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Lecture 20 Notes

These notes correspond to Section 6.4 in the text.

Properties of Sturm-Liouville Eigenfunctions and Eigenvalues

We continue our study of the Sturm-Liouville eigenvalue problem

$$L[v] = -[(p(x)u')' + q(x)u] = \lambda r(x)u, \quad a < x < b, \tag{1}$$

with boundary conditions

$$B_a[v] = B_b[v] = 0. \tag{2}$$

In particular, we state and prove several useful properties of their solutions, which are eigenfunctions, and their corresponding values of λ, which are eigenvalues.

Proposition 1 *Let u_λ and v_λ be linearly independent solutions of (1) for the same value of λ. Then λ is an eigenvalue of the Sturm-Liouville problem (1), (2) if and only if*

$$\det \begin{bmatrix} B_a[u_\lambda] & B_a[v_\lambda] \\ B_b[u_\lambda] & B_b[v_\lambda] \end{bmatrix} = 0. \tag{3}$$

Proof: By assumption, the general solution of (1) is

$$w(x) = cu_\lambda(x) + dv_\lambda(x).$$

Let w be a nontrivial solution of (1). Then w is an eigenfunction of (1), (2) with eigenvalue λ if and only if

$$\begin{bmatrix} B_a[w] \\ B_b[w] \end{bmatrix} = \begin{bmatrix} cB_a[u_\lambda] + dB_a[v_\lambda] \\ cB_b[u_\lambda] + dB_b[v_\lambda] \end{bmatrix} = \begin{bmatrix} B_a[u_\lambda] & B_a[v_\lambda] \\ B_b[u_\lambda] & B_b[v_\lambda] \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system of linear equations has a nontrivial solution if and only if (3) is satisfied. □

Example 1 Consider the Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < L,$$

$$v(0) = v'(L) = 0.$$

The general solution of this ODE is

$$v(x) = c\cos(\sqrt{\lambda}x) + d\sin(\sqrt{\lambda}x).$$

Therefore λ is an eigenvalue of this Sturm-Liouville problem if and only if

$$\det \begin{bmatrix} \cos 0 & \sin 0 \\ -\sqrt{\lambda}\sin(\sqrt{\lambda}L) & \sqrt{\lambda}\cos(\sqrt{\lambda}L) \end{bmatrix} = \sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0.$$

It follows that

$$\sqrt{\lambda}L = \frac{2n-1}{2}\pi, \quad n = 1, 2, \ldots,$$

which yields the eigenvalues

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad n = 1, 2, \ldots.$$

□

Symmetry

Let L be a regular Sturm-Liouville operator as defined in (1) and let u, v satisfy the boundary conditions (2). We then have

$$\begin{aligned} uL[v] - vL[u] &= u[-(p(x)v')' - q(x)v] - v[-(p(x)u')' - q(x)u] \\ &= -u(p(x)v')' - q(x)uv + v(p(x)u')' + q(x)vu \\ &= v(p(x)u')' - u(p(x)v')' \end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned} \int_a^b uL[v] - vL[u]\,dx &= \int_a^b v(p(x)u')' - u(p(x)v')'\,dx \\ &= p(x)(vu' - uv')\Big|_a^b - \int_a^b v'p(x)u' - u'p(x)v'\,dx \\ &= p(x)(vu' - uv')\Big|_a^b, \end{aligned}$$

which is known as *Green's identity*.

From the fact that u and v satisfy the boundary conditions,

$$\alpha u(a) + \beta u'(a) = 0, \quad \alpha v(a) + \beta v'(a) = 0,$$

$$\gamma u(b) + \delta u'(b) = 0, \quad \gamma v(b) + \delta v'(b) = 0.$$

Multiplying the boundary conditions for u by v and vice versa, we obtain

$$\alpha[v(a)u(a) - u(a)v(a)] + \beta[v(a)u'(a) - u(a)v'(a)] = \beta[v(a)u'(a) - u(a)v'(a)] = 0,$$

$$\gamma[v(b)u(b) - u(b)v(b)] + \delta[v(b)u'(b) - u(b)v'(b)] = \delta[v(b)u'(b) - u(b)v'(b)] = 0,$$

and therefore

$$\int_a^b uL[v] - vL[u]\,dx = p(b)[v(b)u'(b) - u(b)v'(b)] - p(a)[v(a)u'(a) - u(a)v'(a)] = 0.$$

