Welcome to Fourier Analysis and Lebesgue Integration

http://eclass.uoa.gr/courses/MATH121/

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Contents

J. Fourier (1768-1830) H. Lebesgue (1875-1941)

Jean-Baptiste Joseph Fourier

Henri Lebesgue

Reminder

(a) Complex Numbers.

(b) Periodic functions.
If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, it is determined by its restriction to any interval $[a, b] \subseteq \mathbb{R}$ of length 2π. Thus it is enough to study the restriction $g := f|_{[-\pi,\pi]} : [-\pi, \pi] \to \mathbb{R}$. Note: $g(-\pi) = g(\pi)$.

Fourier Series

Let $f : [-\pi, \pi] \to \mathbb{R}$ be integrable (in the Riemann sense, for the first part of the course). The Fourier series of f is the series of functions

$$S[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where the Fourier coefficients a_k and b_k of f are given by

$$a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \qquad k = 0, 1, 2, \ldots$$

and

$$b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \qquad k = 1, 2, \ldots$$

(the integrals exist).

Remark For each $k \in \mathbb{Z}_+$,

$$|a_k| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|\,dx \quad \text{and} \quad |b_k| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|\,dx.$$

Thus, the sequences $\{a_k\}$ and $\{b_k\}$ are bounded.

The n-th partial sum of $S[f]$ is the continuous function

$$s_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Question: Does the sequence $s_n(f)$ "converge"? To f?

NO, "usually"

YES, for "good functions"

YES, "for the appropriate mode of convergence".

Trigonometric polynomials

Trigonometric series :

$$\frac{a_o}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx, \quad a_k, b_k \in \mathbb{R}\,.$$

Trigonometric polynomial:

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx + \sum_{k=1}^{N} b_k \sin kx$$

$a_k = b_k = 0$ when $k > N$. Degree: the largest N so that $|a_N| + |b_N| \neq 0$.

$$\text{Equivalent form} \quad \sum_{k=-N}^{N} c_k \exp(ikx)$$

where $\exp(it) = \cos t + i \sin t\,,$ $\quad c_k = \begin{cases} \frac{1}{2}(a_k - ib_k), & k \geq 1 \\ \frac{1}{2}a_0, & k = 0 \\ \frac{1}{2}(a_{-k} + ib_{-k}), & k \leq -1 \end{cases}$

Example 1.

For each $x \in \mathbb{R}$,

$$s_n(x) = \sum_{k=1}^{n} \sin kx = \sin x + \sin 2x + ... + \sin nx$$

$$= \begin{cases} \frac{\cos\frac{x}{2} - \cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}}, & \frac{x}{2\pi} \notin \mathbb{Z} \\ 0, & \frac{x}{2\pi} \in \mathbb{Z} \end{cases}$$

$$c_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

$$= \begin{cases} \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}, & x \neq 2m\pi \\ n + \frac{1}{2}, & x = 2m\pi \end{cases}$$

Example 1. (continued)

Although the two sequences do not converge (why?),
they are bounded (when $x \neq 2k\pi$).

Proof If $x \in (0, 2\pi)$, for each $n \in \mathbb{N}$ we have

$$\left|\frac{1}{2} + \sum_{k=1}^{n} \cos kx\right| \leq \frac{1}{2\left|\sin\frac{x}{2}\right|} \quad \text{and} \quad \left|\sum_{k=1}^{n} \sin kx\right| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}.$$

Furthermore, for each $\delta > 0$ both sequences are uniformly bounded in the interval $[\delta, 2\pi - \delta]$:
for each $x \in [\delta, 2\pi - \delta]$ and every $n \in \mathbb{N}$ we have

$$\left|\frac{1}{2} + \sum_{k=1}^{n} \cos kx\right| \leq \frac{1}{2\sin\frac{\delta}{2}} \quad \text{and} \quad \left|\sum_{k=1}^{n} \sin kx\right| \leq \frac{1}{\sin\frac{\delta}{2}}.$$

Example 2

$$s_n(x) = \sum_{k=1}^{n} \frac{1}{k^2} \sin kx = \sin x + \frac{1}{4} \sin 2x + \ldots + \frac{1}{n^2} \sin nx$$

$$c_n(x) = \sum_{k=1}^{n} \frac{1}{k^2} \cos kx = \cos x + \frac{1}{4} \cos 2x + \ldots + \frac{1}{n^2} \cos nx$$

converge uniformly to continuous functions $\mathbb{R} \to \mathbb{R}$ because

Theorem

If a sequence (g_n) of functions $g_n : X \to \mathbb{C}$ (where $X \subseteq \mathbb{R}$) is uniformly Cauchy [1]*, then (g_n) converges uniformly on X.*
If in addition the g_n are continuous on X, then their limit is a continuous function.

Proposition (Weierstrass M-test)

If for all $n \in \mathbb{N}$, $f_n : X \to \mathbb{C}$ satisfies $|f_n(t)| \leq M_n \ \forall t \in X$ where $\sum_{n=1}^{\infty} M_n < \infty$ then the sequence (g_n) where $g_n(t) = \sum_{k=1}^{n} f_n(t)$ converges uniformly.

[1] i.e. satisfies: for each $\varepsilon > 0$ there is $n_o \in \mathbb{N}$ so that if $n, m \geq n_o$ then *for each* $x \in X$ we have $|g_n(x) - g_m(x)| < \varepsilon$

Example 3

Example

$$s_n(x) = \sum_{k=1}^{n} \frac{1}{k} \sin kx = \sin x + \frac{1}{2} \sin 2x + ... + \frac{1}{n} \sin nx$$

$$c_n(x) = \sum_{k=1}^{n} \frac{1}{k} \cos kx = \cos x + \frac{1}{2} \cos 2x + ... + \frac{1}{n} \cos nx$$

We will show that both sequences converge for each $x \neq 2k\pi$ and define continuous functions. It suffices to restrict to the interval$(0, 2\pi)$, since both sequences are trigonometric polynomials, hence 2π-periodic functions. (Observe that for $x = 2k\pi$ the sequence $(c_n(x))$ diverges.)

Tools

Proposition (Dirichlet)

Let (a_k) *be a sequence of functions* $a_k : X \to \mathbb{C}$ *and* (b_k) *a sequence of numbers. If*

$$(i) \text{ there is } M < \infty \text{ so that } \forall t \in X,\ \forall n \in \mathbb{N},\ : \left|\sum_{k=1}^{n} a_k(t)\right| \leq M,$$

$$(ii)\ b_1 \geq b_2 \geq ... \geq b_n \geq 0$$

and $(iii)\ b_n \to 0$,

then the series $\sum_k b_k a_k$ *converges* uniformly *on* X.

Lemma (summation by parts)

If $b_1 \geq b_2 \geq ... \geq b_n \geq 0$ *and* $a_k \in \mathbb{C}$, *then setting* $s_0 = 0$ *and* $s_k = a_1 + a_2 + ... + a_k$, *we have for each* $m, n \in \mathbb{N}$ *with* $n > m \geq 1$,

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} s_k(b_k - b_{k+1}) + s_n b_n - s_{m-1} b_m$$

Tools

Proof of Dirichlet (Sketch) If $n, m \in \mathbb{N}$ and $n > m$, for each $t \in X$ we have (from the Lemma)

$$\left|\sum_{k=m}^{n} a_k(t)b_k\right| = \left|\sum_{k=m}^{n-1} s_k(t)(b_k - b_{k+1}) + s_n(t)b_n - s_{m-1}(t)b_m\right|$$

$$\leq \sum_{k=m}^{n-1} |s_k(t)|(b_k - b_{k+1}) + |s_n(t)|b_n + |s_{m-1}(t)|b_m$$

(since $b_n, b_m, b_k - b_{k+1} \geq 0$)

$$\leq \sum_{k=m}^{n-1} M(b_k - b_{k+1}) + Mb_n + Mb_m$$

$$= M(b_m - b_n) + Mb_n + Mb_m = 2Mb_m.$$

... since $b_m \to 0$, we obtain that the sequence of partial sums of $\sum_k b_k a_k$ is uniformly Cauchy, hence unifromly convergent. $\square$

Tools

Proposition

If $V \subseteq \mathbb{R}$ *is open* [2] *and* $f_n : V \to \mathbb{C}$ *satisfies: "for each compact* $K \subseteq V$ *the sequence* $(f_n|_K)$ *converges uniformly on* K*" (we say:* (f_n) *converges uniformly on compact subset of* V*) then for each* $x \in V$ *the sequence* $(f_n(x))$ *converges.*

If additionaly the f_n *are continuous on* V*, then their limit* $f : x \to \lim_n f_n(x)$ *is also a continuous function on* V*.*

[2] or, more generally, V: metric space

Summarising

$\left(\sum_{k=1}^{n} \sin kx\right)$ Not convergent, but $\forall\delta > 0$ uniformly bounded on $[\delta, 2\pi - \delta]$.

$\left(\sum_{k=1}^{n} \frac{1}{k^2} \sin kx\right)$ Converges uniformly on $[0, 2\pi]$, hence to a continuous function.

$\left(\sum_{k=1}^{n} \frac{1}{k} \sin kx\right)$ Converges for each $x \in (0, 2\pi)$ to a continuous function, because $\forall\delta > 0$ it converges uniformly on $[\delta, 2\pi - \delta]$.

Fourier Series

If I know that f is a trigonometric polynomial, how can I determine the coefficients?

Remark

If $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx + \sum_{k=1}^{N} b_k \sin kx,$$

then

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx$$

Fourier Series

Remark (Complex form)

$$\textit{if} \quad f(x) = \sum_{k=-N}^{N} c_k \exp ikx$$

then,

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp(-imx) dx, \quad -N \le m \le N.$$

because if $k \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \exp(ikx) dx = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Fourier Series

Generalisation: Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define

$$a_n = a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \quad (n = 0, 1, 2, \ldots)$$

$$b_m = b_m(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx dx, \quad (m = 1, 2, \ldots)$$

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\exp(-ikx)dx, \quad (k \in \mathbb{Z})$$

It suffices that the integrals exist.

Definition: The Fourier series $S(f)$ of f:

$$S(f, x) := \frac{a_o}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx} \text{ (complex form)}$$

(For now, we are not concerned with convergence or divergence of these series.)

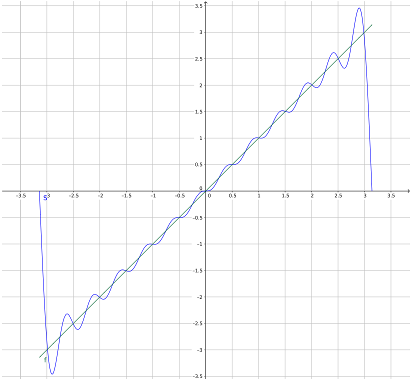
Example

The Fourier series of the function $f(t) = t,\ t \in (-\pi, \pi)$ is

$$f \sim 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + ...\right)$$

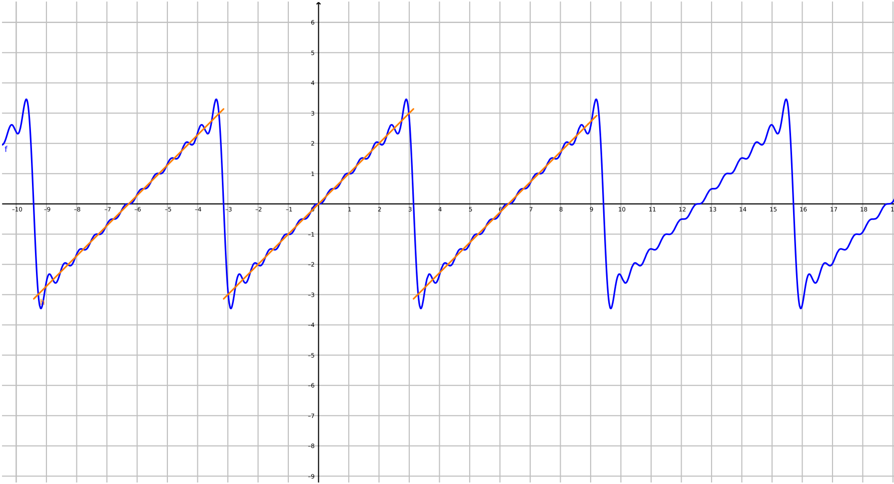
Ic can be shown (Exercise!) that the partial sums of this series form a Cauchy sequence and therefore the series converges.

But does it converge to f?



Parenthesis: periodic extension

$$2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \ldots - \frac{1}{12}\sin 12t\right)$$



Fourier series

Remark

• *The Fourier series of a trigonometric polynomial* p *is the trig. polynomial itself:* $S_n(p) = p$ *when* $n \geq \deg p$*, hence* $S(p) = p$*.*

• *If a trigonometric series* $s(x) = \sum_k c_k e^{ikx}$ *converge uniformly, then the Fourier coefficients* $\hat{s}(k)$ *of* s *are the* c_k*, hence the Fourier series of* s *is* s*.*

• *It is not however always true that every convergent trigonometric series is the Fourier series of some function (see later).*

Fourier series

Proposition (Linearity!)

If f *and* g *are integrable on* $[0, 2\pi]$ *and* $\lambda \in \mathbb{C}$*,*

$$a_n(f + \lambda g) = a_n(f) + \lambda a_n(g),$$

$$b_n(f + \lambda g) = b_n(f) + \lambda b_n(g) \quad (n, m \in \mathbb{N})$$

equivalently $\quad (\widehat{f + \lambda g})(k) = \hat{f}(k) + \lambda \hat{g}(k) \quad (k \in \mathbb{Z})$

therefore $\quad S_n(f + \lambda g) = S_n(f) + \lambda S_n(g) \quad (n \in \mathbb{N}).$

Absolutely convergent Fourier series

Proposition

If f is a continuous and 2π-periodic function and $\sum |\hat{f}(k)| < \infty$ (equivalently $\sum(|a_k(f) + |b_k(f)| < \infty$) then the sequence $(S_N(f))$ converges uniformly (and hence the fumction $S(f) := \lim_N S_N(f)$ is continuous).

Proof Weierstrass' M-test.

But how to conclude that $(S_N(f))$ converges to f?

