RADON INVERSION—VARIATIONS ON A THEME

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1. Introduction. In 1917 Radon published [8] a solution to the problem: recover a function on the plane from its integrals over all lines in the plane. (Functions may be real or complex valued. Radon's paper is reproduced in Helgason's book [5].) This was in fact a variation on a theme of Funk [2], who in 1916 showed how to recover a function on the sphere from its integrals over all great circles. But mathematical terminology has been unkind to Funk, and the name *generalized Radon transform* has been applied to any operator that associates to a function on some geometric space its integrals over some class of geometric objects. The generalized Radon inversion problem is then to recover the function—if possible—from its generalized Radon transform. Perhaps the most direct generalization of Radon's original problem is to consider functions defined on an n-dimensional space and integrals over all k-dimensional affine subspaces, the so-called *k-plane transform*. Special cases of this are $k = n - 1$, called the *Radon transform*, and $k = 1$, called the *X-ray transform* because the exposure at a point on an X-ray picture gives a rough measurement of the integral of the function representing the density of the object being X-rayed along a line. If the X-rays form a parallel beam perpendicular to the photographic plate, then a single picture represents the X-ray transform evaluated on a family of parallel lines. In current practice most X-ray machines produce divergent (or fan) beams, so a single picture represents the X-ray transform evaluated on a family of lines passing through a fixed point. See [4] for a discussion of the mathematical problems that this entails. In either mode, since only a finite number of pictures are taken, the entire X-ray transform is not given. Thus a theoretical inversion of the X-ray transform is not equivalent to the problem of reconstructing an object from a finite set of X-rays. These issues are discussed in detail in the survey articles [9] and [10]. Other examples of generalized Radon transforms are discussed in Helgason's book [5], and in numerous research articles, many of which can be found in the bibliography of [5]. Related mathematical problems are discussed in Zalcman [12].

Generalized Radon transforms have been studied because they have practical applications, because they are of fundamental theoretical significance, or sometimes just because they lead to elegant mathematical problems. It is this last aspect that I want to emphasize in this article, which concludes with a collection of problems and solutions involving the inversion of various generalized Radon transforms. I hope the problems are attractive enough to tempt the reader to take pen in hand and attempt a solution. The problems can be solved by fairly elementary means, the solutions are of a conceptual rather than a technical nature, and in my opinion they are a lot of fun!

Before presenting the problems, I will discuss some solutions to Radon's original problem, since these will reveal some strategies that might prove useful in the sequel. I will also give an especially simple proof of the hole theorem and discuss a class of examples.

Autobiographical sketch of the author prepared at the request of the editors.

I graduated from the Bronx High School of Science in 1960, received a B. A. from Dartmouth College in 1963 and a Ph.D. in mathematics from Princeton University (under E. M. Stein) in 1966. I spent a year in France as a NATO Postdoctoral Fellow, then went to M.I.T. as a C. L. E. Moore Instructor, and in 1969 came to Cornell where I am now Professor of Mathematics. My major area of interest is harmonic analysis and its applications to many areas of mathematics. I have been fortunate to have many fine teachers, including Henrietta Mazen, Richard Williamson, Leon Henkin, Mischa Cotlar, A. Besicovitch, Eli Stein, S. Bochner, Harry Furstenberg and Irving Segal, from whom I learned not just the stuff of mathematics, but something of its spirit. What I love most about mathematics is the joy of discovery, when understanding overcomes confusion. What I like least about mathematics is the way clear and simple ideas tend to become muddy and murky and mystifying when committed to the printed page.

