CONVERGENCE OF RANDOM FOURIER SERIES

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Abstract. This paper will study Fourier Series with random coefficients. We begin with an introduction to Fourier series on the torus and give some of the most important results. We then give some important results from probability theory, and build on these to prove a variety of theorems that deal with the convergence or divergence of general random series. In the final section, the focus is placed on random Fourier series, and we combine results from the previous sections to prove our main theorem. The main result of this paper gives a simple condition for the almost-everywhere convergence or divergence of a random trigonometric series, and proves that divergence implies that the coefficients cannot be the Fourier series of any function.

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1. Introduction

The goal of this paper will be to study the behavior of the random trigonometric series

$$\sum_{n=-\infty}^{\infty} Y_n\, e^{int} \tag{1.1}$$

where the Y_n's are independent (but *not* necessarily identically distributed) complex-valued random variables for $t \in [0,\, 2\pi)$. The partial sums of this series represent random functions on the torus $\mathbb{T}$, and we are interested in exploring when these partial sums converge for some, almost all, or all t. In particular, if these partial sums converge almost-everywhere, then (1.1) represents a function on $\mathbb{T}$ whose Fourier coefficients are given by the Y_n.

To simplify the problem, we will focus on the series

$$\sum_{n=0}^{\infty} X_n \cos(nt + \Phi_n) \tag{1.2}$$

where X_n and Φ_n are real-valued and $X_n e^{i\Phi_n}$ are independent, and the results obtained for this case can be translated to the general case with a few minor modifications.

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The main result of this paper comes in the form of the Paley-Zygmund Theorem, which gives a simple condition for the almost-sure convergence or divergence of (1.2). It is stated below.

Theorem 1.3. (Paley-Zygmund)

If $\sum_{n=0}^{\infty} \mathrm{E}(X_n^2) < \infty$*, (1.2) converges almost-surely almost-everywhere to a function* $f(t)$ *such that* $f \in L^p(\mathbb{T})$ *for* $1 \leq p < \infty$*.*

If $\sum_{n=0}^{\infty} \mathrm{E}\left[X_n^2\right] = \infty$*, (1.2) diverges almost-surely almost-everywhere, and the sequence* $\{X_n\}$ *almost-surely does not represent the Fourier coefficients of a function in* $L^p(\mathbb{T})$ *for* $1 \leq p < \infty$*.*

This theorem divides random Fourier series into two different convergence classes. If the X_n's satisfy the conditions of the Paley-Zygmund Theorem, (1.2) converges to a function which is in L^p for arbitrarily large p, but *not* necessarily to a bounded function. Once boundedness is satisfied, however, the continuity of the function immediately follows, although we will not prove that result in this paper.

2. Fourier Series

We start with some fundamental results about Fourier series.

Throughout this paper we will restrict ourselves to complex-valued functions on $\mathbb{T}$, the one-dimensional torus. Recognizing that $\mathbb{T}$ can be thought of as the group $\mathbb{R}/2\pi\mathbb{Z}$ under addition, we define the Lebesgue measure on $\mathbb{T}$ in the natural way.

Definition 2.1. The integral of a function $f : \mathbb{T} \to \mathbb{C}$ is defined as

$$\int_{\mathbb{T}} f(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

so that the Lebesgue measure dt is the restriction of the Lebesgue measure dx on $\mathbb{R}$ to the interval $[0, 2\pi)$. (The factor $1/2\pi$ is present so that $\int_{\mathbb{T}} dt = 1$)

The most important property of the measure dt is *translation invariance*, which means for any $t_0 \in \mathbb{T}$, we have

$$\int_{\mathbb{T}} f(t - t_0)\, dt = \int_{\mathbb{T}} f(t)\, dt \tag{2.2}$$

This can easily be seen by recognizing that a function on $\mathbb{T}$ can be thought of as a 2π-periodic function on $\mathbb{R}$.

We now define some important Banach spaces on $\mathbb{T}$.

Definition 2.3. The following are Banach spaces:

- $L^p(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} \bigm| \int_{\mathbb{T}} |f(t)|^p\, dt < \infty\}$ for $1 \leq p < \infty$ with norm

$$\|f\|_{L^p} = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^p\, dt \right)^{1/p}$$

- $C(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} \mid f \text{ is continuous}\}$ with norm
$$\|f\|_\infty = \sup_t |f(t)|$$

Note that, for $p < q$, we have $C(\mathbb{T}) \subset L^q(\mathbb{T}) \subset L^p(\mathbb{T})$.

The above Banach spaces are important because they satisfy the following properties, which are not necessarily true in the case of a general Banach space.

Lemma 2.4. *Consider $f \in B$, where B is one of the Banach spaces above, and $\tau, \tau_0 \in \mathbb{T}$. Let $f_\tau(t) = f(t - \tau)$. The following properties are satisfied:*

$$\|f_\tau\|_B = \|f\|_B \qquad \text{Translation Invariance in Norm} \tag{2.5}$$

$$\lim_{\tau \to \tau_0} \|f_\tau - f_{\tau_0}\|_B = 0 \qquad \text{Continuity of Translation in Norm} \tag{2.6}$$

Proof. (2.5) is a consequence of the translation invariance of the measure dt. (2.6) is obvious if f is a continuous function. Now consider $f \in L^p(\mathbb{T})$ for $1 \le p < \infty$. Recalling that continuous functions are dense in $L^p(\mathbb{T})$, take some $\epsilon > 0$ and continuous function g such that $\|f - g\|_B < \epsilon/2$. Then

$$\begin{aligned} \|f_\tau - f_{\tau_0}\|_B &\le \|f_\tau - g_\tau\|_B + \|g_\tau - g_{\tau_0}\|_B + \|g_{\tau_0} - f_{\tau_0}\|_B \\ &= \|(f-g)_\tau\|_B + \|g_\tau - g_{\tau_0}\|_B + \|(g-f)_{\tau_0}\|_B \le \epsilon + \|g_\tau - g_{\tau_0}\|_B \end{aligned}$$

Since $\|g_\tau - g_{\tau_0}\|_B$ vanishes as $\tau \to \tau_0$, and ϵ is arbitrary, (2.6) is proven. □

In this paper we will also refer to $B(\mathbb{T})$, the space of bounded functions with norm $\|f\|_\infty = \sup_t |f(t)|$. However, it is important to note that this space is *not* a Banach space because it is not complete.

Next, we define an important family of functions on $\mathbb{T}$ that are closely related to Fourier series.

Definition 2.7. A *trigonometric polynomial* on $\mathbb{T}$ is a function P of the form

$$P(t) = \sum_{n=-N}^{N} a_n e^{int} \tag{2.8}$$

The *degree of* P is the largest integer m such that $|a_m| + |a_{-m}| \neq 0$.

Definition 2.9. A *trigonometric series* on $\mathbb{T}$ is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int} \tag{2.10}$$

Note that this definition makes no assumptions about the convergence of the series for any t.

Remark 2.11. Recall the random trigonometric series $\sum_{n=-\infty}^{\infty} Y_n\, e^{int}$ introduced at the beginning of the paper. Using the well-known identity $e^{it} = \cos t + i \sin t$ and writing $Y_n = Z_n e^{i\Theta_n}$ with Z_n and Θ_n real, we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} Y_n\, e^{int} &= \sum_{n=0}^{\infty} Z_n\, e^{i(\Theta_n + nt)} + \sum_{n=1}^{\infty} Z_{-n}\, e^{i(\Theta_{-n} - nt)} \\ &= \sum_{n=0}^{\infty} Z_n \cos(nt + \Theta_n) + i \sum_{n=0}^{\infty} Z_n \sin(nt + \Theta_n) \\ &\quad + \sum_{n=1}^{\infty} Z_{-n} \cos(-nt + \Theta_{-n}) + i \sum_{n=1}^{\infty} Z_{-n} \sin(-nt + \Theta_{-n}) \end{aligned}$$

So we see that determining the convergence of (1.1) depends on the convergence of four different cases of (1.2), where $\Phi_n = \Theta_n$ for the first term, $\Phi_n = \Theta_n - \frac{\pi}{2}$ for the second, $\Phi_n = -\Theta_n$ for the third, and $\Phi_n = -\Theta_n + \frac{\pi}{2}$ for the fourth.

