605: Exercises II

1. Let $f : [-\pi, \pi] \to \mathbb{R}$ be an *even* integrable function such that: $a_k(f) \geq 0$ for every $k \geq 0$. Show that $\sum\limits_{k=0}^{\infty} a_k(f) < \infty$. (Reminder: $a_k(f) = \hat{f}(k) + \hat{f}(-k)$.)

2. Every integrable function $f : [-\pi, \pi] \to \mathbb{C}$ can be written uniquely as $f = f_a + f_p$ where f_a is even and f_p is odd. Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_a|^2 + \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_p|^2.$$

3. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous 2π-periodic function. Suppose that $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{k=-n}^{n} |k\hat{f}(k)| = 0$. Show that then $S_n(f) \to f$ uniformly.

4. If $f : \mathbb{R} \to \mathbb{C}$ is a 2π periodic and integrable function, show that

$$\lim_{x\to 0} \int |f(t-x) - f(t)|\, dt = 0.$$

Hint: Consider first the case when f is continuous.

(*From Exercises 2.5 of A. Giannopoulos' lecture notes 2012*:)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function which is Riemann integrable in $[-\pi, \pi]$. For each $m \in \mathbb{N}$, define

$$g_m(x) = f(mx).$$

Describe the graph of g_m compared to that of f. Is g_m periodic? Express the Fourier coefficients of g_m in terms of those of f.

8. Consider the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ which is defined in $[-\pi, \pi]$ by

$$f(x) = |x|.$$

Draw the graph of f, calculate its Fourier coefficients and show that $\widehat{f}(0) = \pi/2$ και

$$\widehat{f}(k) = \frac{-1 + (-1)^k}{\pi k^2}, \qquad k \neq 0.$$

Write the Fourier series $S[f]$ of f as a series of cosines and sines. Setting $x = 0$, show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \qquad \text{και} \qquad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

9. Let $[a, b]$ be a closed interval contained in the interior of $[-\pi, \pi]$. Consider the function $f(x) = \chi_{[a,b]}(x)$ defined in $[-\pi, \pi]$ by: $f(x) = 1$ if $x \in [a, b]$ and $f(x) = 0$ otherwise; extend f 2π-periodically to $\mathbb{R}$. Show that the Fourier series of f is

$$S[f](x) = \frac{b-a}{2\pi} + \sum_{k\neq 0} \frac{e^{-ika} - e^{-ikb}}{2\pi i k} e^{ikx}.$$

Show that, for any $x \in \mathbb{R}$, $S[f](x)$ is not absolutely convergent. Find the points $x \in \mathbb{R}$ for which $S[f](x)$ converges.

11. Let f, f_n $(n \in \mathbb{N})$ be 2π-periodic functions, integrable in $[-\pi, \pi]$, which satisfy

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - f_n(x)|\, dx = 0.$$

Show that

$$\widehat{f_n}(k) \to \widehat{f}(k) \quad \text{as } n \to \infty,$$

uniformly in k. That is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$ and each $k \in \mathbb{Z}$, we have

$$|\widehat{f_n}(k) - \widehat{f}(k)| < \varepsilon.$$