2. Radon Inversion. We begin by choosing a convenient parametric representation of all lines in the plane. Actually, if we want a one-to-one correspondence between parameters and lines, then there is no convenient representation. (This is essentially a consequence of the fact that the lines in the plane have a natural structure of a nontrivial vector bundle over the circle.) However, it is convenient enough to consider each line as given by the equation $x \cdot w = s$, where $w = (w_1, w_2)$ is a unit vector in the plane and s a real number. Using the pair (w, s) as parameters for the line gives a two-to-one correspondence, since the parameters $(-w, -s)$ correspond to the same line. Note that w is just a unit normal to the line ($w = (-\sin\theta, \cos\theta)$ if the line makes an angle θ with the x-axis) and $|s|$ is the distance between the line and the origin. The line $x \cdot w = s$ has a natural parametric representation (with parameter $t \in \mathbb{R}$)

$$x_1(t) = sw_1 + tw_2, \quad x_2(t) = sw_2 - tw_1,$$

and we can write the Radon transform $Rf(w, s) = \int_{x \cdot w = s} f = \int_{-\infty}^{\infty} f(x(t))\, dt$ for suitable functions f defined on the plane.

The quickest way to invert the Radon transform is to note the connection with the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi}\, dx$. If we write ξ in polar coordinates $\xi = |\xi| w$, so $x \cdot \xi = |\xi| x \cdot w$, then $e^{-ix\cdot\xi}$ will be constant on all the lines $x \cdot w = s$. Thus in performing the x-integration in the Fourier transform we want to integrate first along these lines, giving $e^{-is|\xi|} Rf(w, s)$ and then in the perpendicular direction, $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-is|\xi|} Rf(w, s)\, ds$. To paraphrase: *the two-dimensional Fourier transform is a one-dimensional Fourier transform of the Radon transform*. The two-dimensional Fourier inversion formula $f(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi$ then gives a Radon inversion formula,

$$f(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \left(\int_{-\infty}^{\infty} e^{-is|\xi|} Rf\left(\frac{\xi}{|\xi|}, s \right) \right) e^{ix\cdot\xi}\, d\xi. \tag{2.1}$$

By expressing the ξ-integration in polar coordinates this simplifies slightly to

$$f(x) = \int_0^{\infty} \int_{S^1} \left(\int_{-\infty}^{\infty} e^{ir(x\cdot w - s)} Rf(s, w)\, ds \right) r\, dw\, dr. \tag{2.2}$$

Of course we must place hypotheses on f so that the Fourier inversion formula is valid in order to obtain this result.

This first inversion formula is not particularly illuminating since it involves a multiple integral that is not absolutely convergent. (The oscillations of the complex exponential give the s-integral sufficiently rapid decay in r to make the r-integral converge.) We might be tempted to tinker with it further, formally interchanging the orders of integration and trying to make sense out of the divergent expression that results. However, this approach leads to technicalities that can be avoided by another, more direct method. This second method involves looking at the dual Radon transform: Instead of integrating functions defined on points over all points lying on a fixed line, we integrate functions defined on lines over all lines containing a fixed point. Thus let $g(w, s)$ be a suitable function of lines. (Take g to be even; $g(-w, -s) = g(w, s)$, so it is truly defined on the lines $x \cdot w = s$ and not merely on the parameters (w, s) representing the lines.) For a fixed point z, all the lines passing through z have the form $x \cdot w = s$ where $s = z \cdot w$, so that the parameters $(w, z \cdot w)$ describe all these lines as w varies over the circle. Since the circle naturally parametrizes the set of lines we want to integrate over, there is no doubt that we want to define

$$R^*g(z) = \int_{S^1} g(w, z \cdot w)\, dw$$

(or more explicitly

$$R^*g(x, y) = \int_0^{2\pi} g\left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, x\cos\theta + y\sin\theta \right) d\theta)$$

for the dual Radon transform. The terminology is well justified because R^* is the adjoint operator of R, in the sense that

$$\int_{-\infty}^{\infty}\int_{S^1} Rf(w,s)g(w,s)\,dw\,ds = \int_{R^2} f(x)R^*g(x)\,dx$$

as can easily be verified by substituting the definition of R^*g in the right side and performing the x-integration first along the lines $x \cdot w = s$. We will not make use of this fact here; rather we want to emphasize that the composition operator R^*Rf makes sense from a geometric point of view as a first step in inverting R. The reason for this is that the value of f at a fixed point z only influences Rf along those lines passing through z, so the integral of Rf over these lines, $R^*Rf(z)$, is the simplest object we can construct from Rf that might have a connection with $f(z)$.