The Fourier series of a function is a trigonometric series with the a_n chosen in such a way that they "represent" the function in a certain sense. To motivate the correct choice of these coefficients, observe that for a trigonometric polynomial $P(t)$, we have

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} P(t) e^{-int} dt$$

because for $j \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ijt} dt = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases} \tag{2.12}$$

Therefore, we define the n'th Fourier coefficient of a function $f \in L^1(\mathbb{T})$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} dt \tag{2.13}$$

Definition 2.14. The Fourier series $S[f]$ of a function $f \in L^1(\mathbb{T})$ is given by the trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} \tag{2.15}$$

Again, no assumptions about convergence for any t are made.

We write $S_N(f, t)$ for the partial sums of (2.11) up to degree N. Formally,

$$S_n(f, t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt} \tag{2.16}$$

This function is well defined for all $t \in \mathbb{T}$.

Earlier we claimed that the translation invariance of the measure dt was important, and to show why we define the convolution operation on $\mathbb{T}$.

Definition 2.17. For two functions f and g in $L^1(\mathbb{T})$, their convolution is given by

$$(f * g)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t - \tau) g(\tau)\, d\tau \tag{2.18}$$

It can be shown that $f*g \in L^1(\mathbb{T})$, and that the convolution operation is commutative, associative, and distributive with respect to addition of functions. The true importance of the convolution operation, however, comes from its relation to the Fourier coefficients of functions.

Theorem 2.19. *Consider $f, g \in L^1(\mathbb{T})$. Then for $h(t) = (f * g)(t)$, we have*

$$\|h\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1} \tag{2.20}$$

and

$$\hat{h}(t) = \hat{f}(t) \hat{g}(t) \tag{2.21}$$

Proof. Let $F(t,\tau) = f(t-\tau)g(\tau)$. Then we have

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{2\pi}\int_{\mathbb{T}}|F(t,\tau)|\,dt\right)\,d\tau = \frac{1}{2\pi}\int_{\mathbb{T}}|g(\tau)|\,\|f\|_{L^1}\,d\tau = \|f\|_{L^1}\|g\|_{L^1}$$

By Fubini's Theorem, $\frac{1}{4\pi^2}\int\int F(t,\tau)dtd\tau = \frac{1}{4\pi^2}\int\int F(t,\tau)d\tau dt$, and using a well-known integral inequality we get

$$\|h\|_{L^1} = \frac{1}{4\pi^2}\int_{\mathbb{T}}\left|\int_{\mathbb{T}}F(t,\tau)d\tau\right|dt \le \frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|F(t,\tau)|dt\,d\tau = \|f\|_{L^1}\|g\|_{L^1}$$

which establishes (2.15). To prove (2.16), using Fubini's Theorem once more, we get

$$\begin{aligned}\hat{h}(n) &= \frac{1}{2\pi}\int_{\mathbb{T}}h(t)e^{-int}\,dt = \frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}}f(t-\tau)e^{-in(t-\tau)}g(\tau)e^{-in\tau}dt\,d\tau\\ &= \left(\frac{1}{2\pi}\int_{\mathbb{T}}f(t)e^{-int}dt\right)\left(\frac{1}{2\pi}\int_{\mathbb{T}}g(\tau)e^{-in\tau}d\tau\right) = \hat{f}(n)\hat{g}(n)\end{aligned}$$

□

3. Summability Kernels and the Convergence of Fourier Series

With the Fourier series of a function and the important convolution operation defined, the next topic to consider is the convergence of Fourier series. Although we might hope that the Fourier series of a function in $L^1(\mathbb{T})$ will converge to the function, either in the L^1 norm or pointwise, this is not the case in general. Below we define two important kernels, the Fejer kernel and the Dirichlet kernel. These kernels are functions that, after convolution with some function f, give some insight into the convergence of the Fourier series of f.

Definition 3.1. The n'th *Dirichlet kernel* $D_n(t)$ is given by

(3.2) $$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

The n'th *Fejer kernel* $K_n(t)$ is given by

(3.3) $$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikt}$$

Observe that, because of (2.12),

(3.4) $$\frac{1}{2\pi}\int_{\mathbb{T}}D_n(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{T}}K_n(t)\,dt = 1$$

Both of these kernels can be written in closed form, as the following lemma shows.

Lemma 3.5.

(3.6) $$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

(3.7) $$K_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2})t}{\sin\frac{1}{2}t}\right)^2$$

Proof. Recall the formula for finite sums of geometric sequences:

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

Therefore, we can write

$$\sum_{k=-n}^{n} r^k = r^{-n}\sum_{k=0}^{2n} r^k = r^{-n}\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-\frac{1}{2}}}{r^{-\frac{1}{2}}}\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-\frac{1}{2}}-r^{n+\frac{1}{2}}}{r^{-\frac{1}{2}}-r^{\frac{1}{2}}}$$

Here $r = e^{it}$. Recalling that $\sin t = \frac{1}{2i}(e^{it}-e^{-it})$, we get

$$\sum_{k=-n}^{n} e^{ikt} = \frac{e^{-i(n+\frac{1}{2})t}-e^{i(n+\frac{1}{2})t}}{e^{-\frac{1}{2}it}-e^{\frac{1}{2}it}} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

which proves (3.6). To prove (3.7), we write

$$\begin{aligned}
\sum_{k=0}^{n}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t} &= \frac{1}{2i\sin\frac{1}{2}t}\sum_{k=0}^{n}\left(e^{i(k+\frac{1}{2})t}-e^{-i(k+\frac{1}{2})t}\right)\\
&= \frac{1}{2i\sin\frac{1}{2}t}\left(e^{\frac{1}{2}it}\sum_{k=0}^{n}e^{ikt}-e^{-\frac{1}{2}it}\sum_{k=0}^{n}e^{-ikt}\right)\\
&= \frac{1}{2i\sin\frac{1}{2}t}\left[e^{\frac{1}{2}it}\left(\frac{1-e^{i(n+1)t}}{1-e^{it}}\right)-e^{-\frac{1}{2}it}\left(\frac{1-e^{-i(n+1)t}}{1-e^{-it}}\right)\right]\\
&= \frac{1}{2i\sin\frac{1}{2}t}\left(\frac{1-e^{i(k+1)t}}{e^{-\frac{1}{2}it}-e^{\frac{1}{2}it}}-\frac{1-e^{-i(k+1)t}}{e^{\frac{1}{2}it}-e^{-\frac{1}{2}it}}\right) = \frac{1}{2i\sin\frac{1}{2}t}\left(\frac{2-(e^{i(k+1)t}+e^{-i(k+1)t})}{e^{-\frac{1}{2}it}-e^{\frac{1}{2}it}}\right)\\
&= \frac{1}{2i\sin\frac{1}{2}t}\left(\frac{2-2\cos(n+1)t}{-2i\sin\frac{1}{2}t}\right) = \frac{\frac{1}{2}(1-\cos(n+1)t)}{(\sin\frac{1}{2}t)^2}
\end{aligned}$$

Since $\cos 2t = 1-2\sin^2 t$, this proves the lemma. □

Since both kernels are functions on $\mathbb{T}$ (in particular, trigonometric polynomials), we can find their Fourier coefficients, which have a simple form. They are

$$\hat{D}_n(m) = \begin{cases} 1 & \text{if } |m|\le n\\ 0 & \text{otherwise}\end{cases} \qquad \hat{K}_n(m) = \begin{cases} 1-\frac{|m|}{n+1} & \text{if } |m|\le n\\ 0 & \text{otherwise}\end{cases}$$

The importance of the Dirichlet kernel can be seen as follows. By Theorem 2.15,

$$\widehat{D_n * f}(m) = \hat{D}_n(m)\hat{f}(m) = \begin{cases} \hat{f}(m) & \text{if } |m|\le n\\ 0 & \text{otherwise}\end{cases}$$

Moreover,

$$\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{k=-n}^{n} e^{in(k-\tau)}f(\tau)\,d\tau = \sum_{k=-n}^{n} e^{ikt}\left(\frac{1}{2\pi}\int_{\mathbb{T}} f(\tau)e^{-in\tau}\,d\tau\right)$$

which means

$$(D_n * f)(t) = S_n(f,\, t) \tag{3.8}$$

Therefore, the limit as $n\to\infty$ of the convolution of the Dirichlet kernel and a function f is the Fourier series of the function. Recalling that the convolution operation is distributive with respect to addition, and writing $\sigma_n(f,\, t) = (K_n * f)(t)$, we also have

$$\sigma_n(f,\, t) = \frac{1}{n+1}\sum_{k=0}^{n} S_n(f,\, t) \tag{3.9}$$

The convolution of a function and the Fejer kernel gives the mean of the partial sums of its Fourier series up to order n.

