605: Comments on Exercises I

1. *Show that the set $\mathcal{T}$ of trigonometric polynomials is a linear space, a subspace of the space of continuous functions $f:[-\pi,\pi]\to\mathbb{C}$ and that the set $\{e_k : k\in\mathbb{Z}\}$ (where $e_k(x)=e^{ikx}$) is a linear basis of $\mathcal{T}$, as is the set $\{c_0, c_n, s_n, n=1,2,\dots\}$ (where $c_0(x)=1, c_n(x)=\cos nx, s_n(x)=\sin nx$).*

Show also that $\mathcal{T}$ is closed under pointwise multiplication and therefore for example the function $p(x)=(7-2\cos x)^5$ belongs to $\mathcal{T}$. Examine whether $\mathcal{T}$ contains some nonzero polynomial.

Comments: To show that the family $\{e_k : k\in\mathbb{Z}\}$ is linearly independent: Suppose a certain linear combination $q:=\sum\limits_{k=-N}^{N} a_k e_k$ is the zero function, i.e. $q(x)=\sum\limits_{k=-N}^{N} a_k e^{ikx}=0$ for all x. We have to show that all the coefficients $a_m, m=-N,\dots,N$ are 0. For some integer $m\in[-N,N]$, consider

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} q(x)e^{-imx}dx.$$

We know this is equal to $\hat{q}(m)=a_m$ (reason: the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-m)x}dx=\delta_{km}$). But $q=0$, so $\hat{q}(m)=0$ for all m.

The same method shows that the set $\{c_0, c_n, s_n, n=1,2,\dots\}$ is linearly independent: now one must multiply separately by $\cos mx$ and by $\sin mx$.

Another method: consider the function

$$e^{iNx}\sum_{k=-N}^{N} a_k e^{ikx} = a_{-N}+a_{-N+1}e^{ix}+\dots+a_N(e^{ix})^{2N}.$$

This is a polynomial of degree (at most) $2N$ in the complex variable $z:=e^{ix}$. So, if it has more than $2N$ complex roots, it must be identically zero, i.e. $a_k=0$ for all k.

So we have reached a stronger conclusion: if the trig. polynomial $q(x)$ vanishes for more than $2N$ distinct values of x, then it must be identically zero.

To show that $\mathcal{T}$ is closed under multiplication: For the exponential form, this is obvious: the product of two elements of the basis is another element of the basis: $e_k e_m = e_r$ where $r=k+m$.

For the sine-cosine form one must remember the trig. formulae relating products $\cos kx\cos mx$, $\cos kx\sin mx$ and $\sin kx\sin mx$ to linear combinations of sines and cosines. But these formulae are easy to find and prove: use the relations $e^{ix}=\cos x+i\sin x$ to transform them into sums of products of exponentials.

Conclusion: The set $\mathcal{T}$ is not only a linear space, but it is also a ring under pointwise operations; it is an *algebra*.

Question: Does $\mathcal{T}$ contain any polynomial (except zero)? Can a polynomial in x be a linear combination of $\sin nx$ and $\cos nx$?

I will leave this question for the Discussions!

4. *We have seen that for all $\delta>0$ there exists $M(\delta)<\infty$ so that for all $x\in[\delta,2\pi-\delta]$ and all $n\in\mathbb{N}$ we have*

$$\left|\frac{1}{2}+\sum_{k=1}^{n}\cos kx\right|\le M(\delta) \quad \text{και} \quad \left|\sum_{k=1}^{n}\sin kx\right|\le M(\delta).$$

Examine whether the two sequences are uniformly bounded in $(0,2\pi)$.

Comments: They are not: There cannot exist a *constant* M such that $|c_n(x)|\le M$ and $|s_n(x)|\le M$ for all $x\in(0,2\pi)$ and all $n\in\mathbb{N}$ simultaneously.

For the first, $c_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos kx$: this is a continuous function on $[0,2\pi]$ and it takes the value $n+\frac{1}{2}$ at $x=0$. Therefore for each n there must exist a point $x_n>0$ so that $|c_n(x_n)|>n+\frac{1}{3}$. [1]

This argument does not work for s_n, since $s_n(0)=0$. But we can still find a suitable x_n: since $s_n(x)$ is given (for $x\in(0,2\pi)$) by the formula $s_n(x)=\frac{\cos\frac{x}{2}-\cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$, we need a sequence (x_n) with $x_n\to 0$ (so that the denominator $2\sin\frac{x_n}{2}$ goes to 0) while at the same time the nominator $\cos\frac{x_n}{2}-\cos(n+\frac{1}{2})x_n$ does not vanish. For example, we can try $x_n=\frac{\pi}{2n+1}$. We get

[1] Thanks, J. A.-B.

