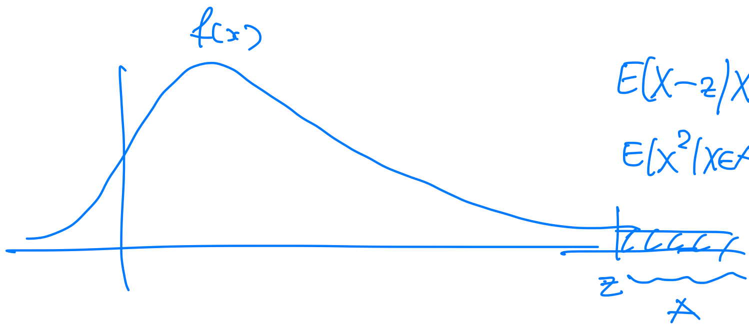
8-5-2023

$X \sim f$

$P(X \in A) \approx 0$

$E(X-z \mid X>z)$

$E(X^2 \mid X\in A)$



$$\theta = E\left(h(X) \mid X \in A\right)$$

$$= \int_{x\in A} h(x)\, f_{X \mid X\in A}(x)\, dx$$

$$f_{X\mid A}(x) = \begin{cases} \dfrac{f(x)}{P(X\in A)}, & x \in A \\ 0, & x \notin A \end{cases}$$

$$= \int_{x\in A} h(x)\, \frac{f(x)}{P(X\in A)}\, dx$$

$$= \frac{\int_{x\in A} h(x) f(x) dx}{\int_{x\in A} f(x)\, dx} = \frac{E_f\Big[\overbrace{1(A)\, h(X)}^{N}\Big]}{E_f\Big[\underbrace{1(A)}_{D}\Big]} = \frac{E_f(N)}{E_f(D)}$$

$$E_f(N) = E_g\left(N \frac{f(X)}{g(X)}\right)$$

$$E_f(D) = E_g\left(D \frac{f(X)}{g(X)}\right)$$

$X_1, \dots, X_N \sim g$

$$\hat{\theta} = \frac{\frac{1}{N} \sum_{j=1}^{N} h(x_i)\, 1(x_i \in A)\, \frac{f(x_i)}{g(x_i)}}{\frac{1}{N} \sum_{j=1}^{N} 1(x_i \in A)\, \frac{f(x_i)}{g(x_i)}} = \frac{\hat{N}}{\hat{D}}$$

$E(\hat{\theta}) \neq \theta .$

$$E_g(\hat{N}) = E_g(N) = E_g\left(h(x)\, 1(x \in A) \frac{f(x)}{g(x)}\right] = E_f\left(h(x)\, 1(x \in A)\right)$$

$$E(\hat{D}) = E(D) = \dots E_f\left(1(x \in A)\right)$$

$$\frac{E(\hat{N})}{E(\hat{D})} = \theta$$

Όμως $\hat{\theta} = \frac{\hat{N}}{\hat{D}}$, $E(\hat{\theta}) = E\left(\frac{\hat{N}}{\hat{D}}\right) \neq \frac{E(\hat{N})}{E(\hat{D})} = \theta .$

$Var(\hat{\theta})$ irrelevant

Ενδιαφέρει $E(\hat{\theta} - \theta)^2 = MSE(\hat{\theta})$

bootstrap

Παράδειγμα

X_1, X_2, X_3, X_4 indep

$X_i \sim Exp\left(\frac{1}{i+2}\right) \qquad E(X_i) = i+2, \quad i=1,\dots,4$

$S = X_1 + \dots + X_4$

$E(S) = 3+4+5+6 = 18$

$\theta = E(S \mid S > 62) \qquad \left(P(S>62) \approx 0\right)$

$$\hat{\theta} = \frac{\sum_{j=1}^{N} S_j \, 1(S_j > 62) \frac{f(S_j)}{g(S_j)}}{\sum_{j=1}^{N} 1(S_j > 62) \frac{f(S_j)}{g(S_j)}}$$

$f_j(x_j) = \lambda_j e^{-\lambda_j x_j} \quad , \quad \lambda_j = \frac{1}{j+2}$

$f(x_1, x_2, x_3, x_4) = \left(\prod_{j=1}^{4} \frac{1}{j+2}\right) e^{-(\lambda_1 x_1 + \dots + \lambda_4 x_4)}$

$g_j(x)$: tilted density $f_{t,j}(x) \sim Exp(\lambda_j - t)$

$g(x_1, \dots, x_4) = \prod_{j=1}^{4} \left(\frac{1}{j+2} - t\right) \cdot e^{-(\lambda_1 x_1 + \dots + \lambda_4 x_4)} \, e^{+tS}$

$= \dots = C(t) \, e^{+tS} f(x)$

$$\frac{f(x_1,\dots,x_4)}{g(x_1,\dots,x_4)} = C(t)e^{-tS}$$

Τελικά (πράξεις)

$$\hat{\theta} = \frac{\sum_{j=1}^{k} S_j \, 1(S_j > 62)\, e^{-tS_j}}{\sum_{j=1}^{k} 1(S_j > 62)\, e^{-tS_j}}$$