Observe that for each $k \in \mathbb{Z}$ we have $\widehat{S_N(f)}(k) = \hat{f}(k)$ when $N \geq |k|$, hence $\widehat{S(f)}(k) = \hat{f}(k)$ for each $k \in \mathbb{Z}$ (why?).

It suffices therefore to prove the following Uniqueness Theorem:

The Uniqueness Theorem

Theorem

If f and g is continuous and 2π-periodic functions with $\hat{g}(k) = \hat{f}(k)$ for each $k \in \mathbb{Z}$ (equivalently $a_n(f) = a_n(g)$ and $b_n(f) = b_n(g)$ for each $n \in \mathbb{N}$), then $f = g$.

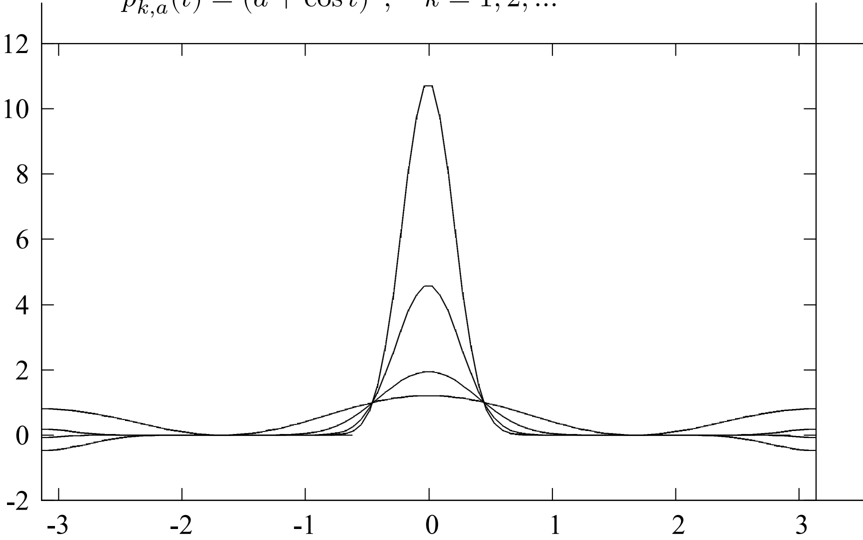
Sketch of Proof We will show that if $f \neq g$ there exists a trig. polynomial p with $\int_{-\pi}^{\pi} fp \neq \int_{-\pi}^{\pi} gp$. Then, there must exist k so that $\int_{-\pi}^{\pi} fe_k \neq \int_{-\pi}^{\pi} ge_k$, i.e. $\hat{f}(-k) \neq \hat{g}(-k)$.

Let $\psi := f - g$. In the special case: $\psi(0) > 0$, we will show there is a trigonometric polynomial of the form $p_{k,a}(t) = (a + \cos t)^k$ for appropriate a, k such that $\int_{-\pi}^{\pi} \psi p \neq 0$.

General case: If $\psi(t_0) := h(t_0) \neq 0$, there is a θ so that $e^{i\theta}\psi(t_0) > 0$, hence the function ϕ given by $\phi(s) = e^{i\theta}\psi(s + t_0)$ satisfies $\phi(0) > 0$. Thus some $\hat{\phi}(k)$ must be nonzero. But then $\hat{\psi}(k) = e^{-i\theta}e^{ikt_0}\hat{\phi}(k) \neq 0$.

The trigonometric polynomials $p_{k,a}$

$$p_{k,a}(t) = (a + \cos t)^k, \quad k = 1, 2, \ldots$$



with $a = \frac{1}{10},\ k = 2, 7, 16, 25.$

The Uniqueness Theorem

Continuity was used only at the point t_0 :

Theorem

If f and g are integrable on $[-\pi, \pi]$ and $\hat{g}(k) = \hat{f}(k)$ for each $k \in \mathbb{Z}$ (equivalently $a_n(f) = a_n(g)$ and $b_n(f) = b_n(g)$ for each $n \in \mathbb{N}$), then $f(t_0) = g(t_0)$ at each point where $f - g$ is continuous.

Simple cases of convergence

Proposition

If f continuous, 2π-periodic and $\sum |\hat{f}(k)| < \infty$ (equivalently $\sum(|a_k(f) + |b_k(f)| < \infty$) then $(S_N(f))$ converges uniformly to f.

Proposition

If f continuous, 2π-periodic and its derivative f' exists and is integrable,

$$S(f', x) = \sum_{k=1}^{\infty} (kb_k \cos kx - ka_k \sin kx).$$

Complex form:

$$\widehat{f'}(k) = ik\hat{f}(k) \quad (k \in \mathbb{Z}).$$

Simple cases of convergence

Proposition

If $f : \mathbb{R} \to \mathbb{C}$ is continuous, 2π-periodic and $\sum |k\hat{f}(k)| < \infty$, then f is continuously differentiable and the series $\sum ik\hat{f}(k) \exp ikx$ converges to f' uniformly.

Lemma

If f and its derivatives $f', f'', \dots, f^{(n-1)}$ are continuous 2π-periodic functions and $|f^{(n)}|$ is integrable then $|\hat{f}(k)| \leq \frac{\|f^{(n)}\|_1}{|k|^n}$ for each $k \neq 0$ (where $\|g\|_1 = \frac{1}{2\pi}\int |g|$).

Proposition

If f, f' and f'' are continuous and 2π-periodic, the series $\sum \hat{f}(k) \exp ikx$ converges uniformly to f.

Fejér' s Theorem

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and 2π-periodic.
Reminder: $S_n(f,t) = \sum_{|k| \leq n} \hat{f}(k) e^{ikt}$.
The sequence $(S_n(f))$ is not always always convergent (not even pointwise). However,

Theorem (Fejér)

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous and 2π-periodic function, then the sequence $(\sigma_n(f))$ where

$$\sigma_m(f) = \frac{1}{m+1} \sum_{n=0}^{m} S_n(f) \quad (m \in \mathbb{N})$$

converges to f uniformly.

Summability

$$S_n(f)(t) = \sum_{k=-n}^{k=n} \hat{f}(k) \exp(ikt)$$

$$= \sum_{k=-n}^{k=n} \left(\int_{-\pi}^{\pi} f(s) \exp(-iks) \frac{ds}{2\pi} \right) \exp(ikt)$$

$$= \int_{-\pi}^{\pi} \left(\sum_{k=-n}^{k=n} \exp(ik(t-s)) \right) f(s) \frac{ds}{2\pi} := \int_{-\pi}^{\pi} D_n(t-s) f(s) \frac{ds}{2\pi}.$$

hence

$$\sigma_m(f)(t) = \frac{1}{m+1} \sum_{n=0}^{m} S_n(f)(t)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{m+1} \sum_{n=0}^{m} \sum_{k=-n}^{k=n} \exp(ik(t-s)) \right) f(s) ds$$

$$:= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_m(t-s) f(s) ds.$$

Two kernels: Dirichlet against Fejér

Dirichlet: $$D_n(x) = \sum_{k=-n}^{k=n} \exp(ikx) = \begin{cases} \frac{\sin\left(\frac{2n+1}{2}x\right)}{\sin(x/2)}, & x \neq 0, \\ 2n+1, & x = 0 \end{cases} \qquad (d)$$

Fejér: $$K_m(x) = \frac{1}{m+1} \sum_{n=0}^{m} \left(\sum_{k=-n}^{n} \exp(ikx) \right) = \begin{cases} \frac{1}{m+1} \left(\frac{\sin\left(\frac{m+1}{2}x\right)}{\sin(x/2)} \right)^2, & x \neq 0, \\ m+1, & x = 0 \end{cases} \qquad (k)$$

Proof of (d) for $x \neq 0$

$$
\begin{aligned}
\sin(\frac{x}{2})D_n(x) &= \sin(\frac{x}{2}) \sum_{k=-n}^{k=n} \exp(ikx) \\
\Rightarrow \quad (e^{\frac{ix}{2}} - e^{-\frac{ix}{2}})D_n(x) &= (e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}) \sum_{k=-n}^{k=n} \exp(ikx) \\
\Rightarrow \quad (e^{ix} - 1)D_n(x) &= (e^{ix} - 1) \sum_{k=-n}^{k=n} \exp(ikx) \\
&= \sum_{k=-n}^{k=n} (\exp(i(k+1)x) - \exp(ikx)) \\
&= \exp(i(n+1)x) - \exp(-inx) \\
&= e^{\frac{ix}{2}} (\exp(i(n+\frac{1}{2})x) - \exp(-i(n-\frac{1}{2}x)) \\
&= e^{\frac{ix}{2}} 2i \sin((n+\frac{1}{2})x) \,.
\end{aligned}
$$

Proof of (k)

If $x \neq 0$,

$$\begin{aligned}
\frac{1}{\sin\frac{x}{2}}\sum_{n=0}^{m}\sin(n+\frac{1}{2})x =& \frac{1}{2\sin^2\frac{x}{2}}\sum_{n=0}^{m} 2\sin\frac{x}{2}\sin(n+\frac{1}{2})x \\
=& \frac{1}{2\sin^2\frac{x}{2}}\sum_{n=0}^{m}(\cos nx - \cos(n+1)x) \\
=& \frac{1}{2\sin^2\frac{x}{2}}(1-\cos(m+1)x)
\end{aligned}$$

Therefore $K_m(x) = \frac{1}{m+1}\sum_{n=0}^{m} D_n(x) = \frac{1}{m+1}\sum_{n=0}^{m}\frac{\sin(n+\frac{1}{2})x}{\sin(x/2)}$,

$$K_m(x) = \frac{1}{m+1}\cdot\frac{1}{\sin^2\frac{x}{2}}\frac{1-\cos(m+1)x}{2} = \frac{1}{m+1}\cdot\frac{\sin^2(\frac{m+1}{2}x)}{\sin^2\frac{x}{2}}.$$

If $x = 0$,

$$K_m(0) = \frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=-n}^{k=n}\exp 0 = \frac{1}{m+1}\sum_{n=0}^{m}(2n+1) = m+1. \quad \square$$

Lemma

Claim: $K_m = \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) e_k\,.$ Proof:

$$K_m = \frac{1}{m+1} \sum_{n=0}^{m} \left(\sum_{k=-n}^{n} e_k\right) = \frac{1}{m+1} \sum_{n=0}^{m} (e_0 + (e_1 + e_{-1}) + \cdots + (e_n + e_{-n}))$$

$$= \frac{1}{m+1}(e_0 \qquad (n=0)$$

$$+ e_0 + (e_1 + e_{-1}) \qquad (n=1)$$

$$+ e_0 + (e_1 + e_{-1}) + (e_2 + e_{-2}) \qquad (n=2)$$

$$+ \ldots \ldots$$

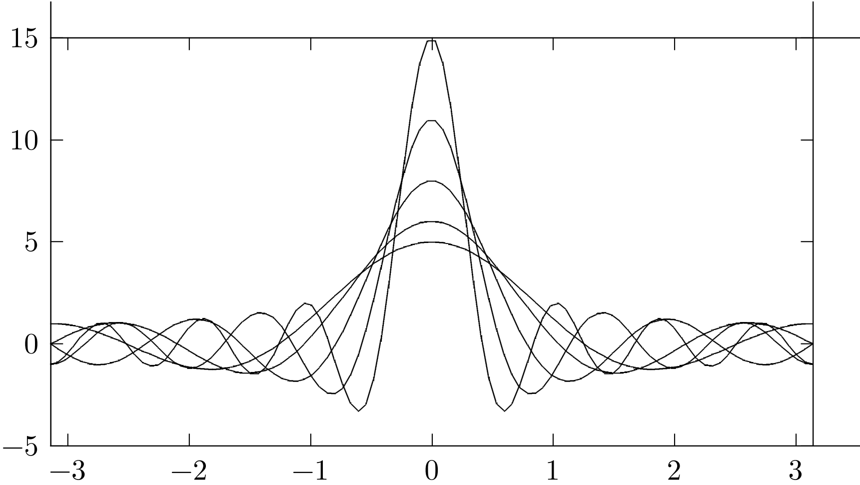
$$+ e_0 + (e_1 + e_{-1}) + (e_2 + e_{-2}) + \cdots + (e_m + e_{-m})) \qquad (n=m)$$

$$= e_0 + \frac{m}{m+1}(e_1 + e_{-1}) + \frac{m-1}{m+1}(e_2 + e_{-2}) + \cdots + \frac{1}{m+1}(e_n + e_{-n})$$

$$= \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) e_k\,.$$

The Dirichlet kernel

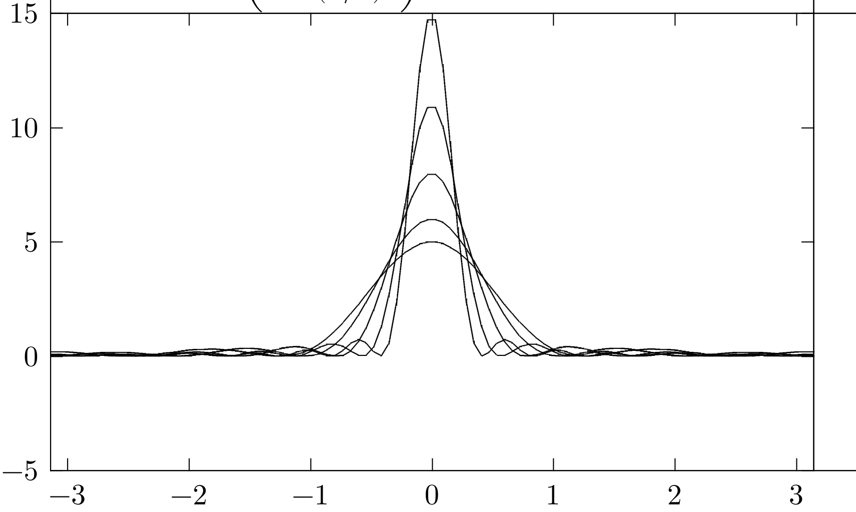
$$D_m(x) = \frac{\sin\left(\frac{2m+1}{2}x\right)}{\sin(x/2)},\ x \neq 0, \quad D_m(0) = 2m+1.$$



$m = 4, 5, 7, 10, 14.$

The Fejér kernel

$$K_m(x) = \frac{1}{m+1}\left(\frac{\sin(\frac{m+1}{2}x)}{\sin(x/2)}\right)^2, \; x \neq 0, \quad K_m(0) = m+1.$$



$m = 4, 5, 7, 10, 14.$

Properties of Fejér's kernel K_m

Remark

The Fejér kernel has the following properties:

(α) *There exists M so that $\|K_m\|_1 \leq M$ for each m.*

(β) *If $\delta \in (0, \pi)$ and $E_\delta = [-\pi, -\delta] \cup [\delta, \pi]$, then $\lim_m \int_{E_\delta} |K_m| = 0$.*

(γ) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_m(x)dx = 1$ *for every m.*

• Property (γ) holds by the definition of K_m, since $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt}dt = 1$ if $k = 0$ and 0 otherwise.