Now we are in a position to prove some important results about the convergence of Fourier series.

Theorem 3.10. *Let B be a Banach space from Definition (2.3), and consider some $f \in B$. Then*

$$\lim_{n\to\infty} K_n * f = f \tag{3.11}$$

in the norm of B.

Proof. Choose $\epsilon > 0$. It is easy to show that

$$K_n * f = \frac{1}{2\pi}\int_{\mathbb{T}} K_n(\tau) f(t-\tau)\, d\tau$$

in the norm of B. Now, for $0 < \delta < \pi$, using (3.4), we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} K_n(\tau) f(t-\tau)\, d\tau - f(t) = \frac{1}{2\pi}\left(\int_{-\delta}^{\delta} + \int_{\delta}^{2\pi-\delta}\right) K_n(\tau)[f(t-\tau) - f(t)]\, d\tau$$

$$\begin{aligned}\left\|\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)[f(t-\tau)-f(t)]\, d\tau\right\|_B &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} |K_n(\tau)|\, \|f(t-\tau)-f(t)\|_B\, d\tau \\ &\leq \sup_{|\tau|\leq\delta} \|f(t-\tau)-f(t)\|_B \|K_n\|_{L^1}\end{aligned}$$

By Lemma 2.4, we have $\lim_{\tau\to 0}\|f(t-\tau)-f(t)\|_B = 0$, so we can find $\delta > 0$ such that $\sup_{|\tau|\leq\delta}\|f(t-\tau)-f(t)\|_B\|K_n\|_{L^1} < \epsilon$. Using this same δ, we write

$$\left\|\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)[f(t-\tau)-f(t)]\, d\tau\right\|_B \leq \sup_{\tau\in\mathbb{T}}\|f(t-\tau)-f(t)\|_B\, \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |K_n(\tau)|\, d\tau$$

By (3.7), for $t \in (0, 2\pi)$, $\lim_{n\to\infty}|K_n(t)| = \lim_{n\to\infty}\frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2})t}{\sin\frac{1}{2}t}\right)^2 = 0$, so

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|K_n(\tau)|\, d\tau = 0$$

which means we can bound $\sup_{\tau\in\mathbb{T}}\|f(t-\tau)-f(t)\|_B\,\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|K_n(\tau)|\, d\tau$ by ϵ as well, finishing the proof. □

Corollary 3.12. (Uniqueness of Fourier Series)

If $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{N}$, then $f = g$.

Proof. Again writing $(K_n * f)(t) = \sigma_n(f, t)$, we have

$$\sigma_n(f-g, t) = \sum_{k=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)(\hat{f}(k) - \hat{g}(k))e^{ikt} = 0$$

for all n. Since $\sigma_n(f-g) \to f-g$, we have $f - g = 0$, or $f = g$. □

The Fejer sums of a function in a Banach space with certain important properties converge to the function in the norm of the Banach space, and this is an important foundation for many of the results that follow. Convergence of Dirichlet sums, however, is more problematic. We start with a lemma that gives a condition for the convergence of Dirichlet sums.

Definition 3.13. A Banach space B in definition (2.3) admits convergence in norm if

$$\lim_{n\to\infty} \|S_n(f) - f\|_B = 0$$

for all $f \in B$.

Lemma 3.14. *A Banach space B admits convergence in norm if there exists some $K > 0$ such that*

$$\|S_n(f)\|_B \leq K\|f\|_B \tag{3.15}$$

for all $f \in B$ and $n \in \mathbb{N}$.

Proof. Choose $\epsilon > 0$. Theorem (3.10) implies that trigonometric polynomials are dense in B, since $K_n * f$ is a trigonometric polynomial for all n. So choose a trigonometric polynomial $P(t)$ such that $\|f - P\|_B < \epsilon/2K$. For n greater than the degree of P, $S_n(P) = P$. Then we have

$$\begin{aligned} \|S_n(f) - f\|_B &= \|S_n(f) - S_n(P) + P - f\|_B \\ &\leq \|S_n(f - P)\|_B + \|P - f\|_B \leq K\frac{\epsilon}{2K} + \frac{\epsilon}{2K} \end{aligned}$$

□

Recalling that $S_n(f) = D_n * f$, we obtain the inequality

$$\|D_n * f\|_B \leq \|D_n\|_{L^1}\|f\|_B$$

which means

$$\frac{\|S_n(f)\|_B}{\|f\|_B} \leq \|D_n\|_{L^1} \tag{3.16}$$

The numbers $\|D_n\|_{L^1} = L_n$ are called the Lebesgue constants. If they were bounded, then any Banach space from Definition (2.3) would admit convergence in norm, but as the next lemma shows, this is not the case.

Lemma 3.17. *$L_n \to \infty$ as $n \to \infty$*

Proof.

$$L_n = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right| dt = \frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right| dt$$

Let $f(x) = x - \sin x$. $f(0) = 0$ and $f'(x) = 1 - \cos x \geq 0$ for $x \in [0, \pi/2]$ which means that $\sin t/2 \leq t/2$ on $[0, \pi]$. So we have

$$\frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right| dt \geq \frac{2}{\pi}\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{t}\right| dt \geq \frac{2}{\pi}\sum_{k=1}^{n-1}\int_{\frac{k\pi}{n+\frac{1}{2}}}^{\frac{(k+1)\pi}{n+\frac{1}{2}}} \frac{|\sin(n+\frac{1}{2})t|}{t}\, dt$$

Now,

$$\int_{\frac{k\pi}{n+\frac{1}{2}}}^{\frac{(k+1)\pi}{n+\frac{1}{2}}} \frac{|\sin(n+\frac{1}{2})t|}{t}\, dt = \int_{j\pi}^{(j+1)\pi} \frac{|\sin u|}{u}\, du \geq \frac{1}{\pi}\frac{1}{j+1}\int_{j\pi}^{(j+1)\pi} |\sin u|du = \frac{2}{\pi}\frac{1}{j+1}$$

Therefore,

$$L_n \geq \frac{4}{\pi^2}\sum_{k=1}^{n-1}\frac{1}{j+1}$$

which can be made arbitrarily large as $n \to \infty$. □

We now prove that in the case of $L^1(\mathbb{T})$ and $C(\mathbb{T})$, (3.16) becomes an equality, which means that these spaces do not admit convergence in norm.

Theorem 3.18. *$L^1(\mathbb{T})$ and $C(\mathbb{T})$ do not admit convergence in norm.*

Proof. First consider $L^1(\mathbb{T})$. Recalling that $\|K_N\|_{L^1} = 1$,

$$\sup_{f\in L^1(\mathbb{T})} \frac{\|S_n(f)\|_{L^1}}{\|f\|_{L^1}} \geq \frac{\|S_n(K_N)\|_{L^1}}{\|K_n\|_{L^1}} = \|K_N * D_n\|_{L^1}$$

Since $\lim_{N\to\infty} K_N * D_n = D_n$,

$$\sup_{f\in L^1(\mathbb{T})} \frac{\|S_n(f)\|_{L^1}}{\|f\|_{L^1}} \geq \|D_n\|_{L^1} \quad \Rightarrow \quad \sup_{f\in L^1(\mathbb{T})} \frac{\|S_n(f)\|_{L^1}}{\|f\|_{L^1}} = \|D_n\|_{L^1}$$

By the previous lemma, $\|D_n\|_{L^1} = L_n$ can be made as large as we like, so by Lemma (3.14), $L^1(\mathbb{T})$ does not admit convergence in norm. Now consider $C(\mathbb{T})$. Consider a set of functions $\psi_n \in C(\mathbb{T})$ such that $\|\psi_n\| \leq 1$ and $\psi_n(t) = \operatorname{sgn}(D_n(t))$ except near points of discontinuity of $\operatorname{sgn}(D_n(t))$. If the sum of the lengths of the intervals where $\psi_n(t) \neq \pm 1$ is less than $\epsilon/2n$, then

$$\sup_{f\in C(\mathbb{T})} \frac{\|S_n(f)\|_\infty}{\|f\|_\infty} \geq |S_n(\psi_n, 0)| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} D_n(t)\psi_n(t)\,dt\right| > L_n - \epsilon$$

□

While these two important spaces do not admit convergence in norm, it can be shown that $L^p(\mathbb{T})$ *does* admit convergence in norm for $1 < p < \infty$. We will not prove this result, but the proof has to do with the a properties of a function's conjugate Fourier series in $\mathbb{C}$.

