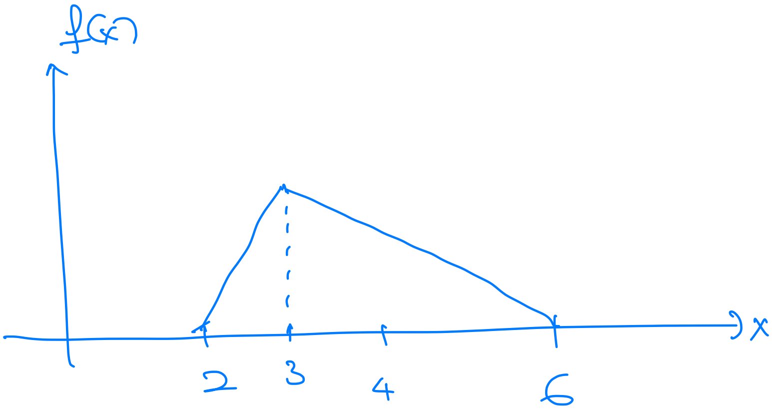
20-3-2023

Κεφ. 5

Άσκ. 2 X : pdf $f(x) = \begin{cases} \frac{x-2}{2}, & 2 \le x \le 3 \\ \frac{2-x/3}{2}, & 3 \le x \le 6 \end{cases}$

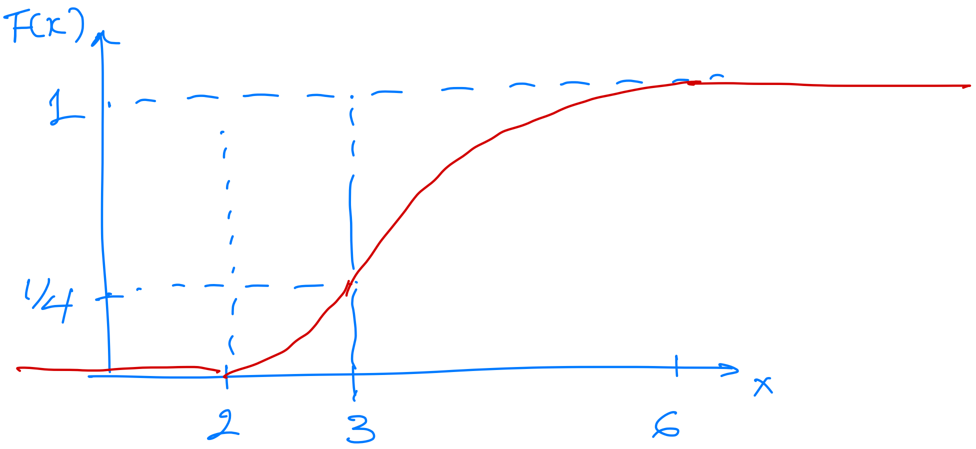


Inverse Transform

$$F(x) = \int_0^x f(y)\,dy$$

1) $x \in [2,3]$: $F(x) = \int_2^x \frac{y-2}{2}\,dy = \frac{(x-2)^2}{4}$

2) $x \in [3,6]$: $F(x) = \frac{1}{4} + \int_3^x \frac{2-y/3}{2}\,dy = \frac{-x^2+12x-24}{12}$



$F(X) = u \quad , \quad u \in [0, 1]$

1) $u \leq 1/4$: $\dfrac{(X-2)^2}{4} = u \Rightarrow X = 2 + 2\sqrt{u}$

$$\left.\begin{array}{ll} u = 0 & X = 2 \\ u = 1/4 & X = 3 \end{array}\right\} \quad 2 \leq X \leq 3$$

2) $u > 1/4$: $\dfrac{-X^2 + 12X - 24}{12} = u \Rightarrow$

$$\Rightarrow X^2 - 12X + 24 + 12u = 0$$

$$X = \frac{12 \pm \sqrt{48 - 48u}}{2} =$$

$$= \frac{12 \pm 4\sqrt{6(1-u)}}{2}$$

$$= 6 \pm 2\sqrt{6(1-u)}$$

$$= 6 - 2\sqrt{6(1-u)}$$

$$\left.\begin{matrix} u = 1/4 & x = 3 \\ u = 1 & x = 6 \end{matrix}\right\} \quad 3 \le X \le 6$$

$$\Rightarrow \boxed{X = \begin{cases} 2+2\sqrt{u} & 0 \le u \le 1/4 \\ 6-2\sqrt{6(1-u)}, & 1/4 \le u \le 1 \end{cases}}$$