• Since $K_m(t) \geq 0$, (γ) implies (α) with $M = 1$.

• Property (β) follows from the remark that if $\delta \leq |x| \leq \pi$, then $|K_m(x)| = K_m(x) \leq \frac{1}{m+1} \frac{1}{\sin^2 \frac{\delta}{2}}$, hence $\lim_m K_m(x) = 0$ uniformly in E_δ and hence $\lim_m \int_{E_\delta} |K_m| = 0$.

Fejér's Theorem: Sketch of the proof

If $\delta > 0$, for large enough $m \in \mathbb{N}$, the value $K_m(s)$ is almost 0 outside the interval $[-\delta, \delta]$ (by (β)). Therefore

$$\sigma_m(f)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)K_m(s)ds \approx \frac{1}{2\pi}\int_{-\delta}^{\delta} f(t-s)K_m(s)ds$$

where the symbol $\approx$ means "nearly equal" here. But f is uniformly continuous, hence if δ is small enough, when $|s| < \delta$ we have $f(t-s) \approx f(t)$. Therefore

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} f(t-s)K_m(s)ds \approx f(t)\left(\frac{1}{2\pi}\int_{-\delta}^{\delta} K_m(s)ds\right)$$

and, again from (β),

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} K_m(s)ds \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} K_m(s)ds = 1$$

by (γ). Thus finally $\sigma_m(f)(t) \approx f(t)$.

First consequences of Fejér's Theorem

- Uniqueness. If f, g are continuous, 2π-periodic and $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$, then $f = g$.

Second Proof. We have $\sigma_n(f) = \sigma_n(g)$ for each $n \in \mathbb{N}$, hence $f = \lim_n \sigma_n(f) = \lim_n \sigma_n(g) = g$ by Fejér.

- Proposition [Fejér] Let $f : \mathbb{R} \to \mathbb{C}$ be Riemann integrable in $[-\pi, \pi]$ and 2π-periodic. If f is continuous at some $t \in [-\pi, \pi]$, then $\sigma_n(f, t) \to f(t)$. [The proof is a variation of the previous one: now δ will depend on t, and convergence is shown at t.]

[Remark: More generally, if the one-sided limits $f(t_+)$ and $f(t_-)$ exist, then $\sigma_n(f, t) \to \frac{f(t_+)+f(t_-)}{2}$. (Proof omitted).]

- Corollary Under the conditions of the Proposition, if $(S_n(f, t_0))$ converges, then it must converge to $f(t_0)$.

- Remark For every f, Riemann integrable in $[-\pi, \pi]$ and 2π-periodic, we have $\|\sigma_n(f)\|_\infty \leq \|f\|_\infty$.

Sections 7 to 10

For the following, see the file `not60520en.pdf`

7. Mean square convergence

8. The Poisson kernel

9. Pointwise convergence and the localisation principle

10. Complements: Divergent Fourier series

Example

$$\text{If} \quad f(t) = \begin{cases} -i(\pi + t), & -\pi \le t < 0 \\ i(\pi - t), & 0 \le t < \pi \end{cases}$$

$$\text{then} \quad S_n(f,t) = \left(\sum_{k=-n}^{-1} + \sum_{k=1}^{n} \right) \frac{1}{k} e^{ikt}$$

$$|S_n(f,t)| \le \|f\|_\infty + 2 = \pi + 2$$

is uniformly bounded, but its ‘negative’ (co-analytic) part $g_n(t) = \sum_{k=-n}^{-1} \frac{1}{k} e^{ikt}$ is not : $g_n(0) = \sum_{m=1}^{n} \frac{1}{-m}$.
hence there cannot exist any *Riemann*-integrable g so that $g_n = S_n(g)$.
We will see later that there exists a *Lebesgue*-integrable g with $g_n = S_n(g)$!

Example: A continuous f with $\limsup |S_n(f,0)| = \infty$

If

$$p_N(x) = e^{i2Nx} \sum_{1 \le |k| \le N} \frac{e^{ikx}}{k}$$

we have shown that there exists M so that $|p_N(x)| \le M$ for all $N \in \mathbb{N}$ and every $x \in \mathbb{R}$. For a subsequence (N_k), define

$$f(x) = \sum_{k=1}^{\infty} a_k p_{N_k}(x)$$

where $a_k = \frac{1}{k^2}$: the series converges uniformly, hence f is continuous. But if $N_k = 3^{2^k}$, $k = 1, 2, \ldots$, then

$$|S_{2N_m}(f)(0)| \to +\infty$$

because $|S_{2N_m}(f)(0)| \ge |g_{N_m}(0)| - c \ge c a_m \log |N_m|$ for a suitable $c > 0$.

Part II

The Lebesgue integral

Behaviour with regard to limits:

Example

Consider the Dirichlet function $f = \chi_{\mathbb{Q}} : [0,1] \to \mathbb{R}$.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

It is not Riemann integrable. But if $\{q_n : n \in \mathbb{N}\}$ is an enumeration of $\mathbb{Q} \cap [0,1]$ and

$$f_n(x) = \begin{cases} 1, & x \in \{q_1, \ldots, q_n\} \\ 0, & x \notin \{q_1, \ldots, q_n\}, \end{cases}$$

then $f_n \nearrow f$ in $[0,1]$ and each f_n is Riemann integrable, being a bounded function with a finite number of discontinuities.

The Riemann integral and the Lebesgue integral

Let $f : [a,b] \to \mathbb{R}$ be bounded.

Riemann: Partition $[a,b]$: $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$

$$L(f,P) = \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k) \text{ and } U(f,P) = \sum_{k=0}^{n-1} M_k(x_{k+1} - x_k)$$

where

$$m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\} \text{ and } M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\}.$$

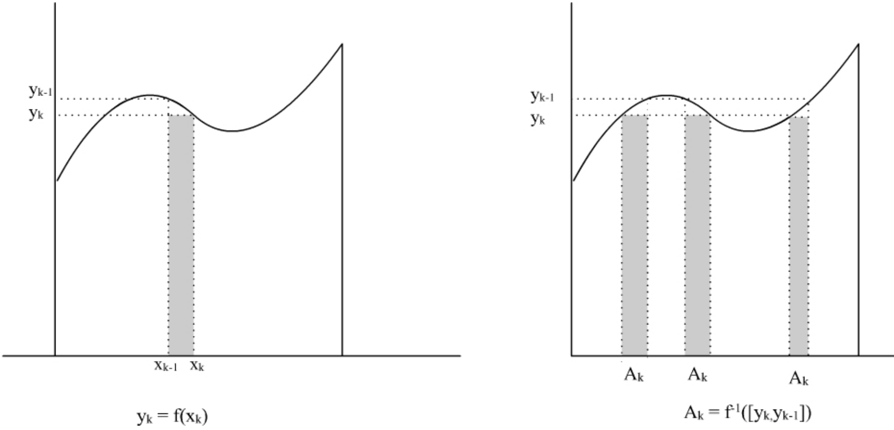
Lebesgue: Partition the range $[m, M]$ of f

$$Q = \{m = y_0 < y_1 < y_2 < \ldots < y_t = M\}.$$

$$\widetilde{L}(f,Q) = \sum_{k=0}^{t-1} y_k \mu(f^{-1}([y_k, y_{k+1}))) \text{ and } \widetilde{U}(f,Q) = \sum_{k=1}^{t-1} y_{k+1} \mu(f^{-1}([y_k, y_{k+1}))$$

μ = "length" (??)

The Riemann integral and the Lebesgue integral



Problem: How to define the "length" of the (possibly complicated) set $f^{-1}([y_k, y_{k-1})) = \{x \in [a,b] : y_k \leq f(x) < y_{k+1}\})$.

Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable. Given $\varepsilon > 0$, choose P with $U(f, P) - L(f, P) < \varepsilon$, pick any $t_k \in [x_k, x_{k+1}]$ and put

$$f_\varepsilon = \sum_{k=0}^{n-1} f(t_k)\chi_k \quad \text{(a step function)}$$

where $\chi_k = \chi_{(x_k, x_{k+1}]}$. Then $\displaystyle\int_a^b |f - f_\varepsilon| < \varepsilon$.

For each χ_k and every $\delta > 0$ there exists a continuous h_k so that $\displaystyle\int_a^b |\chi_k - h_k| < \delta$.

Therefore, if $h_\varepsilon := \displaystyle\sum_{k=0}^{n-1} f(t_k)h_k$ then $\displaystyle\int_a^b |f_\varepsilon - h_\varepsilon| \leq n\delta \, \|f\|_\infty$.

Conclusion: there exists $h_\varepsilon : [a, b] \to \mathbb{R}$ continuous so that $\displaystyle\int_a^b |f - h_\varepsilon| < 2\varepsilon$.

Desirable properties of “length”

(a) $\mu((a,b)) = b - a$

(b) $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mu(E_n)$ when (E_n) are pairwise disjoint

(c) $\mu(E + x) = \mu(E)$ for all $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$

Remark (α) The map $\phi : t \mapsto e^{2\pi it}$ defines a bijective correspondence between $(0,1] \subseteq \mathbb{R}$ and the unit circle $S := \{e^{2\pi it} : t \in \mathbb{R}\} \subseteq \mathbb{C}\}$ which transforms “length” to “arc length”.

(β) If there exists a set $U \subseteq S := \{e^{2\pi it} : t \in \mathbb{R}\}$ such that the sets $U_q := \{e^{2\pi iq}w : w \in U\}$ (where $q \in \mathbb{Q}$) are pairwise disjoint and their union is the circle S, then U cannot be “measured”, hence $\phi^{-1}(U) \subseteq (0,1]$ cannot be “measured”.

$\rightsquigarrow$

There exist sets that cannot be "measured"

For z, w in the circle S define $\quad z \sim w \iff \exists\, q \in \mathbb{Q} : w = e^{2\pi i q} z.$

The equivalence relation $\sim$ splits (partitions) S into (disjoint) classes: $S = \bigcup_{z \in S} [z]$ where $[z] = \{w \in S : w \sim z\}$.
The Axiom of Choice (!) ensures that we may choose one representative $u \in [z]$ from each class. Let $U \subseteq S$ be the set of all these choices, so that $U \cap [z]$ is a singleton for each class $[z]$; thus we have

$$S = \bigcup_{u \in U} [u] \quad \text{(a union of orbits).}$$

For each $q \in \mathbb{Q}$, define $U_q := \{e^{2\pi i q} w : w \in U\}$.

This gives a (different) partition

$$S = \bigcup_{q \in \mathbb{Q}} U_q \quad \text{(countable union of translates of } U\text{).}$$

Suppose that U could be "measured". Then $\mu(U_q) = \mu(U)\ \forall q$, so

$$\mu(S) = \sum_{q \in \mathbb{Q}} \mu(U_q) = \sum_{q \in \mathbb{Q}} \mu(U).$$

But if $\mu(U) = 0$ then $\mu(S) = 0$, while if $\mu(U) > 0$ then $\mu(S) = \infty$. (!)

Strategy: Restrict to "measurable" sets

The strategy will be to define the "length" or "measure" only for a subclass of sets, for which the desirable requirements are fulfilled.

The method to achieve this will be to first define a function (called "outer measure") on all subsets of $\mathbb{R}$ which *partly* satisfies the requirements, and then restrict to the class of sets on which this outer measure satisfies the requirements completely.
We will show that this class (the measurable sets) is large enough.

Definition of Lebesgue outer measure

Let $I = (a, b) \subseteq \mathbb{R}$ be a bounded open interval.

Its length: $\ell(I) := b - a$.

By a cover of a set $A \subseteq \mathbb{R}$ we will mean a *countable* family of bounded open intervals (I_n) with $A \subseteq \bigcup_n I_n$.

Definition (Lebesgue outer measure)

Let $A \subseteq \mathbb{R}$. The outer measure of A is

$$\lambda^*(A) := \inf \left\{ \sum_n \ell(I_n) : (I_n) \text{ cover of } A \right\}.$$

Lebesgue outer measure

Proposition

If $A \subseteq B \subseteq \mathbb{R}$, *then* $\lambda^*(A) \leq \lambda^*(B)$.

Proposition

If $A \subseteq \mathbb{R}$ *is finite or countably infinite, then* $\lambda^*(A) = 0$.

Note But there exist uncountable sets with $\lambda^*(A) = 0$
(for example the Cantor set - see later).

Proposition

$\lambda^*(A + x) = \lambda^*(A)$ *for each* $A \subseteq \mathbb{R}$ *and* $x \in \mathbb{R}$.

Lebesgue outer measure

Proposition

$\lambda^*([a,b]) = b - a.$

Proposition

$\lambda^*((a,b)) = b - a \; (= \ell((a,b)).$

The property

$$\lambda(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \lambda(E_n)$$

when $\{E_n : n \in \mathbb{N}\}$ *are pairwise disjoint (σ-additivity)*

cannot hold for all families $\{E_n : n \in \mathbb{N}\}$, as we saw.

Nevetheless,

Lebesgue outer measure and measurability

Proposition (countable subadditivity)

For each finite or countably infinite family $\{A_n\}$ *of subsets of* $\mathbb{R}$*,*

$$\lambda^*\left(\bigcup_n A_n\right) \le \sum_n \lambda^*(A_n).$$

We want to achieve equality when the $\{A_n\}$ are pairwise disjoint.
We are forced to restrict to sets which "have length":

Definition (Lebesgue measurable set)

A sets $A \subseteq \mathbb{R}$ is called Lebesgue measurable if, for each $X \subseteq \mathbb{R}$,

$$\lambda^*(X) = \lambda^*(X \cap A) + \lambda^*(X \cap A^c).$$

The class of Lebesgue measurable sets is denoted by $\mathcal{M}$.
The restriction of λ^* to $\mathcal{M}$ is called Lebesgue measure.