4. Probability Theory

After establishing some fundamental properties of Fourier series, we now change gears and give some basic results from probability theory that will be important later. We will use the usual definition of probability space, random variables mapping outcomes from the probability space to the real numbers, and the expected value of a random variable. We will use Ω to denote the set of outcomes and ω to denote an element in Ω. For an event $A \subset \Omega$, we write $\mathbb{1}_A$ for the random variable defined so that

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Note that $\mathrm{E}(\mathbb{1}_A) = \mathrm{P}(A)$. We say an event that occurs with probability 1 occurs almost-surely.

Definition 4.1. A Rademacher sequence is a sequence of independent random variables $\{\epsilon_n\}$ such that

$$\mathrm{P}(\epsilon_n = 1) = \frac{1}{2}$$
$$\mathrm{P}(\epsilon_n = -1) = \frac{1}{2}$$

Sometimes we will also use $\{\epsilon_n\}$ to refer to a sequence of constants with value 1 or -1.

Definition 4.2. A symmetric random vector X is a vector such that X and $-X$ have the same distribution. If $\{X_n\}$ is a sequence of independent symmetric random vectors, then it has the same distribution as $\{\epsilon_n X_n\}$.

The next three lemmas that we will prove deal with events that depend on an infinite number of outcomes. These lemmas are very useful since we are trying to study the convergence or divergence of a random series where each term in the series is determined by a random outcome. In particular, these lemmas show that under certain conditions, an event that depends on an infinite number of outcomes occurs either with probability 0 or with probability 1.

Lemma 4.3. *Suppose $\{X_n\}$ is a sequence of random variables with $X_n \geq 0$ and $\sum_{n=1}^\infty E(X_n) < \infty$. Then $\sum_{n=1}^\infty X_n < \infty$ almost-surely.*

Proof. Since the random variables are positive, by Lebesgue's Monotone Convergence Theorem the sum of expected values is the expected value of the sum, so we have

$$\mathrm{E}\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mathrm{E}(X_n) < M$$

for some $M > 0$. Let $Y = \sum_{n=1}^{\infty} X_n$, so $Y \geq 0$. Choose $\epsilon > 0$ and take $N > 1$ large enough so that $M/N < \epsilon$.

$$\mathrm{E}(Y) = \int_0^{\infty} y\, \mu_Y(dy) < M \quad \Rightarrow \quad \int_N^{\infty} y\, \mu_Y(dy) < M$$

Then

$$\mathrm{P}(Y \in (N, \infty)) = \int_N^{\infty} \mu_Y(dy) < \frac{1}{N} \int_N^{\infty} y\, \mu_Y(dy) < \frac{M}{N} < \epsilon$$

Therefore $1 - \epsilon \leq \mathrm{P}(Y \in [0, N]) \leq 1$. Since ϵ is arbitary, this proves the lemma. □

Definition 4.4. Consider an infinite sequence of events $A_1, A_2, \ldots, A_n, \ldots$ Then

$$\overline{\lim}\, A_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n \tag{4.5}$$

Informally, $\overline{\lim}\, A_n$ holds when infinitely many A_n occur.

Lemma 4.6. (Borel-Cantelli) *Consider an infinite sequence $A_1, A_2, \ldots, A_n, \ldots$ of independent events.*

- *If $\sum_1^{\infty} P(A_n) < \infty$, then $P(\overline{\lim}\, A_n) = 0$*
- *If $\sum_1^{\infty} P(A_n) = \infty$, then $P(\overline{\lim}\, A_n) = 1$*

Proof. Recall $\mathrm{E}(\mathbb{1}_{A_n}) = \mathrm{P}(A_n)$. If $\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \mathrm{E}(\mathbb{1}_{A_n}) < \infty$, then by Lemma 4.3, $\sum_{n=1}^{\infty} \mathbb{1}_{A_n} < \infty$ almost-surely. Therefore, it is almost sure that only finitely many A_n hold, so $\mathrm{P}(\overline{\lim}\, A_n) = 0$.

Now suppose $\sum_1^{\infty} \mathrm{P}(A_n) = \infty$.

$$1 - \mathrm{P}(\overline{\lim}\, A_n) = \mathrm{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} A_n^c\right) = \lim_{k \to \infty} \mathrm{P}\left(\bigcap_{n \geq k} (1 - A_n)\right)$$

Since the A_n are independent, we have

$$\begin{aligned} \mathrm{P}\left(\bigcap_{n \geq k} (1 - A_n)\right) &= \prod_{n=k}^{\infty} (1 - \mathrm{P}(A_n)) \\ &\leq \prod_{n=k}^{\infty} \exp(-\mathrm{P}(A_n)) \\ &= \exp\left(-\sum_{n=k}^{\infty} \mathrm{P}(A_n)\right) = 0 \end{aligned}$$

which means $\mathrm{P}(\overline{\lim}\, A_n) = 1$. □

Lemma 4.7. (Zero-One Law) *Let $X = \mathbb{1}_A$ be a random variable defined on $\Omega = \prod_{n=1}^{\infty} \Omega_n$ such that*

$$X(\omega_1, \omega_2, \ldots, \omega_n, \ldots) = X(\omega_1', \omega_2', \ldots, \omega_n', \ldots)$$

whatever the particular values of ω might be. (In other words, X is a tail event that does not depend on any finite number of outcomes.) Then either $P(A) = 1$ or $P(A) = 0$.

Proof. Write

$$\mathrm{E}_n(X) = \int_{\Omega_n} \cdots \int_{\Omega_2} \int_{\Omega_1} X(\omega)\mathrm{P}_1(d\omega)\cdots\mathrm{P}_n(d\omega)$$

Then for each n, $\mathrm{E}_n(X)\mathrm{E}_n(1-X) = 0$ for each n, so $\mathrm{E}(X)\mathrm{E}(1-X) = \mathrm{P}(A)(1-\mathrm{P}(A)) = 0$. □

We now give two inequalities which will be important later.

Lemma 4.8. *For $a > 1$ and $X \geq 0$,*

$$P(X \geq a\mathrm{E}(X)) \leq \frac{1}{a}$$

Proof.

$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}(X\mathbb{1}_{X<a\mathrm{E}(X)} + X\mathbb{1}_{X\geq a\mathrm{E}(X)}) \\ &\geq 0 + \mathrm{E}(a\mathrm{E}(X)\mathbb{1}_{X\geq a\mathrm{E}(X)}) \\ &= a\mathrm{E}(X)\mathrm{P}(X \geq a\mathrm{E}(X)) \end{aligned}$$

□

Lemma 4.9. *For $0 < \lambda < 1$,*

$$P(X \geq \lambda E(X)) \geq (1-\lambda)^2 \frac{E(X)^2}{E(X^2)}$$

Proof. First we define

$$X'(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \geq \lambda\mathrm{E}(X) \\ 0 & \text{if } X(\omega) < \lambda\mathrm{E}(X) \end{cases}$$

The Cauchy-Schwartz inequality gives

$$\begin{aligned} \mathrm{E}(X')^2 &\leq \mathrm{E}(X'^2)\mathrm{P}(X \neq 0) \\ &\leq \mathrm{E}(X^2)\mathrm{P}(X \geq \lambda\mathrm{E}(X)) \end{aligned}$$

We also have $\mathrm{E}(X) \leq \mathrm{E}(X') + \lambda\mathrm{E}(X)$, so

$$(1-\lambda)^2\mathrm{E}(X)^2 \leq \mathrm{E}(X^2)\mathrm{P}(X \geq \lambda\mathrm{E}(X))$$

□

5. Series of Random Vectors

We now prove some results about general series of random vectors that will be needed to prove results about random trigonometric series.