$\cos\frac{\pi}{4n+2} - \cos(n+\frac{1}{2})\frac{\pi}{2n+1} = \cos\frac{\pi}{4n+2} - \cos(\frac{2n+1}{2}\frac{\pi}{2n+1}) = \cos\frac{\pi}{4n+2} - \cos\frac{\pi}{2} \to 1$.
Incidentally, this same sequence $x_n = \frac{\pi}{2n+1}$ also works to show that $(c_n(x_n))$ cannot be bounded.

5. *For which real values of x does the series*

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx$$

converge? Recall (as shown in class) that this is the Fourier series of the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(t) = t$ when $t \in (\pi, \pi]$.
Also find the Fourier series of the 2π-periodic function $g : \mathbb{R} \to \mathbb{R}$ which satisfies $g(t) = t$ when $t \in (0, 2\pi]$.

Comments: It clearly converges at every $x \in 2\pi\mathbb{Z}$ (sum of zeroes). Therefore (periodicity) it is enough to examine what happens when $x \in (0, 2\pi)$. Let us prove that it aleways converges:

Observe that $\sin(n\pi - nx) = \frac{(-1)^{n+1}}{n}\sin nx$ (proof: check when n is even and when n is odd, remembering that sin is 2π-periodic) and hence

$$\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k}\sin kx = \sum_{n=1}^{N}\frac{1}{n}\sin(n\pi - nx)\,.$$

Now you can use the argument used for the series $\sum_{k=1}^{\infty}\frac{1}{k}\sin kx$: indeed the sequence $(\frac{1}{k})$ monotonically decreases to 0, and for each $\delta \in (0,\pi)$ the partial sums $\sum_{n=1}^{N}\sin(n\pi - nx)$ are uniformly bounded in $[-\pi+\delta, \pi-\delta]$ (by $\frac{1}{\sin(\frac{\pi-\delta}{2})}$, from the formula in Exercise 3), so the Dirichlet criterion applies, etcetera...
The Fourier series of the 2π-periodic function $g : \mathbb{R} \to \mathbb{R}$ which satisfies $g(t) = t$ when $t \in (0, 2\pi]$.

$$(n=0) \qquad \hat{g}(0) = \frac{1}{2\pi}\int_0^{2\pi} t dt = \frac{1}{2\pi}\frac{(2\pi)^2}{2} = \pi.$$

$$\begin{aligned}(n\neq 0) \qquad \hat{g}(n) &= \frac{1}{2\pi}\int_0^{2\pi} te^{-int}dt = \frac{1}{-2\pi in}\int_0^{2\pi} t(e^{-int})'dt \\ &= \frac{i}{2\pi n}\int_0^{2\pi} t(e^{-int})'dt = \frac{i}{2\pi n}\left(\left[te^{-int}\right]_0^{2\pi} - \int_0^{2\pi} e^{-int}dt\right) \\ &= \frac{i}{2\pi n}(2\pi e^{-i2n\pi} - (0)e^{in0} - 0) = \frac{i}{n}e^{-i2n\pi} = \frac{i}{n}\end{aligned}$$

So the complex form of $S(f)$ is

$$g \sim \pi + \sum_{n\neq 0}\frac{i}{n}e^{int}.$$

Thus

$$\begin{aligned} a_n(g) &= \hat{g}(n) + \hat{g}(-n) = \frac{i}{n} + \frac{i}{-n} = 0, \quad n = 1,\ldots \\ \text{and } b_m(g) &= \frac{\hat{g}(-m) - \hat{g}(m)}{i} = \frac{1}{-m} - \frac{1}{m} = -2\frac{1}{m}, \quad m = 1,2,\ldots\end{aligned}$$

hence

$$g \sim \pi - 2\sum_{m=1}^{\infty}\frac{1}{m}\sin mt\,.$$

Ia 2. Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and integrable over compact intervals. If $x \in \mathbb{R}$ define $f_x : \mathbb{R} \to \mathbb{C}$ by $f_x(t) = f(t-x)$ $(t \in \mathbb{R})$. Show that $\widehat{f_x}(k) = e^{-ikx}\hat{f}(k)$ for all $k \in \mathbb{Z}$.