With this as motivation, let's compute what $R^*Rf(z)$ actually is. First let $z = (0,0)$. The lines through the origin are just $x \cdot w = 0$, and if we let $w = (-\sin\theta, \cos\theta)$, then the line is given parametrically as $x(t) = t(\cos\theta, \sin\theta)$. Thus

$$\begin{aligned} R^*Rf(0) &= \int_0^{2\pi} Rf((-\sin\theta,\cos\theta),0)\,d\theta \\ &= \int_0^{2\pi}\int_{-\infty}^{\infty} f(t\cos\theta, t\sin\theta)\,dt\,d\theta \\ &= 2\int_{\mathbb{R}^2} f(x)\,|x|^{-1}\,dx. \end{aligned}$$

More generally, the lines through an arbitrary point z are best parametrized $x(t) = z + tw$ (this is a slightly different parametric representation than we gave above where there was no distinguished point on the line, the two differing by an additive change of variable t), so an almost identical computation yields

$$R^*Rf(z) = 2\int f(z+x)\,|x|^{-1}\,dx. \tag{2.3}$$

Note that this bears out well what our motivating reasoning led us to expect since the value of f at z is most emphasized in this integral, being multiplied by the singularity of $|x|^{-1}$. This singularity is relatively mild, so there is no difficulty with the convergence of the integral near $x = 0$ if f is bounded.

Now the right side of (2.3) is a familiar object. Aside from the constant, it is the convolution of f with the function $|x|^{-1}$. The convolutions with negative powers of $|x|$ are called Riesz transforms, and they are known to be essentially the negative powers of the Laplacian. More precisely, for functions f of $\mathbb{R}^n$, the Riesz transforms are defined

$$I_\alpha f(x) = \gamma_\alpha \int_{\mathbb{R}^n} f(x-y)\,|y|^{\alpha-n}\,dy$$

for $0 < \alpha < n$ with

$$\gamma_\alpha = 2^{-\alpha}\pi^{-n/2}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

The formal identity $I_\alpha f = (-\Delta)^{-\alpha/2} f$ is best understood via the Fourier transform formula

$$(I_\alpha f)\hat{}\,(\xi) = |\xi|^{-\alpha}\hat{f}(\xi) \tag{2.4}$$

as compared with

$$(-\Delta f)\hat{}\,(\xi) = |\xi|^{2}\hat{f}(\xi) \tag{2.5}$$

where Δ is the Laplacian $\Sigma_{j=1}^n \partial^2/\partial x_j^2$. The proof of (2.4), aside from the exact determination of the constant γ_α, can be deduced from formal properties of the Fourier transform: any convolution operator must multiply the Fourier transforms, the Fourier transform of $|x|^{\alpha-n}$ must be radial and homogeneous of degree $-\alpha$, and therefore a multiple of $|\xi|^{-\alpha}$. For the determination of the constant, and a precise formulation of the above arguments, we refer the reader to [**3**] or [**11**].

In terms of the above notation, formula (2.3) says $R^*Rf = 4\pi I_1 f$ on $\mathbb{R}^2$. Referring to (2.4) and (2.5) we can invert I_1 by $I_1(-\Delta)$, so

$$f = \frac{1}{4\pi} I_1(-\Delta) R^*Rf. \tag{2.6}$$

This is our second Radon inversion formula.