Definition 5.1. A summation matrix is an infinite scalar matrix $S = (a_{nm})$ with $n, m \in \mathbb{N}$ such that

$$\lim_{n\to\infty} a_{nm} = 1 \quad \text{for all } m \in \mathbb{N}$$

An important summation matrix is $a_{nm} = \sup(0, 1 - \frac{m}{n})$. Note that this matrix is closely related to the Fejer kernel defined earlier.

A series $\sum_1^\infty v_n$ is S-summable if the sequence $w_1, w_2, \ldots, w_n, \ldots$ converges, where

$$w_n = \sum_{m=1}^{\infty} a_{nm} v_m$$

A series is S-bounded if w_n converges for all n and the sequence $w_1, w_2, \ldots, w_n, \ldots$ is bounded. We define $\sup_n \|w_n\|$ as its S-bound.

Just as the Fejer kernel allowed us to study the Fourier series of a function even if the Fourier series itself did not converge, summation matrices allow us to prove results about properties of random series even when the series themselves are difficult to study.

We will soon prove that for a random sequence, S-summability implies almost-sure convergence and S-boundedness implies almost-sure boundedness, but first we will prove some important lemmas.

Lemma 5.2. *Let $X_1, \dots, X_2, \dots, X_n, \dots$ be a sequence of symmetric random vectors. Let*

$$Y_m(\omega) = \sum_1^m X_n(\omega) \quad \text{and} \quad M(\omega) = \sup_m \|Y_m(\omega)\|$$

and let $Y(\omega) = \sum_1^\infty X_n(\omega)$ if the series is convergent. Suppose $\sum_1^\infty X_n$ converges almost surely. Then for each $r > 0$, we have

$$P(M(\omega) > r) \leq 2P(\|Y(\omega)\| > r) \tag{5.3}$$

Proof. Let Ω_0 be the set of ω such that $Y(\omega)$ is defined, and denote by A and B the subsets of Ω_0 where $M(\omega) > r$ and $\|Y(\omega)\| > r$ respectively. A can be divided as follows

$$\begin{array}{lcl} A_1 & : & \|Y_1\| > r \\ A_2 & : & \|Y_1\| \leq r,\ \|Y_2\| > r \\ A_3 & : & \|Y_1\| \leq r,\ \|Y_2\| \leq r,\ \|Y_3\| > r \\ \vdots & & \end{array} \tag{5.4}$$

so that $\bigcup_{m=1}^\infty A_m = A$ and A_m are disjoint. If $\omega \in A_m$, then at least one of the vectors

$$\begin{array}{l} Z = Y_m(\omega) + (\sum_{n=m+1}^\infty X_n(\omega)) \\ Z' = Y_m(\omega) - (\sum_{n=m+1}^\infty X_n(\omega)) \end{array} \tag{5.5}$$

lies outside the ball $\|x\| \leq r$. Since the X_n are symmetric, Y and Y' have the same probability of being outside the ball. The union of these two events is A_m, which means that each has a probability of at least $\frac{1}{2}\mathrm{P}(A_m)$ of occurring. Therefore, $\mathrm{P}(B \cap A_m) \geq \frac{1}{2}\mathrm{P}(A_m)$. Adding the probabilities over all disjoint A_m proves the lemma.

□

For the next lemma, we write Λ for an infinite set of integers and define

$$M_\Lambda(\omega) = \sup_{m \in \Lambda} \|Y_m(\omega)\|$$

Lemma 5.6. *For each $r > 0$ and each set Λ, we have*

$$\mathrm{P}(M > r) \leq 2\mathrm{P}(M_\Lambda > r)$$

Proof. Very similar to the proof above. Let A and B be the events $M > r$ and $M_\Lambda > r$. Let the events A_m be defined as in (5.4). For $\omega \in A_m$, consider

$$\begin{array}{rcl} M_{\Lambda, m} & = & \sup_{v \in \Lambda,\, v \geq m} \|Z_v\| \\ M'_{\Lambda, m} & = & \sup_{v \in \Lambda,\, v \geq m} \|Z'_v\| \end{array}$$

where $Z_v = Y_m + \sum_{v \in \Lambda,\, v \geq m} X_v$ and $Z'_v = Y_m - \sum_{v \in \Lambda,\, v \geq m} X_v$. Again, the subevents of A_m defined by $M_{\Lambda, m}$ and $M'_{\Lambda, m}$ have the same probability, and their union is A_m. Again, $\omega \in B \cap A_m \Rightarrow M_{\Lambda, m} > r$, so $\mathrm{P}(B \cap A_m) \geq \frac{1}{2}\mathrm{P}(A_m)$, and addition of probability of disjoint events proves the lemma. □

Theorem 5.7. *Let X_1, X_2, ..., $X_n, \dots$ be random vectors in a Banach space B, and S be a summation matrix. If the series $\sum_1^\infty X_n$ is almost-surely S-summable, it converges almost-surely. If it is almost-surely S-bounded, then it is almost-surely bounded.*

Proof. Let $S = (a_{nm})$. By assumption, we have $\sum_{m=1}^\infty a_{nm}X_m = Z_n$ almost surely for some $Z_n \in B$, and $lim_{n\to\infty} Z_n = Z$. Since $\lim_{n\to\infty} a_{nm} = 1$, there exists $N_p > 0$ such that

$$\mathrm{P}\left(\left\|\sum_{m\leq p}(1-a_{nm})X_m\right\| > 2^{-p}\right) < 2^{-p}$$

for $n > N_p$. We may also suppose $N_1 < N_2 < \dots$. By assumption, $\sum_{m=1}^\infty a_{nm}X_m$ converges almost surely, so there exists Q_p such that

$$\mathrm{P}\left(\left\|\sum_{m>q}(1-a_{N_p m})X_m\right\| > 2^{-p}\right) < 2^{-p}$$

for $q \geq Q_p$. Now write

$$b_{pm} = \begin{cases} 1 & \text{if } m \leq p \\ a_{N_p m} & \text{if } p < m \leq Q_p \\ 0 & \text{if } m > Q_p \end{cases}$$

This defines a new summation matrix, which we will call T. The finite sums

$$Z'_p = \sum_{m=1}^{\infty} b_{pm}X_m \tag{5.8}$$

satisfy $\|Z'_p - Z_{N_p}\| < 2(2^{-p})$ for $p = v, v+1, \dots$. Therefore, X_n is almost-surely T-summable. Now write $p_1 = 1$, $p_{j+1} = Q_{p_j}$ for $j \in \mathbb{N}$. Assuming $X_m = 0$ when $p_j < m \leq p_{j+1}$ gives

$$Z'_{p_j} = \sum_{m=1}^{p_j} X_m = \sum_{m=1}^{p_{j+1}} X_m \tag{5.9}$$

Therefore, if $X_m = 0$ for when $p_j < m \leq p_{j+1}$ for an infinite set J of values j, (5.9) tends almost surely to a limit when $j \to \infty$ in J.

Now we split $\sum_1^\infty X_n$ into two parts, $\sum_1^\infty X'_n$ and $\sum_1^\infty X''_n$, where

$$\begin{array}{lll} X'_n = X_n & \text{and} \quad X''_n = 0 & \text{if } p_{2j-1} \leq n < p_{2j} \\ X'_n = 0 & \text{and} \quad X''_n = X_n & \text{if } p_{2j} \leq n < p_{2j+1} \end{array}$$

Note that $2X'_n - X_n = \pm X_n$ and $2X''_n - X_n = \pm X_n$, both have the same distribution as X_n since the X_n are symmetric, so $\sum_1^\infty(2X'_n - X_n)$ and $\sum_1^\infty(2X''_n - X_n)$ have the same almost-sure properties as X_n. Therefore $\sum_1^\infty X'_n$ and $\sum_1^\infty X'_n$ are almost-surely T-summable. For these series, (5.9) implies the p_j partial sums are convergent, so for $\sum_1^\infty X_n$, the p_j partial sums are also convergent almost surely.