Άσκ. 18 $\quad X : f(x) = 2xe^{-x^2}, \quad x > 0$

Inverse Transform

$$F(x) = \int_0^x 2y e^{-y^2} dy \qquad \begin{matrix} u = y^2 \\ du = 2y\,dy \end{matrix}$$

$$= \int_0^{x^2} e^{-u} du = \underline{1 - e^{-x^2}}, \quad x > 0.$$

$$F(X) = U \Rightarrow 1 - e^{-X^2} = U \Rightarrow \cdots \boxed{X = \sqrt{-\ln(1-U)}}$$

Qu $Y \sim \mathcal{E}xp(1) \Rightarrow \ldots \boxed{X = \sqrt{Y}} \,\stackrel{?}{=}\, f(x) = 2x e^{-x^2}, \; x > 0$

$X \leq \sqrt{Y}$ $\quad P(X \leq x) = P(\sqrt{Y} \leq x) = P(Y \leq x^2)$

$= 1 - e^{-x^2}$ ✓

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21 $\quad Y \sim \text{Gamma}(a, 1), \quad a < 1$

$X = Y \mid Y > d \qquad (d > 0)$

$$f_Y(y) = \frac{y^{a-1} e^{-y}}{\Gamma(a)}, \quad y \geq 0$$

$$f_X(x) = \begin{cases} \dfrac{f_Y(x)}{\underbrace{P(Y > d)}_{K}}, & x > d \\ 0, & x \leq d \end{cases}$$

Accept/Reject $\mu \in$ $\quad g(x) = \mu e^{-\mu x} \quad (\mathcal{E}xp(\mu))$

$$h(x) = \frac{f_X(x)}{g(x)} = \begin{cases} 0, & x \leq d \\ \dfrac{x^{a-1} e^{-x}}{\Gamma(a) \cdot K \cdot \mu e^{-\mu x}} & \end{cases}$$

$x > d$ $\qquad h(x) = \underbrace{\frac{1}{\Gamma(a) K \cdot \mu}}_{A} \cdot \underbrace{x^{a-1} e^{(\mu-1)x}}_{\max_{x \in (d,\infty)}}$

$$h'(x) = A\, x^{a-2} e^{(\mu-1)x} \left[\underbrace{(\mu-1)}_{?} x + \underbrace{(a-1)}_{<0} \right], \quad x \geq d$$

i) $\mu > 1$

$h'(x) = 0$: $\quad x = \frac{1-a}{\mu-1}$ $\quad$ (μέγιστο ή ελάχιστο?)

$$h'(x) = r(x) \left[(\mu-1)x - (1-a) \right]$$

$r(x) \geq 0 \quad \forall x$

$(\mu-1)x - (1-a) \uparrow x$

$x < \frac{1-a}{\mu-1} \Rightarrow h'(x) < 0$

$x > \frac{1-a}{\mu-1} \Rightarrow h'(x) > 0$

$\Rightarrow$ $\frac{1-a}{\mu-1}$: σημ. ελαχίστου

pseudoconvex

$h'(x) < 0 \quad x < x_0$

$h'(x) > 0 \quad x > x_0$

$$h(x) = A \cdot \frac{e^{(\mu-1)x} \uparrow \infty}{x^{(1-\alpha)} \uparrow \infty} \longrightarrow \infty \quad (x \to \infty)$$