3. The Hole Theorem. One aspect of the inversion of the Radon transform that is not apparent from either (2.1), (2.6), or any of the variants of those explicit formulas, is the "hole" theorem. Suppose the function f is defined on the complement of some bounded convex "hole" K. Then Rf is defined for all lines that do not intersect K. Can the values of f outside the hole be recovered from the values of Rf on all lines outside the hole? The hole theorem gives a qualified assent. The qualification is that we must place strong restrictions on the decay of the function at infinity, say that it has compact support. (It is sufficient to assume that f is rapidly decreasing, but the proof we give below does not show this.) In fact the following simple counterexamples show that such restrictions are necessary. Let $f(x_1, x_2) = (x_1 + ix_2)^{-k}$ for any integer k sufficiently large. Then f is defined and continuous in the complement of any neighborhood of the origin, and it can be made to decay at infinity faster than any fixed polynomial rate, although it is not rapidly decreasing. But by the Cauchy integral formula and a simple limiting argument (closing the contour to exclude the origin) the integral of f over any line not passing through the origin is zero. Therefore Rf does not determine f.

In view of the counterexamples, the hole theorem is surprising. In terms of X-rays it can be interpreted as saying that if a convex portion of the object being examined cannot be X-rayed, we can still obtain all the information we need about the rest of the body from X-ray pictures that avoid that portion. (See [**9**] and [**10**] for a full discussion of this interpretation.) The object being X-rayed is assumed to be finite, so the hypothesis of compact support is verified.

To prove the hole theorem note that by considering differences it is equivalent to the following

THEOREM. *Let f be a continuous function of compact support. Suppose Rf is zero on every line not intersecting a fixed compact convex set K. Then f is zero outside K.*

We will give a proof that is a simplification of Helgason's proof [**5**]. (The theorem is sometimes attributed to Ludwig [**7**], who gave an independent proof, and it also follows from an inversion formula of Cormack [**1**].) The idea of the proof is to show by induction that any polynomial times f satisfies the same condition. For then we can apply the Weierstrass approximation theorem to every line not intersecting K to conclude that f is zero on that line, hence f is zero outside K. To simplify notation let us assume the line is vertical, $x_1 = s$ fixed, $s > 0$ and that K lies to the left of the line. Again for simplicity we will show that for any polynomial p of degree one, the integral of pf along the line is zero, for then this argument can be used for the induction step.

Actually it suffices to handle the polynomial $p(x_1, x_2) = x_2$, since $a + bx_1$ is constant along the line $x_1 = s$. In other words

$$\int_{-\infty}^{\infty} tf(s,t)\,dt = 0. \tag{3.1}$$

The gist of the proof is that (3.1) can be established by "rocking the boat." We look at lines tangent to the circle of radius s about the origin. These lines are parametrized by

$$x(t) = (s\cos\theta + t\sin\theta, -s\sin\theta + t\cos\theta).$$

For $\theta = 0$ this is the vertical line $x_1 = s$. Because K is compact and convex the lines $x(t)$ do not intersect K for θ sufficiently close to zero. Thus we know that the integral of f over these lines is equal to zero:

$$\int_{-\infty}^{\infty} f(s\cos\theta + t\sin\theta, -s\sin\theta + t\cos\theta)\, dt = 0.$$

Now we "rock the boat" by differentiating this identity with respect to θ and then setting $\theta = 0$. The result is

$$\int_{-\infty}^{\infty} \left(t\frac{\partial f}{\partial s}(s,t) - s\frac{\partial f}{\partial t}(s,t) \right) dt = 0.$$

However $\int_{-\infty}^{\infty} (\partial f)/(\partial t)(s,t)\, dt = 0$ by the fundamental theorem of the calculus, since f has compact support, so

$$\int_{-\infty}^{\infty} t\frac{\partial f}{\partial s}(s,t)\, dt = 0. \tag{3.2}$$

At first glance it appears that we have failed to obtain (3.1) because of the s-derivative in (3.2). However there is a magic trick for making the derivative disappear. Recall that the convex set K lies to the left of the line $x_1 = s$. This means that the same argument can be applied to any line $x_1 = r$ for $r \geqslant s$. Integrating (3.2) over all such lines we obtain

$$\int_{s}^{\infty}\int_{-\infty}^{\infty} t\frac{\partial f}{\partial r}(r,t)\, dt\, dr = 0.$$

Finally we interchange the order of integration, and apply the fundamental theorem of calculus to obtain (3.1), using the compact support of f to justify both steps.

