605: Exercises I

1. Show that the set $\mathcal{T}$ of trigonometric polynomials is a linear space, a subspace of the space of continuous functions $f:[-\pi,\pi]\to\mathbb{C}$ and that the set $\{e_k : k\in\mathbb{Z}\}$ (where $e_k(x)=e^{ikx}$) is a linear basis of $\mathcal{T}$, as is the set $\{c_0, c_n, s_n, n=1,2,\dots\}$ (where $c_0(x)=1, c_n(x)=\cos nx, s_n(x)=\sin nx$).

Show also that $\mathcal{T}$ is closed under pointwise multiplication and therefore for example the function $p(x)=(7-2\cos x)^5$ belongs to $\mathcal{T}$. Examine whether $\mathcal{T}$ contains some nonzero polynomial.

2. If a_n, b_n are (real or complex) numbers, show that

$$\frac{a_0}{2}+\sum_{n=1}^{N} a_n\cos nx+\sum_{n=1}^{N} b_n\sin nx=\sum_{k=-N}^{N} c_k\exp(ikx)$$

where $\exp(it)=\cos t+i\sin t$ and $a_n=c_n+c_{-n},\ b_n=i(c_n-c_{-n})\ (n\in\mathbb{Z}_+)$ and

$$c_k=\begin{cases}\frac12(a_k-ib_k), & k\ge 1\\ \frac12 a_0, & k=0\\ \frac12(a_{-k}+ib_{-k}), & k\le -1\end{cases}$$

3. Show that for all $x\in\mathbb{R}$,

$$s_n(x)=\sum_{k=1}^{n}\sin kx=\sin x+\sin 2x+\ldots+\sin nx=\begin{cases}\frac{\cos\frac{x}{2}-\cos(n+\frac12)x}{2\sin\frac{x}{2}}, & x\notin 2\pi\mathbb{Z}\\ 0, & x\in 2\pi\mathbb{Z}\end{cases}$$

$$c_n(x)=\frac12+\sum_{k=1}^{n}\cos kx=\frac12+\cos x+\cos 2x+\ldots+\cos nx=\begin{cases}\frac{\sin(n+\frac12)x}{2\sin\frac{x}{2}}, & x\notin 2\pi\mathbb{Z}\\ n+\frac12, & x\in 2\pi\mathbb{Z}\end{cases}$$

Hint: Examine $c_n(x)+is_n(x)$.

4. We have seen that for all $\delta>0$ there exists $M(\delta)<\infty$ so that for all $x\in[\delta,2\pi-\delta]$ and all $n\in\mathbb{N}$ we have

$$\left|\frac12+\sum_{k=1}^{n}\cos kx\right|\le M(\delta)\ \text{και}\ \left|\sum_{k=1}^{n}\sin kx\right|\le M(\delta)\,.$$

Examine whether the two sequences are uniformly bounded in $(0,2\pi)$.

5. For which real values of x does the series

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx$$

converge? Recall (as shown in class) that this is the Fourier series of the 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ which satisfies $f(t)=t$ when $t\in(\pi,\pi]$.

Also find the Fourier series of the 2π-periodic function $g:\mathbb{R}\to\mathbb{R}$ which satisfies $g(t)=t$ when $t\in(0,2\pi]$.

6. If an integrable function $f:[-\pi,\pi]\to\mathbb{C}$ is even, then its Fourier series is a cosine series (that is, $b_n(f)=0$ for all $n\in\mathbb{N}$). If it is odd, then its Fourier series is a sine series (that is, $a_n(f)=0$ for all $n\in\mathbb{Z}_+$). If f is real valued, then $\hat f(-k)=\overline{\hat f(k)}$ for all $k\in\mathbb{Z}$.

7. We consider the function $f(x)=(\pi-x)^2$ in $[0,2\pi]$ and extend it to a 2π-periodic function defined on $\mathbb{R}$. Show that

$$S[f](x)=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}\,.$$

8. If $0<\delta<\pi$, find the Fourier coefficients of the function $f:[-\pi,\pi]\to\mathbb{R}$ (whose graph is triangular) given by the formula

$$f(x)=\begin{cases}1-\frac{|x|}{\delta} & (|x|\le\delta)\\ 0 & (\delta<|x|\le\pi)\end{cases}$$