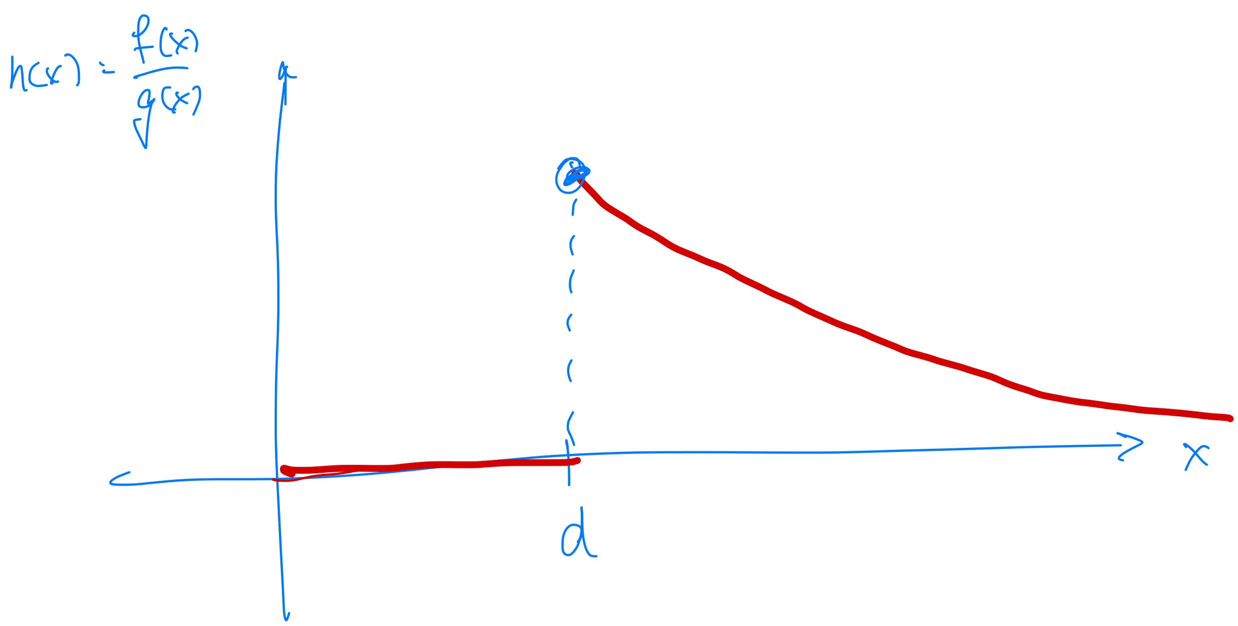
$$\sup_{x>d} h(x) = +\infty$$

δεν μπορούμε να έχουμε $\mu > 1$.

Έστω $\mu < 1$

$$h'(x) = r(x) \cdot \Big[\underbrace{(\mu-1)}_{<0}\, x - \underbrace{(1-\alpha)}_{>0} \Big] \leq 0 \quad \forall x.$$

$h(x)$ φθίνουσα. $(x > d)$



$$\sup_{x>0} h(x) = h(d) = \frac{d^{\alpha-1} e^{(\mu-1)d}}{K\,\Gamma(\alpha) \cdot \mu} = c(\mu)$$

Ποια είναι η καλύτερη επιλογή $Exp(\mu)$?