Now we use Lemma (5.2). Since the p_j partials sums are convergent in probability, for each $\eta > 0$ there exists a j such that

$$\mathrm{P}\left(\left\|\sum_{p_j<m\leq p_k} X_m\right\| > \eta\right) < \eta$$

for $k > j$. According to lemma 1, for each j and k,

$$\mathrm{P}\left(\sup_{p_j<l\leq p_k}\left\|\sum_{p_j<m\leq l} X_m\right\| > \eta\right) < 2\eta \tag{5.10}$$

Writing (5.10) for $\eta = \eta_k = 2^{-k}$, $j = j(\eta_k) = j_k$, and $k = j(\eta_{\kappa+1}) = j_{\kappa+1}$, $\kappa = \mu,\, \mu+1,\, \dots,$ and adding, we obtain

$$\mathrm{P}\left(\left\|\sum_{p_j<m\le l} X_m\right\| \le \eta_k\right) > 1 - 4\eta_k$$

when $p_{j_\kappa} < l \le p_{j_{\kappa+1}}$. Since the p_{j_κ} partial sums converge almost surely, the series $\sum_1^\infty X_m$ converges with probabiliy as near to 1 as we please.

The case for boundedness is similar, and uses lemma 5.6 instead of lemma 5.2. We omit the proof. □

As a consequence of this proof, if $\sum_1^\infty X_m$ is not almost surely convergent, then there exists an $\eta > 0$ and sequences of integers $m_1,\, m_2,\, \dots$ and $m_1',\, m_2',\, \dots$ with $m_1 < m_1' < m_2 < m_2' < \dots$ such that

$$\mathrm{P}\left(\left\|\sum_{m_k<n<m_k'} X_n\right\| > \eta\right) > \eta \quad \text{for } k \in \mathbb{N} \tag{5.11}$$

We now go on to prove a variety of theorems that deal with the conditions under which the random series converge or diverge almost surely. These theorems are the backbone upon which our study of random Fourier series is built, and many of the results given for random Fourier series are just special cases of these more general theorems.

Theorem 5.12. *If $\|X_n\| \in L^2(\Omega)$ and $E(X_n) = 0$ for each n, then*

$$P\left(\sup_{n=1,\,2,\,\cdots,\,N} \|X_1 + X_2 + \cdots + X_n\| > r\right) < \frac{1}{r^2}\left(V(X_1) + V(X_2) + \cdots + V(X_N)\right)$$

Proof. Let $Y_n = \sum_{m=1}^n X_m$, and again consider the disjoint events

$$\begin{array}{lcl} A_1 & : & \|Y_1\| > r \\ A_2 & : & \|Y_1\| \le r,\ \|Y_2\| > r \\ A_3 & : & \|Y_1\| \le r,\ \|Y_2\| \le r,\ \|Y_3\| > r \\ \vdots & & \end{array} \tag{5.13}$$

with $A = \bigcup_1^N A_n$. Our goal is to estimate $\mathrm{P}(A) = \sum_1^N \mathrm{P}(A_n)$. By the definition of A_n, we have

$$r^2\mathrm{P}(A_n) \le \mathrm{E}(\mathbb{1}_{A_n}\|X_1 + \cdots + X_n\|^2)$$

Now, we write $X = \mathbb{1}_{A_n}(X_1 + \cdots + X_n)$ and $Y = (X_{n+1} + \cdots + X_N)$. The events X and Y are independent, since they depend on the sum of sets of events independent from each other. Recall that $\mathrm{Cov}(X,\, Y) = \mathrm{E}(X \cdot Y) - \mathrm{E}(X)\mathrm{E}(Y)$. Therefore, $\mathrm{Cov}(X,\, Y) = 0$. Since $X = 0$ lies outside of A_n, we also have $\mathrm{Cov}(X,\, Y) = \mathrm{Cov}(X, \mathbb{1}_{A_n}Y) = 0$. Therefore,

$$\mathrm{E}(\|X + \mathbb{1}_{A_n}Y\|^2) = \mathrm{E}(\|X\|^2) + \mathrm{E}(\|\mathbb{1}_{A_n}Y\|^2) \ge \mathrm{E}(\|X\|^2)$$

so that

$$r^2\mathrm{P}(A_n) \le \mathrm{E}(\|X + \mathbb{1}_{A_n}Y\|^2) = \mathrm{E}(\mathbb{1}_{A_n}\|X_1 + \cdots + X_N\|^2)$$

Adding over $n = 1,\, \dots\, ,\, N$ gives the desired inequality, since $\mathrm{V}(X_n) = \mathrm{E}(\|X_n\|^2)$ by the assumption $\mathrm{E}(X_n) = 0$. □

Theorem 5.14. *Suppose $X_n \in L^2(\Omega)$ and $E(X_n) = 0$ for all n and moreover $\sum_1^\infty V(X_n) < \infty$. Then the series $\sum_1^\infty X_n$ converges almost-surely. In particular, if $\sum_1^\infty \|u_n\|^2 < \infty$, then $\sum_1^\infty \epsilon_n u_n$ converges almost surely.*

Proof. By the previous theorem, for each $r > 0$,

$$\mathrm{P}\left(\sup_j \|X_m + \cdots + X_{m+j}\| > r\right) < \frac{1}{r^2}\sum_{n=m}^{\infty} \mathrm{V}(X_n)$$

Since $\sum_1^\infty \mathrm{V}(X_n) < \infty$, we have

$$\mathrm{P}\left(\lim_{m\to\infty} \sup_j \|X_m + \cdots + X_{m+j}\| > r\right) = 0$$

Since $r > 0$ can be made arbitrarily small, this proves the theorem. □

Theorem 5.15. (Paley-Zygmund Inequality) *Suppose* $\|X_n\| \in L^4(\Omega)$*,* $E(X_n) = 0$*, and* $E(\|X_n\|^4) \leq C\,V(X_n)^2$ *for each* n*. Let* $0 < \lambda < 1$*. Then*

$$P(\|X_1 + \cdots + X_\nu\| > \lambda(V(X_1) + \cdots + V(X_\nu))^{1/2}) > \eta \tag{5.16}$$

where $\eta = \min(\frac{1}{3}, 1/C)((1-\lambda^2)^2$ *and* ν *is an arbitrary integer. In particular, if* $\{u_n\}$ *is a sequence of vectors and* $\{\epsilon_n\}$ *is a Rademacher sequence,*

$$P(\|\epsilon_1 u_1 + \cdots + \epsilon_\nu u_\nu\| > \lambda(\|u_1\|^2 + \cdots + \|u_\nu\|^2)^{1/2}) > \frac{1}{3}(1-\lambda^2)^2 \tag{5.17}$$

Proof. Consider the random variable $X = \|X_1 + \cdots + X_\nu\|^2$. Using lemma (4.9), we have

$$\mathrm{P}(X \geq \lambda^2 \mathrm{E}(X)) \geq (1-\lambda^2)^2 \frac{\mathrm{E}(X)^2}{\mathrm{E}(X^2)}$$

Since the X_n are independent, we have $\mathrm{E}(X) = \mathrm{V}(X_1) + \cdots + \mathrm{V}(X_\nu)$. Manipulating the properties of the inner products gives

$$\mathrm{E}(X^2) = \sum_{n_1, n_2, n_3, n_4} \mathrm{E}[(\langle X_{n_1}, X_{n_2}\rangle)(\langle X_{n_3}, X_{n_4}\rangle)]$$

If $n_1 \neq n_2$ or $n_3 \neq n_4$ then $\mathrm{E}[(\langle X_{n_1}, X_{n_2}\rangle)(\langle X_{n_3}, X_{n_4}\rangle)] = 0$ by independence. So we can write

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{n=1}^{\nu} \mathrm{E}(\|X\|^4) + 2\sum_{1\leq n<m\leq v} \big[(\mathrm{Re}\,\mathrm{E}(\langle X_n, X_m\rangle^2) \\
&\quad + \mathrm{E}(|\langle X_n, X_m\rangle|^2) + \mathrm{E}(\|X_n\|^2)\mathrm{E}(\|X_m\|^2)] \\
&\leq C\sum_{n=1}^{\nu} \mathrm{V}(X_n)^2 + 6\sum_{1\leq n<m\leq v} \mathrm{V}(X_n)\mathrm{V}(X_m) \\
&\leq \sup(3, c)\left(\sum_{n=1}^{\nu} \mathrm{V}(X_n)\right)^2
\end{aligned}$$

which proves the result. □

Theorem 5.18. *Suppose* $\|X_n\| \in L^4(\Omega)$*,* $E(X_n) = 0$*, and* $E(\|X_n\|^4) \leq C\,V(X_n)^2$ *for each* n*. Given a summation matrix* S*, suppose that* $\sum_1^\infty X_n$ *is* S*-bounded. Then* $\sum_1^\infty V(X_n) < \infty$*. In particular, if* $\sum_1^\infty \epsilon_n u_n$ *is almost-surely* S*-bounded, then* $\sum_1^\infty \|u_n\|^2 < \infty$*.*