$t = ?$

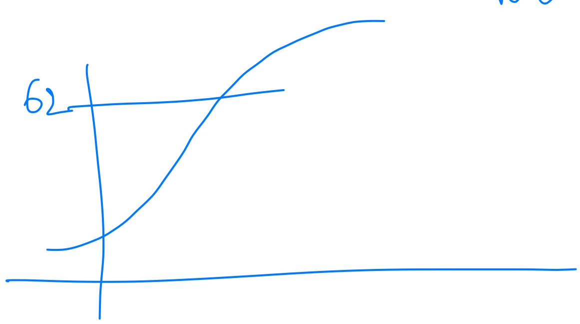
Θέλουμε t τέτοιο ώστε $E_{f_t}(S) = 62$

$$E_{f_t}(S) = E_{f_t}(X_1 + \dots + X_4)$$

$$X_j \sim Exp\left(\frac{1}{j+2} - t\right) \Rightarrow E(X_j) = \frac{1}{\frac{1}{j+2} - t}$$

Επομένως $E_{f_t}(S) = \underbrace{\sum_{j=1}^{4} \frac{1}{\frac{1}{j+2} - t}}_{W(t)} = 62$

$W(t) = 62$



$\Rightarrow \dots \quad t \simeq 0.14$

Παράδειγμα 2

Έστω $X \sim f$ (γνωστή)

$$\theta = P(X > a) \approx 0$$

$$\theta = \underset{f}{E}\left(1(X>a)\right)$$

$$= E_g\left[1(X>a)\frac{f(x)}{g(x)}\right] \qquad g \sim Exp(\lambda)$$

$$= E_g\left(1(X>a)\frac{f(x)}{g(x)} \,\middle|\, X>a\right)\cdot P_g(X>a)$$

$$+ \cancel{E_g\left(1(X>a)\frac{f(x)}{g(x)} \,\middle|\, X\le a\right)}^{\,0}\cdot P_g(X\le a)$$

$$= P_g(X>a)\cdot E_g\left(\cancel{1(X>a)}^{\,1}\frac{f(x)}{g(x)} \,\middle|\, X>a\right)$$

Έστω $g = \lambda e^{-\lambda x}$

$$P_g(X>a) = e^{-\lambda a}$$

$$\frac{f(x)}{g(x)} = \frac{f(x)}{\lambda e^{-\lambda x}} = \frac{1}{\lambda}\cdot f(x)\,e^{\lambda x}$$

$$\theta = \frac{e^{-\lambda a}}{\lambda}\, E_g\left(\frac{f(x)}{g(x)} \,\middle|\, X>a\right) =$$

$$= \frac{e^{-\lambda a}}{\lambda} E_g\left(\underline{e^{\lambda X} f(x) \mid X > a}\right)$$

$$X \sim Exp(\lambda) \qquad E\left(h(X) \mid X > a\right)$$

αμνημοσύνη ιδ.

$$X = Y + a \;, \quad X \mid X > a \sim a + Y \;, \quad Y \sim Exp(\lambda)$$

$$E_g\left(h(X) \mid X > a\right) = E_g\left(h(a + X)\right)$$

$$\theta = \frac{e^{-\lambda a}}{\lambda} E_g\left[e^{\lambda(X+a)} f(X+a)\right]$$

$$\boxed{\theta = \frac{1}{\lambda} E_g\left(e^{\lambda X} f(X+a)\right]} \qquad \underline{X_g \sim Exp(\lambda)}$$

$$\lambda = ? \qquad E_g(X) = \frac{1}{\lambda} = a \Rightarrow \lambda = \frac{1}{a}$$

Μείωση Διασποράς μέσω Common Random Numbers

Παράδειγμα: Έστω n εργασίες πρέπει να εκτελεστούν σε ένα μηχάνημα με τη σειρά.

Διάρκειες εργασιών $T_1, \dots, T_n$

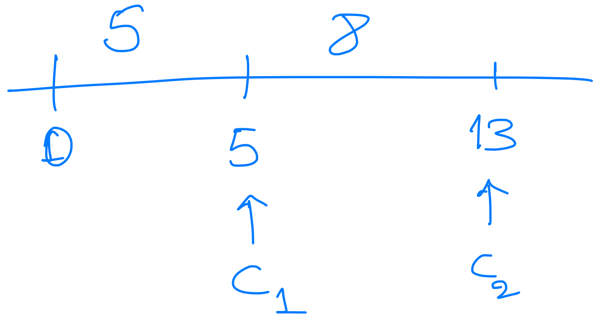
$f(t_1, \dots, t_n)$ από κοινού κατανομή

Σειρά εκτέλεσης ?