Solution

$$\begin{aligned} 2\pi\widehat{f_x}(k) &= \int_{-\pi}^{\pi} f_x(t)e^{-ikt}dt = \int_{-\pi}^{\pi} f(t-x)e^{-ikt}dt \\ &\overset{(s=t-x)}{=} \int_{-\pi-x}^{\pi-x} f(s)e^{-ik(s+x)}ds = e^{-ikx}\int_{-\pi-x}^{\pi-x} f(s)e^{-iks}ds \\ &= e^{-ikx}\int_{-\pi}^{\pi} f(s)e^{-iks}ds\end{aligned}$$

(the last equality holds because the function $s \mapsto f(s)e^{-iks}$ is 2π-periodic hence its integral over any interval of length 2π is the same).

Comment on the previous Exercise Applying the last result to the function $g_{-\pi}(t) = g(t+\pi)$ we see that $\widehat{g_{-\pi}}(k) = e^{ik\pi}\hat{g}(k)$ and so $\widehat{g_{-\pi}}(0) = \pi$ while for $m = 1, 2, \ldots, a_m(g_{-\pi}) = 0$ and

$$b_m(g_{-\pi}) = \frac{1}{i}(\widehat{g_{-\pi}}(-m) - \widehat{g_{-\pi}}(m)) = 2\frac{(-1)^{m+1}}{m} \,.$$

Thus

$$g_\pi \sim \ \pi + 2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mt \,.$$

But notice that when $t \in (-\pi, \pi]$, we have $t + \pi \in (0, 2\pi]$ and so $g_{-\pi}(t) = g(t+\pi) = t + \pi$. Therefore we recover the Fourier series for the 2π-periodic function f satisfying $f(t) = t$ for $t \in (-\pi, \pi]$:

$$f = (g_\pi - \pi) \sim \ 2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mt \,.$$

8. *If $0 < \delta < \pi$, find the Fourier coefficients of the function $f : [-\pi, \pi] \to \mathbb{R}$ (whose graph is triangular) given by the formula*

$$f(x) = \begin{cases} 1 - \frac{|x|}{\delta} & (|x| \le \delta) \\ 0 & (\delta < |x| \le \pi) \end{cases}$$

Solution The function is even. So $b_n(f) = 0$ for all $n \in \mathbb{N}$ by Exercise 6. Also,

$$\int_{-\pi}^{\pi} f(x) \cos nx dx \overset{(e)}{=} 2\int_0^{\pi} f(x) \cos nx dx = 2\int_0^{\delta} (1 - \frac{x}{\delta}) \cos nx dx$$

because $x \mapsto f(x) \cos nx$ is even (e) and $f(x) = 0$ for $x > \delta$. Therefore we have

$$\begin{aligned}
& a_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\delta \ \ (\delta \text{ is the area of the triangle}) \\
(n > 0) \qquad & a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\delta}(1 - \frac{x}{\delta})\left(\frac{\sin nx}{n}\right)' dx \\
& = \frac{2}{n\pi}\int_0^{\pi}\left(\left[(1 - \frac{x}{\delta})\sin nx\right]_0^{\delta} - \int_0^{\delta}(1 - \frac{x}{\delta})' \sin nx dx\right) \\
& = \frac{2}{n\pi}(0 + \frac{1}{\delta}\int_0^{\delta} \sin nx dx) = \frac{2}{n\pi\delta}\left[\frac{-\cos nx}{n}\right]_0^{\delta} = \frac{2}{n^2\pi\delta}(1 - \cos n\delta) \\
& = \frac{2}{n^2\pi\delta} 2\sin^2(\frac{n\delta}{2}) = \frac{\delta}{\pi}\left(\frac{\sin\frac{n\delta}{2}}{\frac{n\delta}{2}}\right)^2
\end{aligned}$$

So

$$f \sim \ \frac{\delta}{2\pi} + \sum_{n \ge 1} \frac{\delta}{\pi}\left(\frac{\sin\frac{n\delta}{2}}{\frac{n\delta}{2}}\right)^2 \cos nx.$$