than assuming its existence on a σ-field. Starting with any nonnegative function defined on a given class of sets, he determined an outer measure defined on all sets of the space considered. This outer measure, in general, is only subadditive. He then isolated sets called measurable which form a σ-field and on which the outer measure is completely additive, that is, it is a measure. In the definition of the integral, Carathéodory continued his stress on measures by pursuing the idea of area under the curve. To this end, he defined product measure (to take the place of area in the plane) and defined the integral in terms of this product measure.

The idea of the integral as a functional, specifically a linear functional, was stressed by F. Riesz (1909) and Daniell (1918). Their work established fundamental connections between integration and functional analysis.

Riesz solved a problem posed previously by J. Hadamard when he showed that the Stieltjes integral $\int f\,dg$ was the most general linear continuous functional on the space $C(I)$ of continuous functions on $[a,b]$. That is, given a linear continuous functional S on $C(I)$, he showed there exists a function g of bounded variation such that $S(f) = \int_a^b f\, dg$, $\forall f \in C(I)$, thereby establishing a fundamental connection between linear continuous functionals and measures.

Daniell disassociated the integral from its dependence on a measure by abstracting the essential properties of the Lebesgue integral. He began by postulating a functional $\int$ defined on a certain class of functions F, for example the continuous functions or step functions. This functional is assumed to be linear $(\int af + bg = a\int f + b\int g)$, nonnegative $(f \geq o \Rightarrow \int f \geq o)$, and to have the monotone convergence property $(f_n \uparrow f \Rightarrow \int f_n \rightarrow \int f)$. Daniell then devised a procedure for extending this functional to a larger class of functions in such a way that it still satisfies the given conditions. If the class F is the continuous functions and $\int$ is the Riemann integral, then the extension procedure will yield the Lebesgue integral for the Lebesgue-integrable functions.

The idea of the integral as a linear functional was further extended beginning in the 1930's with the study of integrals of functions with values in a Banach space. The integral now maps functions into a more general space than the real line.

A third post Lebesgue approach to integration was toward the unification of the ideas of antiderivative and limit of a sum. The Lebesgue integral did not completely combine these two ideas. For example, the derivative of $x^2 \sin \frac{\pi}{x^2}$ has an antiderivative but is not integrable in the Lebesgue sense. To overcome such difficulties Denjoy (1912) and Perron (1914) devised new integrals.

BIBLIOGRAPHY

1. E. T. Bell: The Development of Mathematics. New York: McGraw Hill, 1940.

2. G.A. Bliss: Integrals of Lebesgue. Amer. Math. Soc. Bull., vol. 24, 1917-18, pp 1-47.

3. E. Borel: Leçons Sur La Theorie Des Fonctions, 2nd ed. Paris: Gauthier-Villars, 1914.

4. N. Bourbaki: Elements D'Histoire Des Mathematiques. Paris: Hermann, 1960.

5. C. B. Boyer: The History of the Calculus. New York: Dover, 1949.

6. F. Cajori: A History of Mathematics. New York: MacMillan, 1931.

7. G. Cantor: Ueber unendliche, lineare Punktmannichfaltigkeiten. Mathemat. Annalen, vol. 23, 1884, pp. 453-488.

8. A. Cauchy: Oeuvres Complètes, 2nd series, vol. 4. Paris: Gauthier-Villars, 1899.

9. P. J. Daniell: A General Form of the Integral. Ann. of Math., vol. 19, 1917-18, pp. 279-294.

10. G. M. Darboux: Memoire sur les fonctions discontinues. Ann. Ec. Norm Sup. number 2, vol. 4, 1875, pp. 57-112.

11. G. Lejeune Dirichlet: Werke I. Berlin: Reimer, 1889.

12. Encyclopaedia Britannica. Volume 12. Chicago: Benton, 1965.

13. Encyclopédie Des Sciences Mathematiques. Paris: Gauthier-Villars, 1912.

14. H. Eves: An Introduction to the History of Mathematics. New York: Rinehart, 1963.

15. J. Fourier: The Analytic Theory of Heat. Translated by A. Freeman. New York: Dover, 1955.

16. A. Harnack: Die allgemeinen Sätze über den Zusammenhang der Funktionen einen reellen Variabeln mit ihren Ableitungen. Mathemat Annalen, vol. 24, 1874, pp. 217-252.