Thus, a set is measurable if "it splits correctly" – with respect to outer measure – all other sets.

The class of measurable sets

Remark. In order to prove that $A \in \mathcal{M}$, it suffices to show

$$\lambda^*(X) \geq \lambda^*(X \cap A) + \lambda^*(X \cap A^c).$$

for each $X \subseteq \mathbb{R}$ (in fact, it suffices to assume $\lambda^*(X) < \infty$).

Proposition

If $\lambda^*(A) = 0$, then $A \in \mathcal{M}$.

Proposition

The complement of a measurable set is measurable:
if $A \in \mathcal{M}$ then $A^c = \mathbb{R} \backslash A \in \mathcal{M}$.

Proposition

The union of two measurable sets is measurable:
if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

Hence also the intersection: $(A \cap B) = (A^c \cup B^c)^c$.

The class of measurable sets

Proof

$X \cap (A \cup B) = X \cap (A \cup (A^c \cap B)) = (X \cap A) \cup (X \cap A^c \cap B)$,
hence

$$\begin{aligned}
&\lambda^*(X \cap (A \cup B)) + \lambda^*(X \cap (A \cup B)^c) = \\
&= \lambda^*((X \cap A) \cup (X \cap A^c \cap B)) + \lambda^*(X \cap (A \cup B)^c) \\
&\overset{(sub)}{\leq} \lambda^*(X \cap A) + \lambda^*((X \cap A^c) \cap B) + \lambda^*((X \cap A^c) \cap B^c) \\
&\overset{(B \in \mathcal{M})}{=} \lambda^*(X \cap A) + \lambda^*(X \cap A^c) \\
&\overset{(A \in \mathcal{M})}{=} \lambda^*(X).
\end{aligned}$$

Thus,

$$\lambda^*(X \cap (A \cup B)) + \lambda^*(X \cap (A \cup B)^c) \leq \lambda^*(X).$$

The class of measurable sets

Proposition

If $A, B \in \mathcal{M}$ *and* $A \cap B = \emptyset$ *then, for each* $X \subseteq \mathbb{R}$,

$$\lambda^*(X \cap (A \cup B)) = \lambda^*(X \cap A) + \lambda^*(X \cap B).$$

hence

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B).$$

By induction:

Corollary (Finite aditivity)

If $B_1, \ldots, B_m$ *are pairwise disjoint sets in* $\mathcal{M}$ *then, for each* $X \subseteq \mathbb{R}$,

$$\lambda^*(X \cap (B_1 \cup \cdots \cup B_m)) = \sum_{n=1}^{m} \lambda^*(X \cap B_n)$$

hence

$$\lambda^*(B_1 \cup \cdots \cup B_m) = \sum_{n=1}^{m} \lambda^*(B_n).$$

The class of measurable sets

Proposition

If $(A_n)_{n=1}^{\infty}$ is a countable family of measurable sets, then their union $\bigcup_{n=1}^{\infty} A_n$ is a measurable set.

*** *** ***

Definition (σ-algebra)

Let Ω be a nonempty set. A class $\mathcal{A}$ of subsets of Ω is called a σ-algebra if it satisfies

(i) $\Omega \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$, then $\Omega \backslash A \in \mathcal{A}$.

(iii) If $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

It follows that:

(iv) If $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.

(v) If $A, B \in \mathcal{A}$, then $A \backslash B = A \cap B^c \in \mathcal{A}$.

The class of measurable sets

Theorem

Let $\mathcal{M} = \{A \subseteq \mathbb{R} \mid A \text{ Lebesgue measurable}\}$. Then $\mathcal{M}$ is a σ-algebra and the set function $\lambda : \mathcal{M} \to [0, +\infty]$

$$A \mapsto \lambda(A) := \lambda^*(A)$$

is countably additive (σ-additive). Thus, if $(A_n)_{n=1}^{\infty}$ is a countable family of pairwise disjoint Lebesgue measurable sets ($A_n \in \mathcal{M}$ for all n and $A_n \cap A_m = \emptyset$ if $n \neq m$), then

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Definition (Lebesgue masure)

The set function $\lambda : \mathcal{M} \to [0, +\infty]$

$$A \mapsto \lambda(A) := \lambda^*(A)$$

is called Lebesgue measure.

Borel sets. They are Lebesgue measurable

Proposition

All intervals are Lebesgue measurable sets.

Consider the intersection of all σ-algebras containing the set of intervals:

Definition (The Borel σ-algebra)

The smallest σ-algebra of subsets of $\mathbb{R}$ which contains the set of all intervals is called the σ-algebra of Borel subsets of $\mathbb{R}$ (or the Borel σ-algebra) and is denoted by $\mathcal{B}$.

Proposition

$\mathcal{B} \subseteq \mathcal{M}$ *(we will show later that $\mathcal{B} \neq \mathcal{M}$)*.

Proposition

Every open and every closed subset of $\mathbb{R}$ is a Borel set, hence is measurable.

... hence every countable intersection of open sets (every G_δ) and every countable union of closed sets (every F_σ).

Approximating measurable sets

Proposition

Let $A \subseteq \mathbb{R}$. The following are equivalent:

1. *The set A is measurable.*
2. *For every $\varepsilon > 0$ there exists an open set $G \subseteq \mathbb{R}$ with $A \subseteq G$ and $\lambda^*(G \backslash A) < \varepsilon$.*
3. *There exists a G_δ-set B so that $A \subseteq B$ and $\lambda^*(B \backslash A) = 0$.*

Proposition

Let $A \subseteq \mathbb{R}$. The following are equivalent:

1. *The set A is measurable.*
2. *For every $\varepsilon > 0$ there exists a closed set $F \subseteq \mathbb{R}$ with $F \subseteq A$ and $\lambda^*(A \backslash F) < \varepsilon$.*
3. *There exists an F_σ-set C such that $C \subseteq A$ and $\lambda^*(A \backslash C) = 0$.*

(Exercise)

"Continuity" of measure

Remark

If $X, Y \in \mathcal{M}$, $X \subseteq Y$ *and* $\lambda(X) < \infty$, *then* $\lambda(Y \backslash X) = \lambda(Y) - \lambda(X)$.

Proposition

(i) *If* (A_n) *is an increasing sequence of measurable sets and* $A := \bigcup_{n=1}^{\infty} A_n$, *then*

$$\lambda(A_n) \to \lambda(A).$$

(ii) *If* (B_n) *is a decreasing sequence of measurable sets with* $\lambda(B_1) < +\infty$ *and* $B := \bigcap_{n=1}^{\infty} B_n$, *then*

$$\lambda(B_n) \to \lambda(B).$$

Remark: For (ii), it is enough to have $\lambda(B_k) < +\infty$ for *some* k. But (ii) fails for $B_n = [n, \infty)$, for example.

Regularity of Lebesgue measure

Theorem

Lebesgue measure λ on $\mathbb{R}^k$ is a regular measure.
For each K compact, we have $\lambda(K) < \infty$ and for each $A \in \mathcal{M}$

$$\begin{aligned}\lambda(A) &= \sup\{\lambda(K) : K \text{ compact and } K \subseteq A\} \\ &= \inf\{\lambda(G) : G \text{ open and } G \supseteq A\}\,.\end{aligned}$$

For a proof for $k = 1$ see `regen.pdf`.

It is possible for a measurable set of positive measure to contain no nonempty open intervals (examples later). However,

Theorem (Steinhaus)

If A is a Lebesgue measurable subset of $\mathbb{R}^k$ with $\lambda(A) > 0$, then there is a $\delta > 0$ so that

$$B(0, \delta) \subseteq A - A.$$

Summary: Lebesgue Measure

The outer Lebesgue measure of a subset $A \subseteq \mathbb{R}$ is

$$\lambda^*(A) = \inf \left\{ \sum_n \ell(I_n) : (I_n) \text{ cover of } A \right\}.$$

A set $A \subseteq \mathbb{R}$ is called Lebesgue measurable ($A \in \mathcal{M}$) if, for all $X \subseteq \mathbb{R}$,

$$\lambda^*(X) = \lambda^*(X \cap A) + \lambda^*(X \cap A^c).$$

Equivalently, if for every $\varepsilon > 0$ there is an open $G \subseteq \mathbb{R}$ with $A \subseteq G$ and $\lambda^*(G \backslash A) < \varepsilon$.
Equivalently, if for every $\varepsilon > 0$ there is a closed $F \subseteq \mathbb{R}$ with $F \subseteq A$ and $\lambda^*(A \backslash F) < \varepsilon$.

When $A \in \mathcal{M}$, the Lebesgue measure of A is defined to be its outer measure.
The family $\mathcal{M}$ contains all open sets, and is closed for complements and countable unions (it is a σ-algebra). But there exist non-measurable sets.
The σ-algebra $\mathcal{B} \subseteq \mathcal{M}$ generated by the open sets is called the Borel σ-algebra.
The map $\lambda : \mathcal{M} \to [0, +\infty] : A \mapsto \lambda^*(A)$ is σ-additive: if $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{M}$ are pairwise disjoint,

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n).$$

The measure λ is invariant under translations. It is a regular measure.

Measurable functions

Reminder If $f : \mathbb{R} \to \mathbb{R}$, we want to define $\int f d\lambda$ by approximating it by sums of the form:

$$\sum_{k=0}^{n-1} y_k \lambda(f^{-1}([y_k, y_{k+1})))$$

λ = Lebesgue measure. We need measurability of :
$f^{-1}([y_k, y_{k+1})) = \{x \in [a,b] : y_k \leq f(x) < y_{k+1}\})$.

Definition

Let $X \subseteq \mathbb{R}$, $X \in \mathcal{M}$. A function $f : X \to \mathbb{R}$ is called *(Lebesgue) measurable* if

$$f^{-1}((-\infty, b]) \in \mathcal{M}, \text{ for all } b \in \mathbb{R}.$$

Definition

Let $Y \subseteq \mathbb{R}$ be a Borel set. A function $f : Y \to \mathbb{R}$ is called *Borel measurable* or just *Borel* if

$$f^{-1}((-\infty, b]) \in \mathcal{B}, \text{ for all } b \in \mathbb{R}.$$

Measurable functions

(Notation: $[f \leq b] := f^{-1}((-\infty, b]) = \{x \in X : f(x) \leq b\}$.)

Proposition

Let $X \subseteq \mathbb{R}$, $X \in \mathcal{M}$ and $f : X \to \mathbb{R}$ a function. The following are equivalent:

1. *f is measurable.*
2. *$f^{-1}((-\infty, b)) \in \mathcal{M}$ for all $b \in \mathbb{R}$.*
3. *$f^{-1}([b, +\infty)) \in \mathcal{M}$ for all $b \in \mathbb{R}$.*
4. *$f^{-1}((b, +\infty)) \in \mathcal{M}$ for all $b \in \mathbb{R}$.*

Remark Then, for each interval $J \subseteq \mathbb{R}$ (or $J = \{a\}$) we have $f^{-1}(J) \in \mathcal{M}$.

Proposition

If $\mathrm{B} \subseteq X \subseteq \mathbb{R}$ where $X \in \mathcal{M}$, the function $\chi_B : X \to \mathbb{R}$ with $\chi_B(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B \end{cases}$ *is measurable if and only if $B \in \mathcal{M}$.*

Borel functions

Proposition

Let $X \subseteq \mathbb{R}$, $X \in \mathcal{B}$ and $f : X \to \mathbb{R}$ a function. The following are equivalent:

1. *f is Borel measurable.*
2. *$f^{-1}((-\infty, b)) \in \mathcal{B}$ for all $b \in \mathbb{R}$.*
3. *$f^{-1}([b, +\infty)) \in \mathcal{B}$ for all $b \in \mathbb{R}$.*
4. *$f^{-1}((b, +\infty)) \in \mathcal{B}$ for all $b \in \mathbb{R}$.*

Remark Then, for each interval $J \subseteq \mathbb{R}$ (or $J = \{a\}$) we have $f^{-1}(J) \in \mathcal{B}$.

Proposition

If $\mathrm{B} \subseteq X \subseteq \mathbb{R}$ where $X \in \mathcal{B}$, the function $\chi_B : X \to \mathbb{R}$ with $\chi_B(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B \end{cases}$ is Borel measurable if and only if $B \in \mathcal{B}$.

Remark The Dirichlet function is Borel measurable.

Measurable functions

Proposition

If $f : \mathbb{R} \to \mathbb{R}$ then

$$f \text{ continuous } \Rightarrow f \text{ Borel measurable } \Rightarrow f \text{ Lebesgue measurable.}$$

Example The function $\chi_{[0,1]}$ is Borel but not continuous.
The function χ_A where $A \in \mathcal{M} \setminus \mathcal{B}$ (does there exist such a set?) is Lebesgue measurable, but not Borel measurable.

Proposition

If $X \subseteq \mathbb{R}$ is measurable [resp. Borel] and $f : I \to \mathbb{R}$ is an increasing (or decreasing) function then f is measurable [resp. Borel measurable].

Measurable functions

Proposition

Let X be a measurable subset of $\mathbb{R}$ and $f, g : X \to \mathbb{R}$ measurable functions. Then,

1. *The function $f + g$ is measurable.*
2. *For each $\lambda \in \mathbb{R}$ the function λf is measurable.*
3. *The function $f \cdot g$ is measurable.*
4. *If $f(x) \neq 0$ for all $x \in X$, the function $1/f$ is measurable.*
5. *The functions $\max\{f, g\}$, $\min\{f, g\}$ and $|f|$ are measurable.*

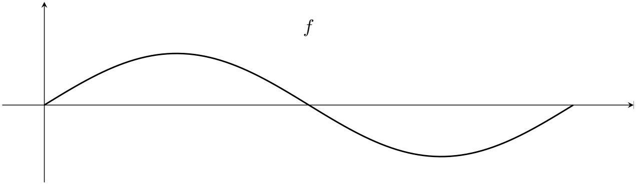
Alternative approach for (1) and (3):

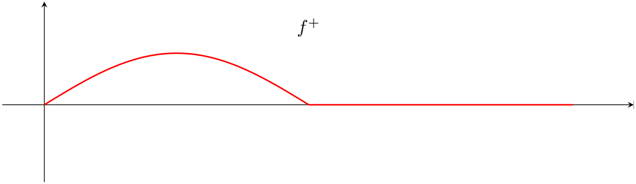
Proposition

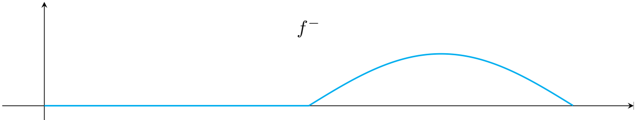
Let X be a measurable subset of $\mathbb{R}$ and $f, g : X \to \mathbb{R}$ measurable functions. If $F : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, the function $h : X \to \mathbb{R} : x \to F(f(x), g(x))$ is measurable.