The above argument used the differentiability of f, but a routine regularization and limiting argument allows us to dispense with this extra assumption. Also the argument generalizes easily to higher dimensions.

The hole theorem can be thought of as a Radon inversion theorem for functions defined on the complement of the hole. It is only a conditional inversion because of the support assumption.

This is analogous to the fact that a function of compact support on the line can be recovered from its derivative by the fundamental theorem $f(x) = \int_{-\infty}^{x} f'(t)\, dt$, but for f constant this is no longer valid. In some of the problems below, the inversion formulas will also be conditional; in others they will be valid whenever they make sense.

4. Variations. We have seen above that the strategy of looking at R^*R leads to an explicit inversion of the Radon transform. There are many other situations where this strategy is useful, and it is instructive to look at a whole class of examples. Consider a fixed "shape" S_0 in the plane, and then all congruent shapes, the images of S_0 under rigid motions. Assuming that there is some natural way of integrating functions over S_0 (technically, we want a measure supported on S_0, but in the examples it will always be obvious what to do), the same integration can be performed over the congruent images. If we let $\mathbb{S}$ denote the class of congruent images, and S an arbitrary element of $\mathbb{S}$, then

$$Rf(S) = \int_S f$$

is the associated Radon transform. If $\mathbb{S}$ is the class of straight lines, then we have the ordinary Radon transform. But we could choose for $\mathbb{S}$ the class of circles of fixed radius, squares of fixed size (or their boundaries), rectangles, triangles, etc. There is no need to consider only the plane, since the notion of congruence and rigid motion makes sense in higher dimensions. But for simplicity let us restrict attention to the plane and allow only orientation preserving rigid motions. These are of the form

$$x \to a(\theta)(x+b) \quad \text{where } a(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is a rotation through angle θ, and b is an arbitrary plane vector. We can parametrize the set $\mathbb{S}$ of congruent images of S_0 by $a(\theta)$ and b, by setting $S(a(\theta), b) = \{x : a(\theta)(x+b) \in S_0\}$. This is not a canonical labeling since there is no distinguished S_0, and it may not be one-to-one if S_0 has any symmetries; but no matter. We write

$$\begin{aligned} Rf(a(\theta), b) &= \int_{S(a(\theta), b)} f \\ &= \int_{S_0} f(a(-\theta)y - b)\, d\mu(y) \\ &= \int_{S_0} f(y-b)\, d\mu(a(\theta)y) \end{aligned}$$

where we let $d\mu(y)$ stand for the measure of integration on S_0. Writing $d\mu_\theta(y) = d\mu(a(\theta)y)$ for the rotated measure, we can interpret this expression in a significant way: *for each fixed* θ, $Rf(a(\theta), b)$ *is the convolution of* $f(-x)$ *with* $d\mu_\theta$ *at the point* b.

The problem of when this Radon transform is invertible is known as the Pompeiu problem (see Zalcman [**12**] and the references there). If no growth conditions are put on the function f, then the problem becomes quite technical. Therefore we will assume that f is bounded by a polynomial. With this assumption, the Fourier transform is the natural tool to illuminate the problem. Indeed since the Fourier transform of a convolution is the product of Fourier transforms, the Fourier transform in the b-variable of $Rf(a(\theta), b)$ is

$$\begin{aligned} \int_{\mathbb{R}^2} Rf(a(\theta), b) e^{-ib\cdot\xi}\, db &= \hat{f}(-\xi)\hat{d\mu}_\theta(\xi) \\ &= \hat{f}(-\xi)\, \hat{d\mu}(a(-\theta)\xi). \end{aligned}$$