π.χ. ① αν $T_1 = 5$, $T_2 = 8$

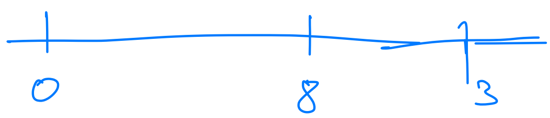
κριτήριο F = flowtime $= \sum_{j=1}^{n}$ (χρόνοι ολοκλήρωσης) $= \sum C_j$

π.χ. 1 – 2



: $F = 5 + 13 = 18$

2 – 1 :



$F = 8 + 13 = 21$

F min για SPT

Shortest Processing Time

② Προθεσμίες : $D_1 = 7$, $D_2 = 9$

Κόστος καθυστέρησης $\sum_j (C_j - D_j)^+ \cdot \alpha_j$

(scheduling theory)

Έστω δύο πολιτικές για τη σειρά εκτέλεσης π_1, π_2

$\pi : (T_1, \dots, T_n) \longrightarrow (j_1, \dots, j_n)$ μετάθεση του $\{1, \dots, n\}$

$\pi : \mathbb{R}_+^n \longrightarrow \{$Σύνολο μεταθέσεων $(1, \dots, n)\}$

$\forall \pi$: μέτρο απόδοσης $h_\pi(T_1, \dots, T_n)$ min ή max.

Γενικό ερώτημα

$$\pi^* : \min_\pi E\left[h_\pi(T_1, \dots, T_n)\right]$$

Έστω Π_1, Π_2 δύο πολητικές:

$$\theta_1 = E_{\Pi_1} h(T_1,\dots,T_n) = E\left(h_1(T_1,\dots,T_n)\right)$$

$$\theta_2 = E_{\Pi_2} h(T_1,\dots,T_n) = E\left(h_2(T_1,\dots,T_n)\right)$$

$$\theta = \theta_1 - \theta_2$$

Έστω ότι υπάρχει γεννήτρια $T = (T_1,\dots,T_n)$

$$\theta_1 = E\left[h_1(T_1,\dots,T_n)\right] : \quad \text{Δείγμα} \quad \underbrace{T^{(1)},\dots,T^{(N_1)}}_{\sim f}$$

$$\hat{\theta}_1 = \frac{1}{N_1}\sum_{j=1}^{N_1} h_1\left(T^{(j)}\right)$$

$$\hat{\theta}_2 = \frac{1}{N_2}\sum_{j=1}^{N_2} h_2\left(\tilde{T}^{(j)}\right) \qquad \underbrace{\tilde{T}^{(1)}, \tilde{T}^{(2)}, \dots, \tilde{T}^{(N_2)}}_{f}$$

$$\hat{\theta} = \hat{\theta}_1 - \hat{\theta}_2 \qquad E(\hat{\theta}) = \theta_1 - \theta_2 \quad \checkmark$$

$$Var(\hat{\theta}_1 - \hat{\theta}_2) = Var(\hat{\theta}_1) + Var(\hat{\theta}_2) - \cancel{2\,Cov(\hat{\theta}_1, \hat{\theta}_2)}^{\,0}$$

Αν $T^{(1)}, \ldots, T^{(N_1)}$ κ' $\tilde{T}^{(1)}, \ldots, \tilde{T}^{(N_2)}$

είναι συσχετισμένα

με τέτοιο τρόπο ώστε $Cov(\hat{\theta}_1, \hat{\theta}_2) > 0$

τότε $Var(\hat{\theta}_1 - \hat{\theta}_2) < Var(\hat{\theta}_1) + Var(\hat{\theta}_2)$

Έστω $h_{1,2}(T_1, \ldots, T_n)$ αύξουσες ως προς $T_j \ \forall j$

(π.χ. scheduling h : κόστος)

τότε $Cov(h_1(T), h_2(T)) > 0$

$[\, Cov(f(U), f(1-U)) < 0 \,]$ είχαμε δει

Common random numbers:

$$\tilde{T}_j = T_j \quad , j = 1, \ldots, N$$

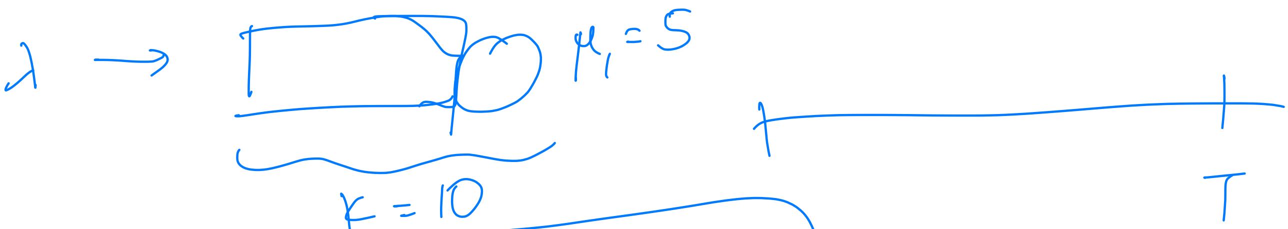
Έλεγχος

: $H_0 : \theta_1 = \theta_2 \qquad H_1 : \theta_1 > \theta_2$

paired test: $H: \delta = 0$ $\quad H_1: \delta > 0$

$$\delta = \theta_1 - \theta_2 = E_\xi (T_1 - T_2)$$

Παρ. 8 ώρα αναμονής δύο ιατρικές



$\boxed{\text{(Coupling Theory)}}$