The functions f^+ and f^-

$$f^+ = \max\{f, 0\}, \quad f^- = -\min\{f, 0\}, \quad f = f^+ - f^-,$$
$$|f| = f^+ + f^-$$







Measurable functions $f : X \to [-\infty, +\infty]$

Definition

Let $X \subseteq \mathbb{R}$ be measurable. A function $f : X \to [-\infty, +\infty]$ is called *(Lebesgue) measurable* if, for every $b \in \mathbb{R}$,

$$f^{-1}([-\infty, b]) = \{x \in X : f(x) \le b\} \in \mathcal{M}.$$

Remark Then, the set

$$\{x \in X : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) \le -n\}$$

is measurable. So is the set $\{x \in X : f(x) = +\infty\}$.

Proposition

A function $f : X \to [-\infty, +\infty]$ is measurable
iff $\forall a \in \mathbb{R}$ the set $f^{-1}([-\infty, a))$ is measurable,
iff $\forall a \in \mathbb{R}$ the set $f^{-1}([a, +\infty])$ is measurable,
iff $\forall a \in \mathbb{R}$ the set $f^{-1}((a, +\infty])$ is measurable.

The notion “almost everywhere”

Definition

Let X be a measurable subset of $\mathbb{R}$. We say that a property $P(x)$ holds almost everywhere in X (or for almost all $x \in X$) if

$$\lambda^*(\{x \in X \mid P(x) \text{ fails }\}) = 0.$$

Proposition

Let X be a measurable subset of $\mathbb{R}$ and $f, g : X \to [-\infty, +\infty]$. If f is measurable and $f(x) = g(x)$ almost everywhere in X (we will write $f = g$ a.e.), then g is measurable.

Reminder: lim sup, lim inf

Let (a_n) be a sequence, $a_n \in [-\infty, \infty]$. If $\sup\{a_k : k \geq 1\} = +\infty$, we set $\limsup_n \alpha_n = +\infty$. If not, for each $n \in \mathbb{N}$, define $b_n = \sup\{a_k : k \geq n\}$.
Observe that $b_n \geq a_n$ for all n and (b_n) is decreasing. Therefore $\lim_n b_n$ exists and equals $\inf_n b_n$.

Definition

If (a_n) is bounded above, $\limsup_n a_n = \lim_n b_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} a_k \right)$ (otherwise, $\limsup_n a_n = +\infty$).

Similarly,

Definition

If (a_n) is bounded below, $\liminf_n a_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} a_k \right)$ (otherwise, $\liminf_n a_n = -\infty$).

Remark Let (a_n) be bounded above, $a \in \mathbb{R}$. Then: $a = \limsup_n a_n \iff$ for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : a_k \geq a + \varepsilon\}$ is finite and the set $\{k \in \mathbb{N} : a - \varepsilon < a_k < a + \varepsilon\}$ is infinite.

Sequences of measurable functions

Let $X \subseteq \mathbb{R}$ be a measurable set and (f_n) a sequence of functions, $f_n : X \to [-\infty, \infty]$.
The function $f = \sup_n f_n$ is defined pointwise:
$f(x) = \sup\{f_n(x) : n \in \mathbb{N}\} \in [-\infty, \infty]$ for all $x \in X$.
Similarly $(\limsup_n f_n)(x) = \limsup_n f_n(x)$ for all x.

Proposition

If every f_n is measurable,

- (α) *The functions $\sup_n f_n$ and $\inf_n f_n$ are measurable.*
- (β) *The functions $\limsup_n f_n$ and $\liminf_n f_n$ are measurable.*
- (γ) *If the sequence $\{f_n\}$ converges pointwise to a function f, then f is also measurable.*

Remark The Proposition does NOT hold for continuous functions, nor for Riemann integrable functions. Examples?

Sequences of measurable functions

Proposition

Let be a X *measurable subset of* $\mathbb{R}$ *and* $f : X \to [-\infty, +\infty]$ *a function. If* $f_n : X \to [-\infty, +\infty]$ *are measurable functions and* $f_n(x) \to f(x)$ *almost everywhere in* X*, then* f *is measurable.*

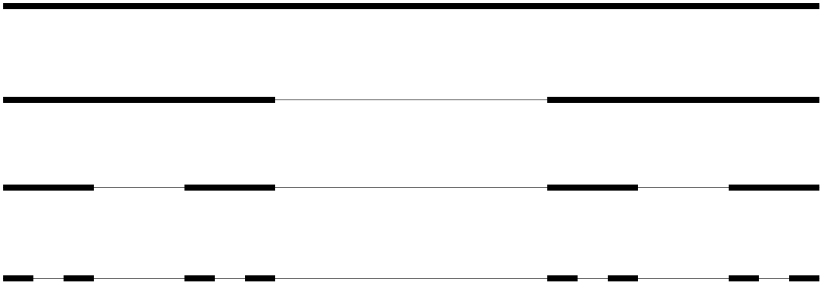
The Cantor set $C = \bigcap_{n=1}^{\infty} C_n$

$$C_0 = [0, 1]$$

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

$$\vdots$$



The Cantor set $C = \bigcap_{n=1}^{\infty} C_n$

Remark

The Cantor set has Lebesgue measure zero and is closed and has empty interior. It is however uncountable.

There exists a 1-1 onto map $\{0,1\}^{\mathbb{N}} \to C$.

The Cantor set

Remark

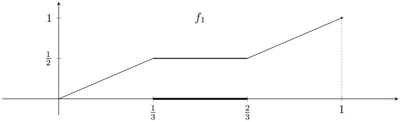
The Cantor set is perfect, i.e. it is closed and has no isolated points.

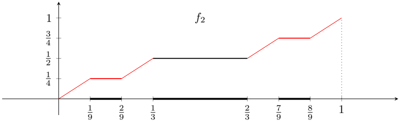
Remark

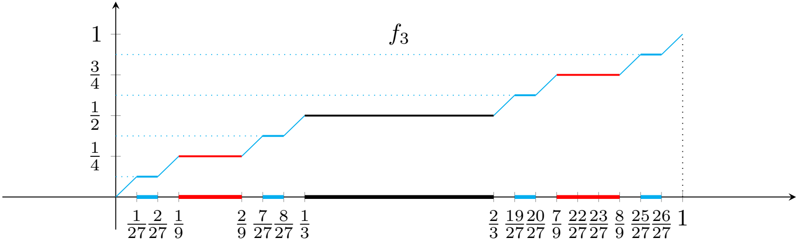
For each $\alpha \in (0, 1)$*, one can construct a "Cantor-like set"* C^{α} *(i.e. a compact set, with empty interior and no isolated points) having measure* α*.*

The Cantor-Lebesgue function or "devil's staircase"

For each $n \in \mathbb{N}$ define $f_n : [0,1] \to [0,1]$ as follows: If $J_1^n, \dots, J_{2^n-1}^n$ denote the consecutive open intervals comprising $[0,1]\backslash C_n$, define: $f_n(0) = 0$, $f_n(1) = 1$ and $f_n(x) = \frac{k}{2^n}$ for all $x \in J_k^n$. In each of the closed intervals comprising C_n, extend linearly so as to obtain a continuous function:







The Cantor-Lebesgue function f

Proposition

The sequence $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to a continuous function* $f : [0,1] \to [0,1]$. *The function* f *is increasing and onto* $[0,1]$. *It is almost everywhere differentiable: For each* x *in the (open) set* C^c, *the derivative* $f'(x)$ *exists, and in fact* $f'(x) = 0$.
The image of C *by* f *has measure* $\lambda(f(C)) = 1$ *(while* $\lambda(C) = 0$*).*

There exists a measurable set which is not Borel:
If $g(x) = \frac{x+f(x)}{2}$, then g is a homeomorphism of $[0,1]$ which maps the set C to a set $g(C)$ of strictly positive measure! It follows that there exists $A \subseteq g(C)$ which is non-measurable (exercise).
Then $B := g^{-1}(A)$ is measurable, since $B \subseteq C$. But it is not Borel: for if it were, then $A = h^{-1}(B)$ where $h = g^{-1}$ (a continuous function) would be Borel, hence measurable.

Simple measurable functions

Definition

Let X be a measurable subset of $\mathbb{R}$. A measurable function $s : X \to \mathbb{R}$ is called simple if its set of values $s(X)$ is finite.

Every simple function can be written in standard form

$$s = \sum_{j=1}^{n} a_j \chi_{A_j}$$

where $s(X) = \{a_1, a_2, ..., a_n\}$ and $A_j = s^{-1}(\{a_j\}) \in \mathcal{M}$. The family $\{A_1, A_2, ..., A_n\}$ is a (measurable) partition of X.

Every linear combination $s = \sum_{j=1}^{n} b_j \chi_{B_j}$ of characteristic (or indicator) functions of measurable sets is a simple measurable function (Exercise).

Example

Let $s = \chi_{[-1,1]} + \chi_{[0,2]} : \mathbb{R} \to \mathbb{R}$. Here $s(\mathbb{R}) = \{0, 1, 2\}$.
Its standard form is $s = 0\chi_A + 1\chi_B + 2\chi_{[0,1]}$ where
$A = [-1, 2]^c,\ B = [-1, 0) \cup (1, 2]$.

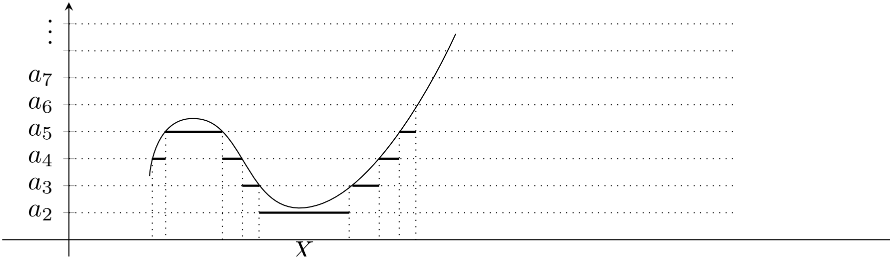
Simple measurable functions

Theorem

Let X be a measurable subset of $\mathbb{R}$ and $f : X \to [0, \infty]$ a non-negative measurable function. Then there is an increasing sequence of simple measurable functions $0 \le s_1 \le s_2 \le \cdots \le f$ so that

$$s_n \nearrow f \quad \textit{(pointwise)}.$$

If f is bounded, the sequence converges uniformly.



Approximation by simple functions: Proof

(a) If f is bounded: Let $N \in \mathbb{N}$ be such that $f(x) < N$ for all $x \in X$. For each $n \in \mathbb{N}$, partition $[0, N)$ into intervals of length $\frac{1}{2^n}$:

$$[0, N) = \left[0, \frac{1}{2^n}\right) \cup \left[\frac{1}{2^n}, \frac{2}{2^n}\right) \cup \ldots \cup \left[\frac{2^n N - 1}{2^n}, \frac{2^n N}{2^n}\right).$$

Consider their inverse images by f :

$$E_{n,i} = \left\{x \in X : \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}\right\}, \qquad i = 1, 2, \ldots, 2^n N.$$

These are measurable sets, and they partition X. If $x \in E_{n,i}$, define

$$s_n(x) = \frac{i-1}{2^n}$$

i.e. put

$$s_n = \sum_{i=1}^{2^n N} \frac{i-1}{2^n} \chi_{E_{n,i}}.$$

This is a simple measurable function and clearly $0 \le s_n \le f$.

Approximation by simple functions: Proof (II)

Claim. $s_n \to f$ uniformly on X.

Proof. Let $x \in X$. Then *for each* n there exists k so that $x \in E_{n,k}$, i.e. $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$ while $s_n(x) = \frac{k-1}{2^n}$, and so

$$0 \leq f(x) - s_n(x) < \frac{1}{2^n}, \quad \forall n\,.$$

Thus $\sup_{x \in X} |f(x) - s_n(x)| \leq \frac{1}{2^n}$, hence $s_n \to f$ uniformly.

(b) If f is not bounded: For each $n \in \mathbb{N}$, partition $[0, +\infty] = [0, n) \cup [n, +\infty]$ and

$$[0, n) = \left[0, \frac{1}{2^n}\right) \cup \left[\frac{1}{2^n}, \frac{2}{2^n}\right) \cup \ldots \cup \left[\frac{2^n n - 1}{2^n}, \frac{2^n n}{2^n}\right).$$

Define: $F_n = \{x \in X : f(x) \geq n\}$

$$E_{n,i} = \left\{x \in X : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\right\}, \qquad i = 1, 2, \ldots, n2^n.$$

These are measurable sets, and they partition X.

Approximation by simple functions: Proof (III)

Define

$$s_n(x) = \begin{cases} n, & \text{if } f(x) \geq n \\ \frac{i-1}{2^n}, & \text{if } \exists i = 1, 2, \ldots, n2^n \text{ so that } \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \end{cases}$$

that is, put

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}.$$

This is a simple measurable function and clearly $0 \leq s_n \leq f$.

Claim. $s_n(x) \to f(x)$ for each $x \in X$.

Proof. If $f(x) < +\infty$, there exists $n_0 = n_0(x)$ so that $f(x) < n_0$. When $n \geq n_0$ we have $f(x) < n$, hence there is a unique k so that $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$ while $s_n(x) = \frac{k-1}{2^n}$, hence

$$0 \leq f(x) - s_n(x) < \frac{1}{2^n}, \quad \forall n \geq n_0(x)$$

and so $s_n(x) \to f(x)$. If on the other hand $f(x) = +\infty$, then $f(x) \geq n$ for all n, hence $s_n(x) = n \to +\infty = f(x)$.

