Regularity of Lebesgue measure

Remark *Every compact* $K \subseteq \mathbb{R}$ *has* $\lambda(K) < \infty$.

Indeed, every compact set is bounded, hence there exists a bounded interval $[a, b]$ with $K \subseteq [a, b]$. It follows that $\lambda(K) \le \lambda([a, b]) = b - a < \infty$.

Theorem 1 *For every* $A \in \mathcal{M}$ *we have*

$$\begin{aligned} \lambda(A) &= \inf\{\lambda(G) : G \text{ open and } G \supseteq A\} \\ &= \sup\{\lambda(K) : K \text{ compact and } K \subseteq A\}\,. \end{aligned}$$

Proof Let $A \in \mathcal{M}$. By the monotonicity of λ, we have

$$\sup\{\lambda(K) : K \text{ compact and } K \subseteq A\} \le \lambda(A) \le \inf\{\lambda(G) : G \text{ open and } G \supseteq A\}\,.$$

To prove that

$$\lambda(A) \ge \inf\{\lambda(G) : G \text{ open and } G \supseteq A\} \tag{*}$$

we may assume that $\lambda(A) < \infty$ (otherwise there is nothing to prove).

For each $\epsilon > 0$ we have shown that there exists an open set G with $G \supseteq A$ and $\lambda(G \backslash A) < \epsilon$, and so $\lambda(G) < \lambda(A) + \epsilon$ (because $\lambda(A) < \infty$). This proves (*).

To prove

$$\lambda(A) \le \sup\{\lambda(K) : K \text{ compact and } K \subseteq A\}\,. \tag{**}$$

Special case. Assume first that A is *bounded*. We have shown (as an exercise) that for each $\epsilon > 0$ there exists a *closed* set $F \subseteq A$ such that $\lambda(A \backslash F) < \epsilon$. It follows that $\lambda(A) < \lambda(F) + \epsilon$ (since A is bounded, hence $\lambda(A) < \infty$). But F is also *bounded* (since $F \subseteq A$) and a subset of $\mathbb{R}$, hence is *compact*. This proves (**) for bounded A.

General case. For $A \in \mathcal{M}$ define $A_n = A \cap [-n, n]$ where $n \in \mathbb{N}$. Then (A_n) is an increasing sequence of measurable sets with $\bigcup_n A_n = A$, hence

$$\lambda(A) = \lim_n \lambda(A_n) = \sup_n \lambda(A_n)\,.$$

But since each A_n is bounded, by the special case we know that

$$\lambda(A_n) = \sup\{\lambda(K) : K \text{ compact and } K \subseteq A_n\}\,.$$

But $\{\lambda(K) : K \text{ compact and } K \subseteq A_n\} \subseteq \{\lambda(K) : K \text{ compact and } K \subseteq A\}$ for all n (since $A_n \subseteq A$) and therefore

$$\begin{aligned} \lambda(A) = \sup_n \lambda(A_n) &= \sup_n \sup\{\lambda(K) : K \text{ compact and } K \subseteq A_n\} \\ &\le \sup\{\lambda(K) : K \text{ compact and } K \subseteq A\}\,. \end{aligned}$$

The proof of (**) is complete.