That is,

$$\langle u, L[v]\rangle = \langle L[u], v\rangle.$$

It follows that L is its own *adjoint*; we say that L is *self-adjoint* or *symmetric*; this is analogous to a square matrix being symmetric. It can be shown that a Sturm-Liouville operator is also self-adjoint in the case of periodic boundary conditions.

Real Eigenvalues

Just as a symmetric matrix has real eigenvalues, so does a (self-adjoint) Sturm-Liouville operator.

> **Proposition 2** *The eigenvalues of a regular or periodic Sturm-Liouville problem are real.*

Proof: As before, we consider the case of a regular Sturm-Liouville problem; the periodic case is similar. Let v be an eigenfunction of the problem (1), (2) with eigenvalue λ. Then

$$\langle v, L[v] \rangle = \langle v, \lambda v \rangle = \lambda \|v\|^2.$$

Similarly,

$$\langle L[v], v \rangle = \overline{\lambda} \|v\|^2.$$

However, by the symmetry of L, $\langle v, L[v] \rangle = \langle L[v], v \rangle$, which means $\lambda = \overline{\lambda}$. We conclude that λ is real. □

Orthogonality

Just as a symmetric matrix has orthogonal eigenvectors, a (self-adjoint) Sturm-Liouville operator has orthogonal eigenfunctions.

> **Proposition 3** *Let v_1 and v_2 be eigenfunctions of a regular Sturm-Liouville operator (1) with boundary conditions (2) corresponding to* distinct *eigenvalues λ_1, λ_2, respectively. Then v_1 and v_2 are orthogonal with respect to the weight function $r(x)$; that is,*
>
> $$\langle v_1, v_2 \rangle_r = \int_a^b \overline{v_1(x)} v_2(x) r(x)\, dx = 0.$$
>
> *This property also holds with periodic boundary conditions.*

Proof: We consider the case of a regular Sturm-Liouville problem; the periodic case is similar. From the relations

$$L[v_1] = \lambda_1 r v_1, \quad L[v_2] = \lambda_2 r v_2,$$

and the symmetry of L, we obtain

$$\begin{aligned} 0 &= \langle v_1, L[v_2] \rangle - \langle L[v_1], v_2 \rangle \\ &= \langle v_1, \lambda_2 r v_2 \rangle - \langle \lambda_1 r v_1, v_2 \rangle \\ &= (\lambda_2 - \lambda_1) \langle v_1, v_2 \rangle_r, \end{aligned}$$

where we have used the fact that the eigenvalues are real. Because $\lambda_2 \neq \lambda_1$, we must have $\langle v_1, v_2 \rangle_r = 0$. □

Example 2 The regular Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < L,$$

with Dirichlet boundary conditions

$$v(0) = v(L) = 0$$

has eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad v_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

It can be verified using product-to-sum identities that for $m, n = 1, 2, \ldots$,

$$\langle v_n, v_m \rangle = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases}.$$

We see that these eigenfunctions are orthogonal, and that the set

$$\left\{ \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \right\}_{n=1}^{\infty}$$

consists of orthonormal eigenfunctions. □

Example 3 The regular Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < L,$$

with Neumann boundary conditions

$$v'(0) = v'(L) = 0$$

has eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad v_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots.$$

It can be verified using product-to-sum identities that for $m, n = 0, 1, 2, \ldots$,

$$\langle v_n, v_m \rangle = \int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases}.$$