Approximation by simple functions: Proof (IV)

(c) Claim. The sequence (s_n) is increasing.

Proof. Let $n \in \mathbb{N}$ and $x \in X$. To show that $s_n(x) \leq s_{n+1}(x)$.

• If $f(x) \geq n+1$ then $s_{n+1}(x) = n+1$, but $f(x) > n$ so $s_n(x) = n$, hence $s_n(x) \leq s_{n+1}(x)$.

• If $n+1 > f(x) \geq n$ then $\exists k : f(x) \in [\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}})$, but $\frac{k}{2^{n+1}} \geq n$ (why?) so $s_{n+1}(x) = \frac{k}{2^{n+1}} \geq n$, while $s_n(x) = n$ since $f(x) \geq n$. Hence $s_n(x) \leq s_{n+1}(x)$.

• If $f(x) < n$, $\leadsto$

Approximation by simple functions: Proof (V)

• If $f(x) < n$ then there exists k so that $\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}$.
Now $s_n(x) = \frac{k}{2^n}$ and

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) = \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right) \cup \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right).$$

There are two cases:

$$f(x) \in \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right) \Rightarrow s_{n+1}(x) = \frac{2k}{2^{n+1}} = s_n(x)$$
$$f(x) \in \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) \Rightarrow s_{n+1}(x) = \frac{2k+1}{2^{n+1}} > s_n(x)$$

In both cases, $s_n(x) \leq s_{n+1}(x)$. □

Approximation by simple functions

Corollary

Let X be a measurable set and $f : X \to [-\infty, \infty]$ a measurable function. Then there exists a sequence $(s_n)_n$ of simple measurable functions with

$$\begin{aligned} & s_n \to f \\ \textit{and} \quad & 0 \leq |s_1| \leq |s_2| \leq \cdots \leq |f|. \end{aligned}$$

In addition, if f is bounded, then the sequence converges uniformly.

Remark: In fact the sequence converges uniformly on any subset $Y \subseteq X$ on which $f|_Y$ is bounded.

Littlewood's three principles

Let $X \subseteq \mathbb{R}$ be measurable with $\lambda(X) < \infty$.

Proposition (measurable sets)

For each $\varepsilon > 0$ there exist intervals $I_1, \ldots, I_n$ so that if $E := I_1 \cup \cdots \cup I_n$ then $\lambda(E \triangle X) < \varepsilon$.

Theorem (Luzin)

If $f : X \to \mathbb{R}$ is measurable, then for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subseteq X$ with $\lambda(X \backslash F_\varepsilon) < \varepsilon$ so that the function $f|_{F_\varepsilon}$ is continuous.

For a proof see `luzinen.pdf`.

Theorem (Egorov)

If $f_n, f : X \to \mathbb{R}$ are measurable with $f_n \to f$ almost everywhere in X, then for every $\varepsilon > 0$ there is a closed set $F_\varepsilon \subseteq X$ with $\lambda(X \backslash F_\varepsilon) < \varepsilon$ so that $f_n \to f$ uniformly on F_ε.

Sketch of proof below.

Littlewood's three principles, intuitive formulation

Let $X \subseteq \mathbb{R}$ be measurable with $\lambda(X) < \infty$.

[Measurable sets] Every such $X \subseteq \mathbb{R}$ "is almost equal" to a finite union of intervals.

[Luzin's theorem] Every measurable function on X "is almost continuous".

[Egorov's theorem] Every sequence of measurable functions on X that converges pointwise, "converges almost uniformly".

Proof of Egorov's theorem

For each k and $m \in \mathbb{N}$, let

$$E_m(k) = \{x : \exists n \geq m : |f_n(x) - f(x)| \geq \frac{1}{k}\}.$$

We have $E_m(k) \supset E_{m+1}(k)$ for each m and

$$\bigcap_{m\geq 1} E_m(k) = \{x : \forall m \, \exists n \geq m : |f_n(x) - f(x)| \geq \frac{1}{k}\}$$
$$\subseteq \{x : |f_n(x) - f(x)| \nrightarrow 0\}$$

But $f_n \to f$ almost everywhere, hence $\lambda\left(\cap_m E_m(k)\right) = 0$.
Since $\lambda(E_1(k)) < +\infty$, it follows that $\lim_m \lambda(E_m(k)) = 0$.
Therefore for each $\delta > 0$ and every $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ so that $\lambda(E_{m_k}(k)) < \dfrac{\delta}{2^k}$.
Define

$$A_\delta = \bigcup_{k=1}^{\infty} E_{m_k}(k)$$

Proof of Egorov's theorem (II)

$$A_\delta = \bigcup_{k=1}^{\infty} E_{m_k}(k)$$

Then $A_\delta \in \mathcal{M}$ and

$$\lambda(A_\delta) \leq \sum_{k=1}^{\infty} \lambda(E_{m_k}(k)) < \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta.$$

Claim : $f_n \to f$ uniformly on $X \backslash A_\delta$.
Proof : Let $\varepsilon > 0$ and $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$. Since $A_\delta \supseteq E_{m_k}(k)$, if $x \notin A_\delta$ we have $x \notin E_{m_k}(k)$; thus for all $n \geq m_k$ we have $|f_n(x) - f(x)| < \frac{1}{k} < \varepsilon$. Since m_k does not depend on x we have $f_n \to f$ uniformly on A_δ^c. □

So, if I choose $F_\delta \subseteq (X \backslash A_\delta)$ with $\lambda((X \backslash A_\delta) \backslash F_\delta)) < \delta$ (regularity), then $\lambda((X \backslash F_\delta)) < 2\delta$ and $f_n \to f$ uniformly on F_δ. □

Counterexample when $\lambda(X) = \infty$: $f_n = \chi_{(n,\infty)} \to 0$ pointwise. But...

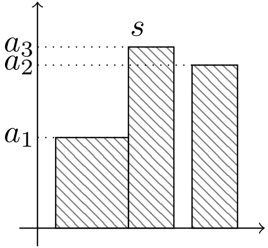
The Lebesgue Integral: Definitions

Let $X \subseteq \mathbb{R}$ be measurable.

(a) If $s : X \to \mathbb{R}^+$ is simple measurable and $s(X) = \{a_1, \dots, a_n\}$ we define

$$\int s d\lambda = \sum_{k=1}^{n} a_k \lambda(A_k) \in [0, +\infty]$$

where $A_k = s^{-1}(\{a_k\})$ (we put $0 \cdot (+\infty) = 0$).



Σχήμα: Integral of a simple function

The Lebesgue Integral: Definitions

(b) If $f : X \to [0, +\infty]$ is measurable, we define

$$\int f d\lambda = \sup \left\{ \int s d\lambda : s \text{ simple measurable, } 0 \leq s \leq f \right\}.$$

For f simple, definitions (a) and (b) coincide.

- If $A \subseteq X$ is a measurable subset then we define $\int_A f d\lambda := \int f \chi_A d\lambda$.
- If f is defined on a measurable subset $A \subseteq X$ and non-negative, we extend f to a (measurable) function $\tilde{f} : X \to [0, +\infty]$ by setting $\tilde{f}(x) = 0$ for $x \in X \backslash A$ and define $\int f d\lambda := \int \tilde{f} d\lambda$.

(c) Let $f : X \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be measurable. The functions $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are non-negative and measurable, hence the integrals $\int f^+ d\lambda$ and $\int f^- d\lambda$ are well defined (in $\overline{\mathbb{R}}$). If at least one of them is finite, we define

$$\int f d\lambda = \int f^+ d\lambda - \int f^- d\lambda \in \overline{\mathbb{R}}.$$

(d) A function $f : X \to \overline{\mathbb{R}}$ is called (absolutely) integrable if is measurable and

$$\int |f| d\lambda < +\infty.$$

The Lebesgue Integral for simple $f \geq 0$

Proposition

If $s_1, s_2 : X \to [0, +\infty)$ *are simple measurable and* $a \geq 0$*, then*

1. $\int a s_1 d\lambda = a \int s_1 d\lambda$ *(positive homogeneity)*
2. $\int (s_1 + s_2) d\lambda = \int s_1 d\lambda + \int s_2 d\lambda$ *(additivity)*
3. *If* $s_1 \leq s_2$ *then* $\int s_1 d\lambda \leq \int s_2 d\lambda$ *(monotonicity).*

For (ii), we will need the (temporary) lemma:

Lemma

If $s : X \to \mathbb{R}^+$ *is simple measurable and* $s = \sum_{k=1}^{m} b_k \chi_{B_k}$ *where* $B_k \cap B_j = \emptyset$ *for* $k \neq j$*, then*

$$\int s d\lambda = \sum_{k=1}^{m} b_k \lambda(B_k).$$

The Lebesgue Integral for measurable $f \geq 0$

Reminder: If $f : X \to [0, +\infty]$ is measurable,

$$\int f d\lambda = \sup\left\{\int s d\lambda : s \text{ simple measurable}, 0 \leq s \leq f\right\}.$$

Recall that if $f : X \to [0, +\infty]$ is measurable and $A \in \mathcal{M}$,
$\int_A f d\lambda := \int \chi_A f d\lambda$.

Proposition

If $f, g : X \to [0, +\infty]$ are measurable and $a \geq 0$, then

- i $\int a f d\lambda = a \int f d\lambda$.
- ii *If* $f \leq g$ *then* $\int f d\lambda \leq \int g d\lambda$.
- iii *If* $A \subseteq B$ $(A, B \in \mathcal{M})$ *then* $\int_A f d\lambda \leq \int_B f d\lambda$.
- iv *If* $A \in \mathcal{M}$ *and* $\lambda(A) = 0$ *or* $f|_A = 0$ *then* $\int_A f d\lambda = 0$.

The Lebesgue Integral for measurable $f \geq 0$

Proposition (Markov's Inequality)

Let $f : X \to [0, +\infty]$ be measurable. For every $a \geq 0$,

$$\int f d\lambda \geq a \cdot \lambda(\{x \in X : f(x) \geq a\})\,.$$

Corollary

If $f : X \to [0, +\infty]$ is integrable (i.e. measurable and $\int |f| d\lambda < \infty$) then $f(x) < \infty$ far almost all $x \in X$.

Additivity of the Lebesgue Integral

The equality $\int f d\lambda + \int g d\lambda = \int (f+g) d\lambda$ holds for $f, g : X \to [0, +\infty]$ simple measurable.

More generally, what if $f, g : X \to [0, +\infty]$ are just measurable? Observe that we can easily prove the *inequality*

$$\int f d\lambda + \int g d\lambda \leq \int (f+g) d\lambda \, .$$

What about equality?? What about approximating by simple functions?

Is it true that $\int \lim f_n d\lambda \stackrel{?}{=} \lim \int f_n d\lambda$??

Examples (a) On $\mathbb{R}$: Let $f_n := \chi_{[n,n+1]}$. We have $f_n \to f = 0$ pointwise, but $\int f_n d\lambda = 1$ for all n while $\int f d\lambda = 0$.
(The mass under the f_n "escapes to infinity horizontally".)

(b) On $\mathbb{R}$: Let $f_n := \frac{1}{n}\chi_{[0,n]}$. This time $f_n \to f = 0$ uniformly but $\int f_n d\lambda = 1$ for each n while $\int f d\lambda = 0$.
(Here the mass "spreads out" over the whole of $\mathbb{R}_+$.)

(c) On $[0, 1]$: Let $f_n := n\chi[\frac{1}{n}, \frac{2}{n}]$. The measure of the space is finite, and $f_n \to f = 0$ pointwise (not uniformly). Again $\int f_n d\lambda = 1$ for all n while $\int f d\lambda = 0$.
(Here the mass "escapes to infinity vertically".)

The Monotone Convergence Theorem

Theorem

Let $X \in \mathcal{M}$ and $f_n : X \to [0, \infty]$ an increasing sequence of non-negative measurable functions. Then

$$\lim_n \left(\int f_n \, d\lambda \right) = \int (\lim_n f_n) \, d\lambda \, .$$

The Monotone Convergence Theorem

Consequence: If $f : X \to [0, +\infty]$ is measurable, then

$$\int f d\lambda = \lim \int s_n d\lambda$$

where (s_n) is any increasing sequence of simple functions $s_n \geq 0$ with $s_n \nearrow f$.

Questions: (α) Does the Monotone Convergence Theorem hold for decreasing sequences?

(b) Perhaps under additional conditions?

First Consequences of the Monotone Convergence Theorem

Proposition (Additivity)

If $f, g : X \to [0, +\infty]$ *are measurable, then*

$$\int (f+g)d\lambda = \int f d\lambda + \int g d\lambda.$$

Theorem (Beppo Levi)

If (f_n) *are measurable,* $f_n : X \to [0, +\infty]$*, then*

$$\int \left(\sum_n f_n \right) d\lambda = \sum_n \left(\int f_n d\lambda \right).$$

Proposition (Fatou's Lemma)

If $f_n : X \to [0, +\infty]$ *are measurable, then*

$$\int (\liminf_n f_n) d\lambda \leq \liminf_n \int f_n d\lambda.$$

Integrable functions

Definition (reminder):

• Let $X \in \mathcal{M}$, $f : X \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be measurable. The functions $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are non-negative and measurable, hence the integrals $\int f^+ d\lambda$ and $\int f^- d\lambda$ are well defined (in $\overline{\mathbb{R}}$). If at least one of them is finite, we define

$$\int f d\lambda = \int f^+ d\lambda - \int f^- d\lambda \in \overline{\mathbb{R}}.$$

• A function $f : X \to \overline{\mathbb{R}}$ is called (absolutely) integrable if is measurable and

$$\int |f| d\lambda < +\infty.$$

Remarks • If f is integrable, then $f(x) \in \mathbb{R}$ for almost all $x \in X$.

• The function f is integrable if and only if both f^+ and f^- are integrable and then $\int f d\lambda = \int f^+ d\lambda - \int f^- d\lambda \in \mathbb{R}$.