min $C(\mu)$

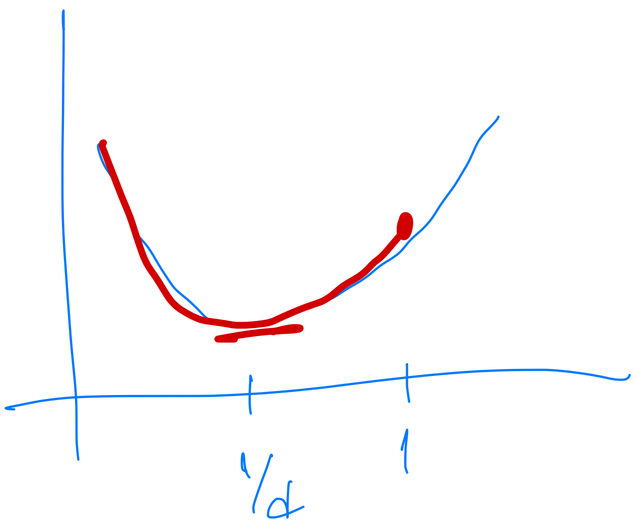
$\mu \leq 1$

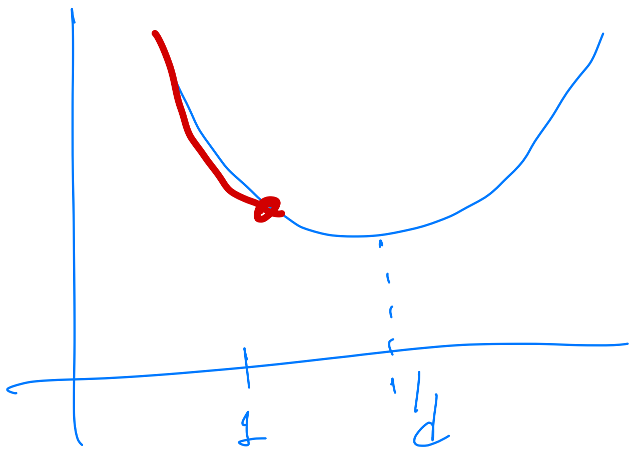
$$c'(\mu) = \frac{d^{a-1}}{K\,\Gamma(a)} \, \frac{e^{(\mu-1)d}(d\mu-1)}{\mu^2}$$

$$= b(\mu)(d\mu-1) \quad , \quad b(\mu)\geq 0$$

$d\mu - 1 = 0 \Rightarrow \mu = \frac{1}{d}$ σημείο ελαχίστου

$1/d < 1$ $\qquad$ $1/d > 1$





$$\mu^* = \begin{cases} 1/d, & \frac{1}{d} < 1 \\ 1, & 1/d > 1 \end{cases} \Rightarrow \boxed{\frac{1}{\mu^*} = \begin{cases} d, & d>1 \\ 1, & d<1 \end{cases}}$$

$$\boxed{\frac{1}{\mu^*} = \max(d,1)}$$

Γεννήτρια Πολυμεταβλητής Κανονικής Κατανομής

$$X \sim \mathcal{N}(\underline{\mu}, \Sigma)$$

$\mu \in \mathbb{R}^n$

$$X = (X_1, \dots, X_n)$$

$\Sigma \in \mathbb{R}^{n \times n}$

συμμετρ. ≥ 0

pos. semi-definite.

$$E(X_i) = \mu_i \quad , \quad i = 1, \dots, n$$

$$\Sigma : \text{pos. semidefinite} \Leftrightarrow x^T \Sigma x \geq 0 \quad \forall x \in \mathbb{R}^n$$

$$\mathrm{Cov}(X_i, X_j) = \sigma_{ij}$$

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

$$X_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$$

X_i, X_j όχι ανεξ.

Έστω $Z = (Z_1, \dots, Z_m)$ iid $\mathcal{N}(0,1)$

$$Z \sim \mathcal{N}(0, I_{m \times m})$$

$A_{m \times n}$, $\boxed{X = A \cdot Z + \mu}$

$\mu \in \mathbb{R}^n$

$$\Rightarrow \quad X_j = \mu_j + \sum_{i=1}^{m} a_{ij} Z_i \quad , \quad j=1,\dots,n$$

$$\underline{X} = AZ + \mu$$

$$E(\underline{X}) = \mu + A\,E(Z) = \mu$$

$$\Sigma_X = E\left((X-\mu)(X-\mu)^T\right) \qquad \left[X-\mu\right]\left[(X-\mu)^T\right]$$

$$= E\left(AZ(AZ)^T\right) =$$

$$= E\left(AZZ^TA^T\right) = A\,\underbrace{E(ZZ^T)}_{I}\,A^T = AA^T$$

$$X \sim \mathcal{N}(\mu, AA^T)$$

$$X \sim \mathcal{N}(\mu, \Sigma)$$

Αρκεί να βρούμε A : $AA^T = \Sigma$

κ' τότε $X = \mu + AZ$, $Z \sim \mathcal{N}(0, I)$

Cholesky decomposition

Όταν Σ συμμετρικός κ' θετ. ορισμένος

$\exists$ A κάτω τριγωνικός : $\boxed{AA^T = \Sigma}$

$n=3$: $A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & 0 & a_{33} \end{pmatrix}$

$AA^T = \begin{pmatrix} a_{11}^2 & a_{11}(a_{21}) & a_{11}a_{31} \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & & \\ & & \end{pmatrix}$

$\Rightarrow \quad a_{11} = \sqrt{\sigma_{11}} \quad , \quad a_{21} = \dfrac{\sigma_{12}}{\sqrt{\sigma_{11}}} \quad , \quad a_{31} = \dfrac{\sigma_{13}}{\sqrt{\sigma_{11}}}$

R : chol(M) $\Longrightarrow$ A .