17. A. Harnack: Ueber den Inhalt von Punktmengen. Mathemat. Annalen, vol. 25, 1885, pp. 241-258.

18. T. L. Heath: A History of Greek Mathematics. vol. II. Oxford: Clarendon Press, 1921.

19. T. L. Heath: A Manual of Greek Mathematics. vol.II. Oxford: Clarendon Press, 1921.

20. T. L. Heath: The Method of Archimedes. Cambridge: University Press, 1912.

21. T. L. Heath: The Thirteen Books of Euclid's Elements. 2nd. ed. vol. III. Cambridge: University Press, 1926.

22. T. L. Heath: The Work of Archimedes. New York: Dover, 1912.

23. E. Heine: Die Elements der Functionenlehre. L. Crelle Journal, vol. 74, 1872, pp. 172-185.

24. R. Henstock: Theory of Integration. London: Butterworths, 1963.

25. T. H. Hildebrandt: On Integrals Related to and Extensions of Lebesgue Integrals. Am. Math. Soc. Bull., vol. 24, 1917-18, pp. 113-144, 177-202.

26. T. H. Hildebrant: Integration in Abstract Spaces. Bull. Am. Math. Soc. vol. 59, number 2, March, 1953, pp. 111-159.

27. T. H. Hildebrandt: Introduction to the Theory of Integration. New York: Academic Press, 1963.

28. E. W. Hobson: The Theory of Functions of a Real Variable. vol. 1, 3rd ed. Cambridge, 1927.

29. A. Hooper: Makers of Mathematics. New York: Random House, 1948.

30. C. Jordan: Cours d'Analyse de l'Ecole Polytechnique. vol. 1. Paris: Gauthier-Villars, 1893.

31. P. E. B. Jourdain: Note on Fourier's Influence on the Conceptions of Mathematics. Proceedings International Congress of Mathematics. vol. 2. Cambridge, 1912, pp. 526-527.

32. P. E. B. Jourdain: The Origins of Cauchy's Concept of the Definite Integral and of the Continuity of a Function. Isis 1, 1913, pp. 661-703.

33. H. Kestelman: Modern Theories of Integration. New York: Dover, 1960.

34. H. Lebesgue: Intégrale, Longeur, Aire. Annali Di Mathematica. series 3, vol. 7, 1902, pp. 231-359.

35. H. Lebesgue: Leçons Sur L'Integration. Paris: Gauthier-Villars, 1928.

36. S. Saks: Theory of the Integral. English Translation. New York: Hafner, 1937.

37. V. Sanford: A Short History of Mathematics. London: Harrap.

38. J. F. Scott: A History of Mathematics. London: Taylor and Francis, 1958.

39. D. E. Smith: History of Math. vol. II. Boston: Ginn, 1925.

40. T. J. Stieltjes: Recherches Sur Les Fractions Continues. Ann. Fac. Sci. de Toulous, vol. VIII, 1894, pp. J 1 - J 122.

41. O. Stolz: Ueber einen zu einer unendlichen Punktmenge gehörigen Grenzwerth. Mathemat. Annalen, vol. 23, 1884, pp. 152-156.

42. D. J. Struik: A Concise History of Math. London: Bell and Sons, 1954.

43. O. Toeplitz: The Calculus. Chicago: University of Chicago Press, 1963.

44. E. J. Townsend: Functions of a Real Variable. New York: Holt, 1928.

45. E. B. Van Vleck: The Influence of Fourier Series Upon the Development of Math. Science, vol. 39, 1914, pp. 113-124.

46. H. Weber: The Collected Works of Bernhard Riemann. 2nd. ed. New York: Dover, 1953.