Integrable functions

Theorem

If $f, g : X \to \overline{\mathbb{R}}$ *are integrable and* $c \in \mathbb{R}$, *then*
(the function $f + cg$ *is well-defined a.e. and)*

$$\int (f + cg) d\lambda = \int f d\lambda + c \int g d\lambda.$$

Proposition

If $f : X \to \overline{\mathbb{R}}$ *is integrable and* $A, B \in \mathcal{M}$ *με* $A \cap B = \emptyset$, *then*

$$\int_{A \cup B} f d\lambda = \int_A f d\lambda + \int_B f d\lambda\,.$$

Integrable functions

Proposition

If $f, g : X \to \overline{\mathbb{R}}$ *are integrable then*

$$(i) \qquad f \leq g \quad \Longrightarrow \quad \int f d\lambda \leq \int g d\lambda.$$

$$(ii) \qquad \left| \int f d\lambda \right| \leq \int |f| d\lambda$$

Integrable functions

Proposition

Let $f, g : X \to \overline{\mathbb{R}}$ *be integrable.*

(i) If $f = g$ *a.e. then* $\int f d\lambda = \int g d\lambda$.

(ii) We have $f = 0$ *a.e. if and only if* $\int_A f d\lambda = 0$ *for every* $A \subseteq X$, $A \in \mathcal{M}$.

Corollary

If $f, g : X \to \overline{\mathbb{R}}$ *are integrable and* $f \leq g$ *a.e. then* $\int f d\lambda \leq \int g d\lambda$.

The Dominated Convergence Theorem

Theorem

Let (f_n) *be a sequence of measurable functions* $f_n : X \to [-\infty, +\infty]$ *which converges for almost all* $x \in X$ *to a function* $f : X \to [-\infty, +\infty]$. *If there exists an integrable function* $g : X \to [0, +\infty]$ *such that* $|f_n| \leq g$ *a.e. for all* n, *then*

$$f \text{ is (absolutely) integrable}$$

$$\text{and} \quad \lim_{n\to\infty} \int |f_n - f| d\lambda = 0$$

$$\text{and} \quad \lim_{n\to\infty} \int f_n d\lambda = \int f d\lambda.$$

See also the counterexamples: [100] when there is no "dominating" integrable g.

Proof. First show that the f_n and f are integrable.
For convergence:

The Dominated Convergence Theorem: Proof of convergence

Put $h_n = |f_n - f|$ and observe that $0 \leq h_n \leq 2g$ a.e. and that $h_n(x) \to 0$ for almost all x. Thus, $2g - h_n \geq 0$ and $2g - h_n \to 2g$ pointwise, almost everywhere.

By Fatou's Lemma we have

$$\int \liminf_n (2g - h_n) d\lambda \leq \liminf_n \int (2g - h_n) d\lambda .$$

But $\displaystyle\int \liminf_n (2g - h_n) d\lambda = \int 2g d\lambda$

and
$$\liminf_n \int (2g - h_n) d\lambda = \int 2g d\lambda + \liminf_n \int (-h_n) d\lambda$$
$$= \int 2g d\lambda - \limsup_n \int h_n d\lambda ,$$

therefore $\displaystyle\limsup_n \int h_n d\lambda \leq 0 .$

On the other hand $0 \leq \int h_n d\lambda$ so $0 \leq \liminf_n \int h_n d\lambda$.

Therefore the limit $\lim_n \int h_n d\lambda$ exists and is 0. □

The Dominated Convergence Theorem

Corollary (The bounded convergence theorem)

Let $X \in \mathcal{M}$ with $\lambda(X) < \infty$, let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions and $f : X \to \mathbb{R}$ with $f_n \to f$ a.e. We assume that, in addition, there exists an $M > 0$ so that $|f_n| \leq M$ a.e. on X for all n. Then the f_n and f are integrable and we have:

$$\int |f_n - f| \, d\lambda \to 0.$$

It also follows that

$$\lim_n \int f_n \, d\lambda = \int f \, d\lambda.$$

The Dominated Convergence Theorem

Corollary

Let $f : \mathbb{R} \to [-\infty, +\infty]$ *be integrable. Then the function*

$$F(x) = \int_{-\infty}^{x} f \, d\lambda := \int_{(-\infty, x]} f \, d\lambda$$

is continuous.

Corollary

Let $f : \mathbb{R} \to [-\infty, +\infty]$ *be integrable. If* $E_n \in \mathcal{M}, \; E_n \subseteq E_{n+1}$ *for every* n *and* $E = \bigcup E_n$, *then*

$$\int_E f \, d\lambda = \lim_n \int_{E_n} f \, d\lambda \, .$$

Reminder: The Riemann integral

A partition $\mathcal{P}$ of $[a, b]$ is a finite set

$$\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$$

If $f : [a, b] \to \mathbb{R}$ is bounded, we set

$$M_i = M_i(f) = \sup\{f(s) : s \in [t_{i-1}, t_i]\}$$
$$m_i = m_i(f) = \inf\{f(s) : s \in [t_{i-1}, t_i]\} \quad (i = 1, \ldots, n).$$

and

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1})$$
$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}).$$

The numbers $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$ are called the lower and upper Riemann sums of f for the partition $\mathcal{P}$.

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then, there is a sequence (P_n) of partitions of $[a, b]$ such that: $P_n \subset P_{n+1}$ (P_{n+1} is a refinement of P_n), $\|P_n\| \to 0$ (the sizes of the partitions P_n tend to 0), and

$$L(f, P_n) \to \int_a^b f(x)\,dx \quad , \quad U(f, P_n) \to \int_a^b f(x)\,dx.$$

Let g_n be the step function with $\int_a^b g_n(x)\,dx = L(f, P_n)$ (that is, if $L(f, P_n) = \sum_{i=0}^{k-1} m_i(x_{i+1} - x_i)$ then put $g_n = \sum_{i=0}^{k-1} m_i \chi_{[x_i, x_{i+1})}$) and let u_n be the step function with $\int_a^b u_n(x)\,dx = U(f, P_n)$. Then $g_n \le f \le u_n$. The sequence (g_n) is increasing and (u_n) is decreasing, hence $\exists g := \lim_n g_n$ and $u := \lim_n u_n$ and $g \le f \le u$. The functions u and g are limits of monotone sequences of integrable functions.

Therefore [3]

$$\int_a^b u \, d\lambda = \lim_n \int_a^b u_n \, d\lambda \overset{(!)}{=} \lim_n \int_a^b u_n(x) \, dx = \int_a^b f(x) \, dx$$

and

$$\int_a^b g \, d\lambda = \lim_n \int_a^b g_n \, d\lambda \overset{(!)}{=} \lim_n \int_a^b g_n(x) \, dx = \int_a^b f(x) \, dx.$$

Hence $g = u$ almost everywhere. Since $g \leq f \leq u$, it follows that $g = f = u$ almost everywhere.
Thus, $f = \lim g_n$ almost everywhere, and hence f is measurable and

$$\int_a^b f \, d\lambda = \lim_n \int_a^b g_n \, d\lambda = \int_a^b g \, d\lambda = \int_a^b f(x) \, dx. \quad \square$$

---

[3] Since u_n is a step function, $\int_a^b u_n d\lambda \overset{(!)}{=} \int_a^b u_n(x) dx$.

Riemann integrable functions

Theorem

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is almost everywhere continuous, that is, if its set of discontinuities has measure zero. Then f is Lebesgue integrable and the two integrals coincide.

Proof Later (if time permits).

Remark The characteristic functions of $[\frac{1}{3}, \frac{2}{3}]$ is almost everywhere continuous, but it is not almost everywhere equal to a continuous function.
On the other hand, the Dirichlet functions is continuous nowhere, but it is almost everywhere equal to the continuous function $f(t) = 0$.

The space $\mathcal{L}^1_{\mathbb{R}}(X)$

Definition

If $X \subseteq \mathbb{R}$ is measurable, the space $\mathcal{L}^1_{\mathbb{R}}(X)$ consists of all functions $f : X \to \mathbb{R}$ which are measurable and satisfy $\int_X |f| d\lambda < +\infty$. The quantity $\int_X |f| d\lambda$ is denoted $\|f\|_1$.

Remarks (i) If $f : X \to [-\infty, +\infty]$ is measurable then $\|f\|_1 < +\infty$ if and only if f takes finite values almost everywhere; therefore it is almost everywhere equal to a function $\tilde{f} \in \mathcal{L}^1_{\mathbb{R}}(X)$.
Abusing terminology, we say $f \in \mathcal{L}^1_{\mathbb{R}}(X)$.

(ii) If $f, g \in \mathcal{L}^1_{\mathbb{R}}(X)$ and $\lambda \in \mathbb{R}$ then $f + \lambda g \in \mathcal{L}^1_{\mathbb{R}}(X)$ and

1. $\|\lambda g\|_1 = |\lambda| \|g\|_1$
2. $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$
3. $\|f\|_1 = 0$ if and only if $f = 0$ almost everywhere.

Completeness: The Riesz–Fischer Theorem

Definition

Let $f_n, f : X \to \mathbb{R}$ be measurable.

(i) The sequence $\{f_n\}$ converges to f in the mean or in L^1 If

$$\int |f_n - f| \, d\lambda \to 0.$$

(ii) The sequence $\{f_n\}$ is Cauchy in the mean if for every $\varepsilon > 0$ there is $n_0(\varepsilon) \in \mathbb{N}$ such that: If $m, n \geq n_0$ then $\int |f_n - f_m| \, d\lambda < \varepsilon$.

Theorem

Let $X \subseteq \mathbb{R}$ be measurable and $\{f_n\}$ a sequence of functions in $\mathcal{L}^1_{\mathbb{R}}(X)$.

- *If $\{f_n\}$ is Cauchy in the mean, then there is a function $f : X \to \mathbb{R}$ in $\mathcal{L}^1_{\mathbb{R}}(X)$ so that $f_n \to f$ in the mean.*
- *In addition, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ so that $f_{n_k} \to f$ almost everywhere.*

The space $(L^1_{\mathbb{R}}(X), \|\cdot\|_1)$

The space $\mathcal{L}^1_{\mathbb{R}}(X)$ is a linear space and $\|\cdot\|_1$ is a *seminorm* on it. Define

$$\mathcal{N} = \{f : X \to \mathbb{R} :\ \text{measurable, } f = 0 \text{ almost everywhere}\}.$$

Remark: $\mathcal{N} = \{f \in \mathcal{L}^1_{\mathbb{R}}(X) : \|f\|_1 = 0\}$.
If $f, g \in \mathcal{L}^1$, then: $f = g$ almost everywhere $\iff$ $f - g \in \mathcal{N}$.
Also, $\mathcal{N}$ is a linear subspace of $\mathcal{L}^1$.
On the quotient space $L^1_{\mathbb{R}}(X) := \mathcal{L}^1_{\mathbb{R}}(X)/\mathcal{N}$, define $\|[f]_{\mathcal{N}}\|_1 := \|f\|_1$, where $[f]_{\mathcal{N}} := \{f + g : g \in \mathcal{N}\}$. This is a well-defined norm.
Thus, the space $L^1_{\mathbb{R}}(X)$ consists of equivalence classes of $\mathcal{L}^1_{\mathbb{R}}(X)$ functions modulo equality almost everywhere.
The Riesz–Fischer Theorem states precisely that the space $(L^1_{\mathbb{R}}(X), \|\cdot\|_1)$ is a complete normed space, that is, a Banach space.

The space $(L^p_{\mathbb{R}}(X), \|\cdot\|_p)$ where $1 \leq p < \infty$

Let $p \in [1, \infty)$. If $X \subseteq \mathbb{R}$ is measurable, the space $\mathcal{L}^p_{\mathbb{R}}(X)$ consists of all functions $f : X \to \mathbb{R}$ which are measurable and satisfy $\int_X |f|^p d\lambda < +\infty$.

The quantity $\left(\int_X |f|^p d\lambda\right)^{1/p}$ is denoted $\|f\|_p$.

On the quotient space $L^p_{\mathbb{R}}(X) := \mathcal{L}^p_{\mathbb{R}}(X)/\mathcal{N}$ (where $\mathcal{N} = \{f : X \to \mathbb{R} : \text{ measurable, } f = 0 \text{ almost everywhere}\}$) define $\|[f]_{\mathcal{N}}\|_p := \|f\|_p$, where $[f]_{\mathcal{N}} := \{f + g : g \in \mathcal{N}\}$. This is a well-defined norm.

Theorem (Riesz-Fischer)

The space $(L^p_{\mathbb{R}}(X), \|\cdot\|_p)$ is a complete normed space, that is, a Banach space: If a sequence $\{f_n\}$ in $\mathcal{L}^p_{\mathbb{R}}(X)$ is Cauchy with respect to $\|\cdot\|_p$, there exists $f : X \to \mathbb{R}$ in $\mathcal{L}^p_{\mathbb{R}}(X)$ so that $\|f_n - f\|_p \to 0$.

Approximation in $(L^p_{\mathbb{R}}(X), \|\cdot\|_p)$

Let $X \subseteq \mathbb{R}$ be measurable. We write $\mathcal{S}(X)$ for the set of all (equivalence classes, modulo equality almost everywhere, of) simple measurable functions $s : X \to \mathbb{R}$ such that $\lambda(\{x \in X : s(x) \neq 0\}$.

Proposition

The space $\mathcal{S}(X)$ is a linear subspace of $L^p_{\mathbb{R}}(X)$ which is dense in $(L^p_{\mathbb{R}}(X), \|\cdot\|_p)$.

We write $C_c(\mathbb{R})$ for the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ which have compact support, that is, there exists a *compact* $K(f) \subseteq \mathbb{R}$ so that $f(x) = 0$ when $x \notin K(f)$.

Proposition

The space $C_c(\mathbb{R})$ is a linear subspace of $L^p_{\mathbb{R}}(X)$ which is dense in $(L^p_{\mathbb{R}}(X), \|\cdot\|_p)$.

Part III

Fourier series for functions of class $\mathcal{L}^1$ and $\mathcal{L}^2$

Complex-valued functions on the unit circle (Reminder)

Denote by $\mathbb{T}$ the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

If $\phi : \mathbb{T} \to \mathbb{C}$, define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(\theta) = \phi(e^{i\theta}).$$

The function f is 2π-periodic.
Conversely, if $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, then the function $\phi : \mathbb{T} \to \mathbb{C}$ given by $\phi(e^{i\theta}) = f(\theta)$ is well defined.
Thus we have a $1 - 1$ correspondence between functions $\phi : \mathbb{T} \to \mathbb{C}$ and 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$.
In what follows we shall make no distinction between ϕ and f.