Κεφ. 7 Discrete Event Simulation

Παράδ. 1 $\{X_n, n=1,2,\dots\}$ ΜΑΔΧ

$$X_n \in \{0, 1, \dots, M\} \qquad P = (P_{ij})$$

$$P_{ij} = P(X_{n+1}=j \mid X_n = i), \quad P_{0j} = P(X_0 = j)$$

Simulation of a sample path $(X_0, X_1, \dots, X_n)$

1) X_0 : discrete $P(X_0 = j) = p_0(j)$, $j = 0, 1, \dots, M$

2) Για $t = 1, 2, \dots, n$

   Έστω $X_{t-1} = i$

   X_t : discrete $P(X_t = j) = P_{ij}$, $j = 0, 1, \dots, M$

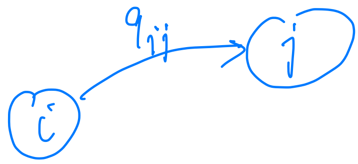
3) Επιστρ. 2.

$$P_{ij}(A) = P(X_n = j,\ X_t \notin A,\ t = 1, 2, \dots, n \mid X_0 = i)$$

Παραδ. 2 $\{X(t),\ t \geq 0\}$ ΜΔΣΧ

Προσομοίωση (1)

$$Q = (q_{ij})$$



T_i : χρόνος παραμονής στην i

$\sim \mathcal{E}xp(q_i)$

$$q_i = \sum_j q_{ij}$$

$$P(X(t)=j \mid X(t^-)=i, \text{ μετάβαση}) = \frac{q_{ij}}{q_i}$$

(2)

Όταν $X(t)=i$

$Y_{ij} \sim \mathcal{E}xp(q_{ij})$ $j=0,1,\dots,m$ ανεξ.

$T = \min\{Y_{ij},\ j=0,1,\dots,M\}$: χρόνος αλλαγής κατάστασης

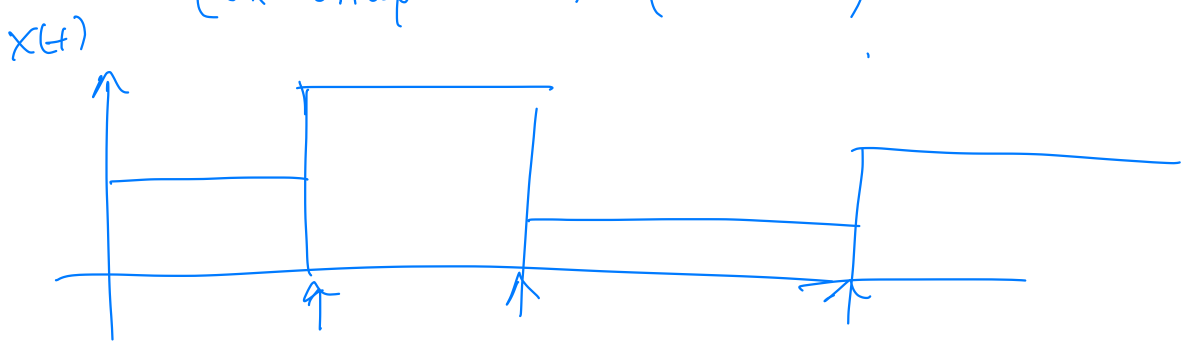
Επόμενη κατάσταση $= j$ αν $T = Y_{ij}$

discrete-event simulation.

Σύστημα Διακριτών Γεγονότων

Δυναμικό σύστημα όπου η κατάσταση αλλάζει μόνο σε διακριτές χρονικές στιγμές όταν συμβεί κάποιο γεγονός

(όχι απαραίτητα Μαρκοβιανή)



$$\theta = E(X)$$

$X \sim f$ αρκεί $E(X) = \theta$

$$X_1, \dots, X_n \rightarrow \hat{\theta} = \overline{X}_n \qquad \Delta \in [\qquad]$$

$$\begin{cases} X^{(1)} \sim f^{(1)} & E(X^{(1)}) = \theta \\ X^{(2)} \sim f^{(2)} & E(X^{(2)}) = \theta \end{cases}$$