Proof. Write $S = (a_{nm})$. Also write

$$\begin{aligned}
E_{n\nu} &= \left\{\left\|\sum_{m=1}^{\nu} a_{nm} X_m\right\| > \lambda\left(\sum_{m=1}^{\nu} a_{nm}^2 \mathrm{V}(X_m)\right)^{1/2}\right\} \\
E_n &= \overline{\lim_{\nu\to\infty}} E_{n\nu} = \bigcap_{p=1}^{\infty} \bigcup_{\nu\geq p} E_{n\nu} \\
E &= \overline{\lim_{n\to\infty}} E_n
\end{aligned}$$

By the previous theorem, $P(E_{n\nu}) > \eta$, so $P(E_n) > \eta$ and $P(E) > \eta$. Since by assumption $\sum_1^\infty X_n$ is S-bounded, there is some $\omega \in E$ and $b > 0$ such that

$$\sum_{m=1}^{\infty} a_{nm} X_m(\omega) \text{ converges} \quad \text{and} \quad \left\| \sum_{m=1}^{\infty} a_{nm} X_m(\omega) \right\| < b$$

for $n \in \mathbb{N}$. For each n with $\omega \in E_n$, we also have $\omega \in E_{n\nu}$ for infinitely many ν, so

$$\lambda^2 \sum_{m=1}^{\infty} a_{nm}^2 \mathrm{V}(X_m) < b^2$$

which holds for infinitely many n. By the definition of a summation matrix, $\lim_{n\to\infty} a_{nm} = 1$, which completes the proof. □

Theorem 5.19. *Suppose that $U_n \in L^2(\Omega)$ and $\sup_n E(U_n^2)/E(U_n)^2 < \infty$. Then $\sum_1^\infty U_n$ converges or diverges almost-surely according to whether $\sum_1^\infty E(U_n)$ converges or diverges.*

Proof. When $\sum_1^\infty \mathrm{E}(U_n) < \infty$ we already know the result by lemma (4.3). If $\sum_1^\infty U_n = \infty$, we use lemma (4.9) and write

$$P(U_1 + \cdots + U_\nu > \lambda \mathrm{E}(U_1 + \cdots + U_\nu)) > (1-\lambda)^2 \frac{\mathrm{E}(U_1 + \cdots + U_\nu)^2}{\mathrm{E}((U_1 + \cdots + U_\nu)^2)}$$

Since $\mathrm{E}(U_n U_m) = \mathrm{E}(U_n)\mathrm{E}(U_m)$ and by assumption, $\mathrm{E}(U_n^2) \le C\mathrm{E}(U_n)^2$ for some C, $\frac{\mathrm{E}(U_1+\cdots+U_\nu)^2}{\mathrm{E}((U_1+\cdots+U_\nu)^2)}$ is bounded below as $\nu \to \infty$. Therefore $P(\sum_1^\infty U_n = \infty) > 0$, so it is true almost surely by the Zero-One law. □

6. The Paley-Zygmund Theorem

We are finally in a position to prove the first of the two main results of this paper, the Paley-Zygmund Theorem. Suppose you have a random sequence of independent random vectors $\{X_n e^{i\Phi_n}\}$, with $X_n \ge 0$ and $\Phi_n \in [0, 2\pi)$. Consider the random trigonometric series

$$\sum_{n=0}^{\infty} X_n \cos(nt + \Phi_n) \tag{6.1}$$

and the Fejer sums of this series, given by

$$\sigma_N(t) = \sum_{n=0}^{N} \left(1 - \frac{n}{N}\right) X_n \cos(nt + \Phi_n) \tag{6.2}$$

By Theorem (3.10), if (6.1) represents a function in $L^p(\mathbb{T})$ then (6.2) converges to (6.1) in $L^p(\mathbb{T})$. Since the sums σ_N represent a summation matrix for each N, by Theorem 5.7 the convergence or boundedness of (6.2) implies the convergence or boundedness of (6.1) almost surely. We state this in the following proposition.

Proposition 6.3.

- *(6.1) $\in L^p(\mathbb{T})$ almost-surely $\Leftrightarrow \lim_{N\to\infty} \sigma_N$ converges in $L^p(\mathbb{T})$ for $1 \le p < \infty$ almost-surely*
- *(6.1) $\in L^p(\mathbb{T})$ almost-surely $\Leftrightarrow sup_N \|\sigma_N\|_{L^p} < \infty$ for $1 \le p < \infty$ almost-surely*

Lemma 6.4. *If $\{x_n\}$ and $\{\phi_n\}$ are two real sequences and $\sum_1^\infty x_n^2 = \infty$, then $\sum_1^\infty x_n^2 \cos^2(nt+\phi_n) = \infty$ for almost every t.*

Proof. If the lemma is false, then there exists a set E of positive measure $|E|$ where the series is bounded, so we can can write

$$\sum_{1}^{\infty} x_n^2 \cos^2(nt + \phi_n) < b \quad \text{for } t \in E$$

Integrating gives

$$\sum_{1}^{\infty} x_n^2 \int_E \cos^2(nt + \phi_n)\, dt < b|E|$$

Now, $\cos^2(nt + \phi_n) = \frac{1}{4}(2 + e^{2i(nt+\phi_n)} + e^{-2i(nt+\phi_n)})$, so

$$\int_E \cos^2(nt + \phi_n)\, dt = \int_E \frac{1}{2}\, dt + \frac{1}{4}\left(\int_E e^{2i(nt+\phi_n)}\, dt + \int_E e^{-2i(nt+\phi_n)}\, dt\right)$$

For any set E, $\lim_{n\to\infty} \int_E e^{int}\, dt = 0$, so $\lim_{n\to\infty} \int_E \cos^2(nt + \phi_n)\, dt = \frac{1}{2}|E|$. Therefore, there exists an N such that, for $n \geq N$,

$$\int_E \cos^2(nt + \phi_n)\, dt > \frac{1}{3}|E|$$

which means $\sum_{n=N}^{\infty} x_n < 3b$, contradicting our assumption and proving the lemma.

□

The results of the following three lemmas will be based on the assumption that either $\sum_0^\infty X_n^2 = \infty$ almost surely or $\sum_0^\infty X_n^2 < \infty$ almost surely. We already know conditions under which the X_n^2 converge or diverge by Theorem 5.19 of the previous section, and we will use these conditions in the final proof of this section.

Lemma 6.5. *If $\sum_0^\infty X_n^2 = \infty$ almost surely, (6.1) $\notin L^p(\mathbb{T})$ almost surely for $1 \leq p < \infty$. Moreover, the sequence $\{X_n\}$ almost-surely does not represent a Fourier series of a function in $L^p(\mathbb{T})$ for $1 \leq p < \infty$.*

Proof. Consider the Fejer sums

$$\sigma_N(t) = \sum_{n=0}^{N} \left(1 - \frac{n}{N}\right) \epsilon_n X_n \cos(nt + \phi_n)$$

Writing

$$p_N(t) = \left(\sum_{n=0}^{N} \left(1 - \frac{n}{N}\right)^2 X_n^2 \cos^2(nt + \phi_n)\right)^{1/2}$$

and choosing $0 < \lambda < 1$, $\eta = \frac{1}{3}(1 - \lambda^2)^2$, the Paley-Zygmund inequality gives

$$\mathrm{P}(\sigma_N(t) \geq \lambda p_N(t)) \geq \eta$$

By the previous lemma, there exists a sequence $\rho_N \to \infty$ and a subset T of the circle such that $\int_T dt = \pi$ and $p_N(t) \geq \rho_N$ for each $t \in T$. Consider some probability space Ω. Let $E = E_N$ be the set of $(\omega, t) \in \Omega \times T$ such that $|\sigma_N(t)| \geq \lambda\rho_N$. For fixed ω, let E_ω by the set of t such that $(\omega, t) \in E$, and for fixed t let E_t be the set of ω such that $(\omega, t) \in E$. Let $F = F_N$ be the event $|E_\omega| \geq \eta$. Then we have

$$|E| = \int_T \mathrm{P}(E_t)\, dt \geq \pi\eta$$

$$|E| = \left(\int_F + \int_{\Omega\setminus F}\right) |E_\omega|\, \mathrm{P}(d\omega)$$