Here we have used the fact that the Fourier transform of the rotation of $d\mu$ is the rotation (through the negative angle) of the Fourier transform of $d\mu$. For fixed ξ the points of $a(-\theta)\xi$ describe a circle about the origin, so if $\hat{d\mu}(\xi)$ does not vanish identically on any circle about the origin, we can recover $\hat{f}$ from Rf and hence f by the Fourier inversion formula. (It never vanishes at the origin since $\hat{d\mu}(0) = \int_{S_0} d\mu > 0$.) Normally we would not expect a random function on the plane to vanish on a circle unless there is some good reason for it to do so— for instance if it is radial. An example of this is produced by taking S_0 to be the unit circle. Then $\hat{d\mu}$ is known exactly (see [**3**] or [**11**]),

$$\begin{aligned} \hat{d\mu}(\xi) &= \int_0^{2\pi} e^{-i(\xi_1\cos\theta + \xi_2\sin\theta)}\, d\theta \\ &= J_0(|\xi|) \end{aligned}$$

where J_0 is the Bessel function of order zero. Since it is known that the Bessel function has isolated zeroes, the Radon transform is not invertible. (In fact, $J_0(\lambda^{-1}|x|)$, where λ is any such zero, is a nonzero function whose integrals over all circles of radius one are zero.) Of course in this case we can still hope for a conditional inversion since Rf determines $\hat{f}$ except on isolated circles, so if we assume say $f \in L^1$ or L^2, f is determined. (Of course neither condition is at all natural for the problem.) Such a conditional inversion problem is discussed in John [**6**], but we shall not consider it further here. Let us observe, however, that no such problem arises if we let $\mathbb{S}$ be the class of *circles* of radius one in three-space. The Fourier transform of the circle measure is constant in the direction perpendicular to the circle, hence it vanishes on isolated cylinders, and it is an obvious geometric fact that no sphere is contained in a countable union of cylinders!

Next let's look at what the dual transform does for us in this context. Obviously we want to define R^* by integrating over all sets $S \in \mathbb{S}$ which contain a fixed point x. If we fix the angular variable θ, then $x \in S(a(\theta), b)$ if and only if $b \in S(a(\theta), x)$; in other words the set of all values of the b parameters for which $S(a(\theta), b)$ contains x is the set $S(a(\theta), x)$, so it is natural to define

$$R^*g(x) = \int_0^{2\pi}\left(\int_{S(a(\theta),x)} g(a(\theta), b)\right) d\theta$$
$$= \int_0^{2\pi}\int_{S_0} g(a(\theta), y-x)\, d\mu(a(\theta)y)\, d\theta.$$

Then we compute

$$R^*Rf(x) = \int_0^{2\pi}\int_{S_0} Rf(a(\theta), y-x)\, d\mu(a(\theta)y)\, d\theta$$
$$= \int_0^{2\pi}\int_{S_0}\int_{S_0} f(z+x-y)\, d\mu(a(\theta)z)\, d\mu(a(\theta)y)\, d\theta.$$

If we ignore the θ integration, the expression we have is the triple convolution $f * d\mu_\theta * d\mu_\theta(-x)$ which goes under Fourier transformation into the triple product $\hat{f}(\xi)\,\hat{d\mu_\theta}(\xi)\,\widehat{d\mu_\theta(-\xi)}$. But since $d\mu_\theta$ is real-valued $\widehat{d\mu_\theta(-\xi)} = \overline{\hat{d\mu_\theta}(\xi)}$, so

$$\int R^*Rf(x)e^{-ix\cdot\xi}\,dx = \int_0^{2\pi}\hat{f}(\xi)\,|\hat{d\mu}(a(-\theta)\xi)|^2\,\hat{d}\theta$$
$$= \hat{f}(\xi)\int_0^{2\pi}|\hat{d\mu}(a(\theta)\xi)|^2\,d\theta.$$

