Composition of measurable functions

$$f \circ \phi : \mathbb{R} \xrightarrow{\phi} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

When is the composition measurable?

Case 1 If ϕ is measurable and f is continuous, then $f \circ \phi$ is measurable.

Proof For each $a \in \mathbb{R}$,

$$(f \circ \phi)^{-1}((a, +\infty)) = \phi^{-1}(f^{-1}((a, \infty)))$$

is measurable because $B := f^{-1}((a, \infty))$ is open (f continuous)
so $\phi^{-1}(B) \in \mathcal{M}$ (ϕ measurable).

Case 2 If ϕ is continuous and f is measurable, then $f \circ \phi$ is not necessarily measurable.

If we repeat the argument of Case 1, then $B := f^{-1}((a, \infty))$ is measurable (f measurable); but how to conclude that $\phi^{-1}(B) \in \mathcal{M}$?

Example [1] Let $g : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function. This is continuous, increasing and onto, but it has the property that it maps the Cantor set C onto $[0, 1]$.

Let $\psi : [0, 1] \to [0, 1]$ be given by $\psi(x) = \frac{1}{2}(g(x) + x)$, and extend ψ to the whole of $\mathbb{R}$ by setting $\psi(x) = x$ when $x \notin [0, 1]$. This is now continuous and 1-1 onto, so it has a continuous inverse, $\phi : \mathbb{R} \to \mathbb{R}$.

Note that $\psi(C)$ is a subset of $[0, 1]$ which has measure 1/2. So, we know that it must contain a non-measurable set A. Then $B := \psi^{-1}(A) \subseteq C$ has measure zero, so is measurable.

Conclusion There is $B \in \mathcal{M}$ such that $\phi^{-1}(B) = A \notin \mathcal{M}$.

Now take for $f = \chi_B$. This is a measurable function since $B \in \mathcal{M}$.

But $f \circ \phi$ is *not* measurable, although ϕ is continuous.

Indeed, $f^{-1}((\frac{1}{2}, \infty)) = \{x \in \mathbb{R} : \chi_B(x) > \frac{1}{2}\} = B$, so

$$(f \circ \phi)^{-1}((\frac{1}{2}, +\infty)) = \phi^{-1}(f^{-1}((\frac{1}{2}, \infty))) = \phi^{-1}(B) = A \notin \mathcal{M}$$

[1] Thanks to D. Gatzouras

The problem is that it is possible for a continuous function to invert measurable sets to non-measurable sets. To avoid this problem, we restrict the class of continuous functions:

Case 3 If ϕ is continuous, with the additional property that $\lambda^*(N) = 0 \Rightarrow \lambda^*(\phi^{-1}(N)) = 0$, and f is measurable, then $f \circ \phi$ is measurable.

Proof For every $a \in \mathbb{R}$, the set $B := f^{-1}((a, \infty))$ is measurable (f measurable). But as we know, there exists an F_σ-set $C \subseteq B$ and a null set $N \subseteq B$ so that $B = C \cup N$. It follows that

$$\phi^{-1}(B) = \phi^{-1}(C) \cup \phi^{-1}(N)\,.$$

Now $\phi^{-1}(N)$ is a null set by assumption, so it is measurable.

Also, $\phi^{-1}(C) \in \mathcal{M}$. Indeed, C is a countable union of closed sets, $C = \bigcup_n F_n$, and so $\phi^{-1}(C) = \bigcup_n \phi^{-1}(F_n)$ is an F_σ (each $\phi^{-1}(F_n)$ is closed since ϕ is continuous), so it is also measurable.

Thus $\phi^{-1}(B)) = \phi^{-1}(C) \cup \phi^{-1}(N) \in \mathcal{M}$ and this concludes the proof.