Then we can write

$$\pi\eta \geq \pi \mathrm{P}(F) + \eta(1 - \mathrm{P}(F)) \quad \Rightarrow \quad \mathrm{P}(F_N) \geq \frac{(\pi-1)\eta}{\pi - \eta}$$

Also,

$$\int_{\mathbb{T}} |\sigma_N(t)|\, dt \geq \int_{E_\omega} |\sigma_N(t)|\, dt \geq \lambda \rho_N |E_\omega|$$

Since $\mathrm{P}(F_N) > c$ for some $c > 0$ for all N, we have $\sum_1^\infty P(F_N) = \infty$, so by Borel-Cantelli, $\overline{\lim}\, F_N$ happens almost surely. Therefore, we can write

$$\overline{\lim}_{N\to\infty} \int_{\mathbb{T}} |\sigma_N(t)|\, dt = \infty$$

Therefore by Proposition 6.3, we have (6.1) $\notin L^1(\mathbb{T})$ almost surely, so (6.1) $\notin L^p(\mathbb{T})$ almost surely for $1 \leq p < \infty$.

To prove the second part of the lemma, assume that $\{X_n\}$ represents the Fourier series of a function $f \in L^p(\mathbb{T})$. Then by Theorem 3.10, $\lim_{N\to\infty} \sigma_N = f$ in the L^p norm, but we have already seen that $\|\overline{\lim}_{N\to\infty} \sigma_N\|_{L^1} = \infty$, which means that it cannot converge to f in $L^p(\mathbb{T})$, since by definition $\|f\|_{L^p}$ is finite.

□

Next we will show that $\sum_0^\infty X_n^2 = \infty$ almost surely implies divergence for almost every t.

Lemma 6.6. *If $\sum_0^\infty X_n^2 = \infty$ almost surely, (6.1) diverges almost surely almost everywhere.*

Proof. By lemma 6.4, we have almost surely that

$$\sum_1^\infty X_n^2 \cos^2(nt + \phi_n) = \infty$$

Let $a_{nm} = r_n^m$ for $0 < r_n < 1$, and $\lim_{n\to\infty} r_n = 1$. Then $S = (a_{nm})$ is a summation matrix. Since $\sum_1^\infty x_n^2 \cos^2(nt + \phi_n) = \infty$, by Theorem 5.18 we have $\sum_{m=1}^\infty \pm r_N^m X_m \cos(mt + \phi_m) = \infty$ almost-surely for each r_N. Taking the limit as $N \to \infty$ proves the result. □

Lemma 6.7. *If $\sum_0^\infty X_n^2 < \infty$ almost surely, then (6.1) converges almost surely almost everywhere to a function in $L^p(\mathbb{T})$ for $1 \leq p < \infty$.*

Proof. By Theorem 5.14, (6.1) converges almost surely for any given t, and therefore converges almost-surely almost-everywhere.

Let $F(t)$ be the function that (6.1) converges to. Then for $\lambda > 0$, we have

$$\begin{aligned} \mathrm{E}(e^{\lambda F(t)} &= \mathrm{E}\left(\exp\left(\sum_1^\infty \lambda X_n \epsilon_n \cos(nt + \phi_n)\right)\right) \\ &= \prod_1^\infty \mathrm{E}(\exp(\lambda X_n \epsilon_n \cos(nt + \phi_n))) \\ &= \prod_1^\infty \cosh(\lambda X_n \cos(nt + \phi_n)) \end{aligned}$$

Since $\cosh u \leq e^{u^2/2}$, we have

$$\mathrm{E}(e^{\lambda F(t)}) \leq e^{\lambda^2 r/2}$$

where $r = \sum_1^\infty X_n^2$. By symmetry, we have $\mathrm{E}(F^{(2n+1)}(t)) = 0$ for $n = 0, 1, 2, \ldots$, so

$$\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \mathrm{E}(F^{(2n)}(t)) \leq e^{\lambda^2 r/2}$$

Choosing $\lambda^2 = 2n/r$ gives

$$\mathrm{E}(F^{(2n)}(t)) \leq (2n)! \left(\frac{2n}{r}\right)^{-n} e^n \leq Cn!(2r)^n$$

for some constant C. Therefore

$$\mathrm{E}(e^{\lambda F^2(t)}) \leq C \sum_0^\infty (2\lambda r)^n < \infty$$

whenever $\lambda < 1/2r$. Since this holds for almost every t, we have

$$\mathrm{E}\left(\int_0^{2\pi} e^{\lambda F^2(t)}\, dt\right) = \int_0^{2\pi} \mathrm{E}(e^{\lambda F^2(t)})\, dt < \infty$$

which means $\int_{\mathbb{T}} e^{\lambda F^2(t)}\, dt < \infty$ almost surely whenever $\lambda < 1/2r$. This implies that $F \in L^p(\mathbb{T})$ almost surely for $1 \leq p < \infty$, since the exponential function grows faster than any power of $F(t)$. □

It is not possible to go further and claim that $F(t)$ is bounded, which is shown as follows.

Construction 6.8. There exists a trigonometric series $g(t) = \sum_0^\infty p_n cos(nt + \phi_n)$ such that $g \in L^p(\mathbb{T})$ for $1 \leq p < \infty$, but g is unbounded.

Let $n_1, n_2, n_3, \ldots$ be an increasing sequence of integers such that, $n_{j+1} > 6n_j$ for some $\epsilon > 0$, and let $\phi_1, \phi_2, \ldots$ be a sequence of real numbers.

Claim: For $k \in \mathbb{N}$, there exists a connected interval $I_k \subset \mathbb{T}$ such that meas $I_k = \frac{3\pi}{4n_k}$ and $\cos(n_k t + \phi_k) > \alpha$ for $t \in I_k$ for some small $\alpha > 0$.

Proof. We will use a proof by induction. Clearly this is true for $k = 1$. Assume the claim is true for some $j \in \mathbb{N}$. Consider $g_{j+1}(t) = \cos(n_{j+1}t + \phi_{k+1})$.

$$\mathrm{Period}(g_{j+1}) = \frac{2\pi}{n_{j+1}} < \frac{\pi}{3n_j} < \frac{1}{2}|I_j|$$

Therefore, g_{j+1} completes two entire periods on I_k, and the conclusion follows from the properties of cosine. □

Taking the limit as $k \to \infty$ shows that there is some t_0 such that $\cos(n_k t_0 + \phi_k) > \alpha$ for all $k \in \mathbb{N}$. Letting $p_k = 1/k$, it follows that $\sum_1^\infty p_k \cos(n_k t_0 + \phi_k) = \infty$ even though $\sum_1^\infty p_k^2 < \infty$, which shows that $\sum_1^\infty p_k \cos(n_k t + \phi_k)$ is not bounded even though it is almost surely in $L^p(\mathbb{T})$ for $1 \leq p < \infty$ by Lemma 6.7.

Now we combine all the results from this section into our main result, the Paley-Zygmund Theorem.

Theorem 6.9. *Suppose that* $\sup_n E(X_n^4)/E(X_n^2) < \infty$. *Then,*

- *If* $\sum_0^\infty E(X_n^2) < \infty$, *(6.1) converges almost surely almost everywhere to a function* $F(t)$ *such that* $F(t) \in L^p(\mathbb{T})$ *for* $1 \leq p < \infty$.

- *If* $\sum_0^\infty E(X_n^2) = \infty$, *(6.1) diverges almost surely almost everywhere and* $\{X_n\}$ *almost surely does not represent the Fourier series of a function in* $L^p(\mathbb{T})$ *for* $1 \leq p < \infty$.

Proof. If $\sum_0^\infty \mathrm{E}(X_n^2) < \infty$, then by Theorem 5.19 $\sum_0^\infty X_n^2 < \infty$ almost surely, and the conclusion follows from Lemma 6.7.

If $\sum_0^\infty \mathrm{E}(X_n^2) = \infty$, then by Theorem 5.19 $\sum_0^\infty X_n^2 = \infty$ almost surely, and the conclusion follows from Lemma 6.5 and Lemma 6.6. □

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