The spaces $L^p(\mathbb{T})$

For $p \in [1, \infty)$, the symbol $\mathcal{L}^p(\mathbb{T})$ denotes the set of all measurable (*) functions $f : \mathbb{T} \to \mathbb{C}$ satisfying

$$\int_{-\pi}^{\pi} |f(t)|^p d\lambda(t) < \infty \qquad \text{(Lebesgue measure).}$$

We write

$$\|f\|_p := \left(\int_{-\pi}^{\pi} |f(t)|^p \frac{d\lambda(t)}{2\pi} \right)^{1/p} .$$

Observe that $\|f\|_p = 0$ if and only if $f(t) = 0$ for almost all t.

(*) A function $h : \mathbb{T} \to \mathbb{C}$ is called measurable iff both $u := \operatorname{Re} h = \frac{1}{2}(h + \bar{h})$ and $v := \operatorname{Im} h = \frac{1}{2i}(h - \bar{h})$ are measurable functions $\mathbb{T} \to \mathbb{R}$.

Notice that then $|h| = (u^2 + v^2)^{1/2}$ is measurable (why?).

The spaces $L^p(\mathbb{T})$

The symbol $L^p(\mathbb{T})$ denotes the space of equivalence classes $[f]$, of $f \in \mathcal{L}^p(\mathbb{T})$, modulo equality almost everywhere (we write f instead of $[f]$).

The space $L^p(\mathbb{T})$ is a linear space and $\|\cdot\|_p$ is a norm on $L^p(\mathbb{T})$ with respect to which $L^p(\mathbb{T})$ is a Banach space (Riesz-Fischer Theorem). The space $L^2(\mathbb{T})$ is a Hilbert space with respect to

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}d\lambda(t).$$

Here $\displaystyle\int h d\lambda := \int \mathrm{Re} h\, d\lambda + i\int \mathrm{Im} h\, d\lambda$ where $\mathrm{Re} h = \frac{1}{2}(h+\bar{h})$, $\mathrm{Im} h = \frac{1}{2i}(h-\bar{h})$.]

Remark

The mapping $\mathcal{I} : f \to \int f d\lambda : L^1(\mathbb{T}) \to \mathbb{C}$ *is linear, and it is positive: if* $f \in L^1(\mathbb{T})$ *and* $f(t) \in \mathbb{R}_+$ *a.e. then* $\mathcal{I}(f) \geq 0$.

It follows that $\displaystyle\left|\int g d\lambda\right| \leq \int |g| d\lambda \quad \forall g \in L^1(\mathbb{T})$.

The spaces $L^p(\mathbb{T})$

If $1 \leq p \leq q < \infty$ and f is measurable, we have $\|f\|_p \leq \|f\|_q \leq \|f\|_\infty$ and hence

$$C(\mathbb{T}) \subseteq L^q(\mathbb{T}) \subseteq L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$$

(recall that we identify $C(\mathbb{T})$ with the space of continuous 2π periodic functions $f : \mathbb{R} \to \mathbb{C}$.)

Proposition

If $p \in [1, \infty)$, the space of simple measurable functions, the space of step functions and $C(\mathbb{T})$ are dense in $(L^p(\mathbb{T}), \|\cdot\|_p)$.

Fourier series for functions of class $\mathcal{L}^1$

Definition (Fourier coefficients)

Let $f \in \mathcal{L}^1(\mathbb{T})$. Define

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f e_{-k} d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt} d\lambda(t) \qquad (k \in \mathbb{Z})\,.$$

Here $\displaystyle\int f d\lambda := \int \mathrm{Re} f \, d\lambda + i \int \mathrm{Im} f \, d\lambda$ where $\mathrm{Re} f = \frac{1}{2}(f + \bar{f})$, $\mathrm{Im} f = \frac{1}{2i}(f - \bar{f})$.

Remark. The function $S_n(f) = \sum_{|k| \leq n} \hat{f}(k) e_k$ is a trigonometric polynomial, hence a continuous (and 2π-periodic) function, for every $f \in \mathcal{L}^1(\mathbb{T})$.

Fourier series for functions of class $\mathcal{L}^1$

Remark

Let $f \in \mathcal{L}^1(\mathbb{T})$. *Then*

$$|\hat{f}(k)| \le \|f\|_1 \text{ for all } k \in \mathbb{Z}$$

$$\text{thus } \|\hat{f}\|_\infty \le \|f\|_1 \,.$$

Proposition (the Riemann-Lebesgue lemma)

Let $f \in \mathcal{L}^1(\mathbb{T})$. *Then*

$$\lim_{|k|\to\infty} \hat{f}(k) = 0 \,.$$

Thus $(\hat{f}(k)) \in c_0(\mathbb{Z})$.

Equivalently,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = 0 \qquad \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = 0 \,.$$

The Uniqueness Theorem for $\mathcal{L}^1$

Remark If we change the values of an $\mathcal{L}^1$ function on a set of measure zero, its Fourier coefficients remain the same. In other words,
If $f = g$ almost everywhere, then $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. The converse also holds:

Theorem

For $f, g \in L^1(\mathbb{T})$ the following are equivalent:
(i) $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$
(ii) $f = g$ a.e.. That is, f and g determine the same element of $L^1(\mathbb{T})$.

Proposition

For all $f \in L^1(\mathbb{T})$, we have $\|\sigma_n(f)\|_1 \leq \|f\|_1$.

Proposition

For all $f \in L^1(\mathbb{T})$, we have $\lim_n \|\sigma_n(f) - f\|_1 = 0$.

Conclusion : The space of trigonometric polynomials is dense in $L^1(\mathbb{T})$.
For proofs, see `L1uniq.pdf`.

Fourier series for functions of class $\mathcal{L}^2$

Best mean square approximation Lemma (see also Prop. 9.1 in not60520en.pdf): Let $f \in \mathcal{L}^2(\mathbb{T})$, $n \in \mathbb{N}$ and p a trigonometric polynomial of degree $\deg(p) \leq n$. Then $\|f - p\|_2 \geq \|f - S_n(f)\|_2$. In fact:

$$\|p\|_2^2 = \sum_{|k| \leq n} |\widehat{p}(k)|^2$$

$$\begin{aligned} \|f - p\|_2^2 &\overset{(!)}{=} \|f - S_n(f)\|_2^2 + \|S_n(f) - p\|_2^2 \\ &= \|f - S_n(f)\|_2^2 + \sum_{|k| \leq n} |\widehat{f}(k) - \widehat{p}(k)|^2 \end{aligned}$$

$$(p = S_m(f)) : \quad \|f - S_m(f)\|_2^2 \geq \|f - S_n(f)\|_2^2 \quad \text{if } m \leq n$$

$$(p = 0) : \quad \|f\|_2^2 = \|f - S_n(f)\|_2^2 + \|S_n(f)\|_2^2 \geq \|S_n(f)\|_2^2$$

Hence $\displaystyle\sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2 \leq \|f\|_2^2$ (Bessel).

Corollary $\|f - \sigma_n(f)\|_2 \geq \|f - S_n(f)\|_2$. (put $p = \sigma_n(f)$)

Reminder Fejér: If $g \in C(\mathbb{T})$, then $\lim_n \|g - \sigma_n(g)\|_\infty = 0$.
Hence $\lim_n \|g - \sigma_n(g)\|_2 = 0$. Hence $\lim_n \|g - S_n(g)\|_2 = 0$.
Since $C(\mathbb{T})$ is dense in $\mathcal{L}^2(\mathbb{T})$ and $\|f\|_2 \geq \|S_n(f)\|_2$, it follows that

Proposition

If $f \in \mathcal{L}^2([-\pi, \pi])$, then $S_n(f) \xrightarrow{\|\cdot\|_2} f$, that is

$$\lim_n \frac{1}{2\pi} \int_{-\pi}^{\pi} |S_n(f) - f|^2 d\lambda = 0.$$

Therefore $|\, \|S_n(f)\|_2 - \|f\|_2 \,| \leq \|S_n(f) - f\|_2 \to 0$, that is

$$\lim_{n \to \infty} \sum_{|k| \leq n} |\hat{f}(k)|^2 = \|f\|_2^2 .$$

Fourier series for functions of class $\mathcal{L}^2$

Proposition (Parseval's equality)

If $f, g \in \mathcal{L}^2(\mathbb{T})$, *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 d\lambda = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \quad \textit{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g} d\lambda = \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}.$$

Corollary

The map

$$\mathcal{F}_2 : (L^2(\mathbb{T}), \|\cdot\|_2) \to (\ell^2(\mathbb{Z}), \|\cdot\|_2) : f \to \hat{f}$$

is a well defined linear isometry.
(Uniqueness) In particular, the map $f \to \hat{f}$ *is 1-1 on* $L^2(\mathbb{T})$*: If* $\hat{f}(k) = \hat{g}(k)$ *for every* $k \in \mathbb{Z}$*, then* f *and* g *determine the same element of* $L^2(\mathbb{T})$*, i.e. they are equal almost everywhere.*

Fourier series for functions of class $\mathcal{L}^2$

The map $\mathcal{F}_2 : (C(\mathbb{T}), \|\cdot\|_{L^2}) \to (\ell^2(\mathbb{Z}), \|\cdot\|_{\ell^2}) : f \to (\hat{f}(k))_k \in \mathbb{Z}$ is isometric, hence 1-1, and has dense range, but it is not onto (why?).
Completeness of $L^2(\mathbb{T})$ yields:

Proposition

The map $\mathcal{F}_2$ sends $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$:
If $\sum_{n\in\mathbb{Z}} |c_n|^2 < +\infty$ then there exists an $f \in \mathcal{L}^2(\mathbb{T})$ so that $\hat{f}(k) = c_k$ for every $k \in \mathbb{Z}$. In fact, if $s_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$ then $\|f - s_n\|_2 \to 0$.

Sketch of proof Since

$$\|f_n\|_{L^2}^2 = \sum_k |\hat{f}_n(k)|^2 = \sum_{|k|\le n} |c_k|^2$$

the sequence (f_n) is Cauchy in the norm of $L^2(\mathbb{T})$.
Hence (completeness!) there exists $f \in \mathcal{L}^2(\mathbb{T})$ so that $\|f - f_n\|_{L^2} \to 0$.
Then $\hat{f}(k) = \langle f, e_k\rangle = \lim_n \langle f_n, e_k\rangle = \lim_n \hat{f}_n(k) = c_k$ for all $k \in \mathbb{Z}$.

Reminder

Given a function $f \in \mathcal{L}^1(\mathbb{T})$,

$$a_n = a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)d\lambda(x), \quad (n = 0, 1, 2, \ldots)$$

$$b_m = b_m(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)d\lambda(x), \quad (m = 1, 2, \ldots)$$

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(-ikx)d\lambda(x) = \langle f, e_k \rangle, \quad (k \in \mathbb{Z})$$

$$\begin{array}{rlll} \widehat{f}(n) = & \frac{1}{2}(a_n - ib_n), & (n > 0) & a_n = & \widehat{f}(n) + \widehat{f}(-n) \\ \widehat{f}(0) = & \frac{1}{2}a_0, & (n = 0) & a_0 = & 2\widehat{f}(0) \\ \widehat{f}(-n) = & \frac{1}{2}(a_n + ib_n), & (n > 0) & b_n = & i(\widehat{f}(n) - \widehat{f}(-n)) \end{array}$$

A trigonometric series which is not the Fourier series of an $f \in L^1(\mathbb{T})$

The trig. series $\sum_{k=1}^{\infty} \frac{1}{k} e_k$ converges for every $t \neq 2k\pi$ (Dirichlet) but is not the Fourier of a Riemann-integrable function, because its partial sums are not uniformly bounded. However, it is the Fourier series of an $f \in \mathcal{L}^2(\mathbb{T})$ since $\sum_{k=1}^{\infty} \left|\frac{1}{k}\right|^2 < \infty$.
We will prove that the convergent trigonometric series

$$\sum_{n=2}^{\infty} \frac{\sin nt}{\log n}$$

(sine series) is not the Fourier series of any Lebesgue-integrable function, while the corresponding cosine series

$$\sum_{n=2}^{\infty} \frac{\cos nt}{\log n}$$

is a Fourier series!
Proofs in `nofou.pdf`.

A trigonometric series which is not the Fourier series of an $f \in L^1(\mathbb{T})$

Proposition

If $f \in \mathcal{L}^1(\mathbb{T})$ and for every $n \in \mathbb{N}$ we have $-\hat{f}(-n) = \hat{f}(n) \geq 0$ then

$$\sum_{n=1}^{\infty} \frac{1}{n} \hat{f}(n) < \infty.$$

... hence f cannot have $\sum \frac{\sin nt}{\log n}$ as its Fourier series.

Proposition

Let $a_n \geq 0,\ a_n \to 0$ and suppose $a_n \leq \frac{1}{2}(a_{n-1} + a_{n+1})$ for all $n \in \mathbb{N}$. Then there exists $f \in \mathcal{L}^1(\mathbb{T})$ such that

$$\hat{f}(k) = a_{|k|} \quad \textit{for all} \;\; k \in \mathbb{Z}.$$

... hence there exists an $f \in \mathcal{L}^1(\mathbb{T})$ whose Fourier series is $\sum \frac{\cos nt}{\log n}$.

We have used two Lemmas:

Lemma

If $f \in \mathcal{L}^1(\mathbb{T})$ *and* $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)d\lambda(t) = 0$, *then the indefinite integral* g *of* f,

$$g(x) = \int_{-\pi}^{x} f(t)d\lambda(t), \qquad x \in [-\pi, \pi]$$

satisfies $ik\hat{g}(k) = \hat{f}(k)$ *for all* $k \in \mathbb{Z}$ *and* $g(-\pi) = g(\pi)$ *and is continuous (it belongs to* $C(\mathbb{T})$*).*

Lemma

If (a_n) *is a null sequence of nonnegative real numbers with the property* $2a_n \leq a_{n-1} + a_{n+1}$ *for all* $n \in \mathbb{N}$ *then*

$$\sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) = a_0.$$