Now observe that $m(\xi) = \int_0^{2\pi}|\hat{d\mu}(a(\theta)\xi)|^2\,d\theta$ is nonvanishing if and only if $\hat{d\mu}(\xi)$ does not vanish identically on any circle about the origin. Thus we lose no information in considering R^*Rf instead of Rf; if we can express f in terms of Rf, then we can express it in terms of R^*Rf, and the expression will be simpler. In fact, by the Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^2}\int m(\xi)^{-1}\int R^*Rf(y)e^{-iy\cdot\xi}\,dy e^{ix\cdot\xi}\,d\xi$$

and $m(\xi)$ is a radial function. In some cases it may be possible to further simplify this expression, but in the examples in problem **E** I have not seen any way to do this.

5. Problems.

A. Consider any *finite plane geometry*, that is, a finite set X of points and a set Y of lines, each line $y \in Y$ being a subset of X subject to the single axiom "two points determine a unique line." In other words, for any two points $x_1, x_2 \in X$, there exists a unique $y \in Y$ such that $x_1 \in y$ and $x_2 \in y$. For a function f on X, define the Radon transform $Rf(y) = \Sigma_{x\in y} f(x)$. Can you find an explicit inversion formula for f in terms of Rf? Assume that there are at least two lines.

There is a natural generalization of this problem to *finite n-dimensional geometries*. Such a geometry is a finite set X_0 of points, and sets X_k of k-planes, $k = 1,\ldots,n-1$, such that each k-plane $x_k \in X_k$ is a subset of X_0. We need two axioms: (1) any $k+1$ points belong to a unique k-plane; (2) if $k+1$ points in a k-plane x_k belong to a single m-plane x_m for $m > k$, then all points of x_k belong to x_m. The k-plane transform for fixed k is then defined as $R_k f(x_k) = \Sigma_{x\in x_k} f(x)$. Can you invert this transform? Assume that there are at least two k-planes for each k.

B. Let L denote the lattice of integer points in the plane, points (k, m) where k and m are integers. Let f be any absolutely summable function on L; in other words

$$\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|f(k,m)|$$

is finite. Let Y denote the set of lines in the plane which intersect the lattice in more than one point. Then it makes sense to define the Radon transform

$$Rf(y) = \sum_{(k,m)\in y} f(k,m),$$

the series being absolutely convergent. Can you invert this transform?

Whatever solution you find, you will have to use the hypothesis that f is absolutely summable, because there exist nonzero functions f such that the series for Rf is absolutely convergent for each line and $Rf \equiv 0$. Can you find such a function?

C. Let N denote the positive integers. For an absolutely summable function on N, define a Radon transform

$$Rf(m) = \sum_{k=1}^{\infty} f(km)$$

for all $m \in N$. Can you invert this transform, assuming some more rapid decay of f, say $|f(n)| \leqslant cn^{-2-\varepsilon}$?

D. Let T^2 denote the torus, which is the plane modulo the lattice L of problem **B**. Some lines in the plane, when projected on the torus, wrap around infinitely often, and others come back upon themselves after a finite length. These can be distinguished according as the slope is irrational or rational. Using the lines of finite length, we can define the Radon transform of a continuous function f on T^2 to be the mean value of f on each such line. (It is actually possible to define the mean value on the lines of infinite length using the Bohr mean from the theory of almost periodic functions, but the mean will always be the usual mean value of f on T^2, and this information is available from the mean value of f on say all horizontal lines.) Can you invert this transform? Part of the problem is to choose appropriate representations for the finite lines on T^2. You may assume that f has an absolutely convergent double Fourier series.

E. Let $\mathbb{S}$ be the set of squares (interiors) with side length one in the plane, and $Rf(S) = \iint_S f(x_1, x_2)\, dx_1\, dx_2$. Can you prove that R is invertible? Suppose instead of the interiors of the squares we consider the boundaries of the squares, with $Rf(S)$ equal to the line integral? Finally, what if we consider the vertices of the square?

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The solutions of the problems occur on page 420.