We see that these eigenfunctions are orthogonal, and that the set

$$\left\{ \sqrt{\frac{1}{L}} \right\} \cup \left\{ \sqrt{\frac{2}{L}} \cos\frac{n\pi x}{L} \right\}_{n=1}^{\infty}$$

consists of orthonormal eigenfunctions. □

Example 4 The regular Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < L,$$

with periodic boundary conditions

$$v(0) = v(L), \quad v'(0) = v'(L)$$

has eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{2n\pi}{L}\right)^2, \quad u_n(x) = \cos\frac{2n\pi x}{L}, \quad n = 0, 1, 2, \ldots,$$

$$v_n(x) = \sin \frac{2n\pi x}{L}, \quad n = 1, 2, \ldots.$$

It can be verified using product-to-sum identities that for $m, n = 0, 1, 2, \ldots$,

$$\langle u_n, u_m \rangle = \int_0^L \cos \frac{2n\pi x}{L} \cos \frac{2m\pi x}{L} \, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases},$$

and for $m, n = 1, 2, \ldots,$,

$$\langle v_n, v_m \rangle = \int_0^L \sin \frac{2n\pi x}{L} \sin \frac{2m\pi x}{L} \, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases},$$

and for $n = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$,

$$\langle u_n, v_m \rangle = \int_0^L \cos \frac{2n\pi x}{L} \sin \frac{2m\pi x}{L} \, dx = 0.$$

We see that these eigenfunctions are orthogonal, and that the set

$$\left\{ \sqrt{\frac{1}{L}} \right\} \cup \left\{ \sqrt{\frac{2}{L}} \cos \frac{2n\pi x}{L} \right\}_{n=1}^{\infty} \cup \left\{ \sqrt{\frac{2}{L}} \sin \frac{2n\pi x}{L} \right\}_{n=1}^{\infty}$$

consists of orthonormal eigenfunctions. □

Real Eigenfunctions

The eigenfunctions of a Sturm-Liouville problem can be chosen to be real.

> **Proposition 4** *Let λ be an eigenvalue of a regular or periodic Sturm-Liouville problem. Then the subspace spanned by the eigenfunctions corresponding to λ admits an orthonormal basis of real-valued functions.*

Proof: The result is trivially true if λ is a simple eigenvalue. If λ has multiplicity 2, which is the maximum possible since the Sturm-Liouville ODE is second-order, then λ has two linearly independent eigenfunctions

$$v_1 = a_1 + ib_1, \quad v_2 = a_2 + ib_2.$$

Because L has real coefficients, it can easily be shown that a_1, a_2, b_1, b_2 are all eigenfunctions of L corresponding to λ.

Suppose that from the set $\{a_1, a_2, b_1, b_2\}$, there are not two linearly independent functions. Then all four functions are scalar multiples of one another, but then it follows that v_1 is a scalar multiple of v_2, which contradicts the assumption that v_1 and v_2 are linearly independent. Thus two functions from $\{a_1, a_2, b_1, b_2\}$ are linearly independent, and by applying Gram-Schmidt orthogonalization to these two functions, two real-valued orthonormal eigenfunctions can be obtained. □

Simple Eigenvalues

The following property regarding the multiplicity of eigenvalues greatly simplifies their numerical computation.

Proposition 5 *The eigenvalues of a regular Sturm-Liouville problem are simple.*

Proof: Let v_1 and v_2 be eigenfunctions of the regular Sturm-Liouville problem (1), (2) with eigenvalue λ. Then we have

$$\begin{aligned} v_1L[v_2] - v_2L[v_1] &= -v_1[(p(x)v_2')' + q(x)v_2] + v_2[(p(x)v_1')' + q(x)v_1] \\ &= v_2(p(x)v_1')' - v_1(p(x)v_2')' \\ &= v_2p'(x)v_1' + v_2p(x)v_1'' - v_1p'(x)v_2' - v_1p(x)v_2'' \\ &= p(x)[v_2v_1'' - v_1v_2''] + p'(x)[v_2v_1' - v_1v_2'] \\ &= [p(x)(v_2v_1' - v_1v_2')]'. \end{aligned}$$

However, we also have

$$v_1L[v_2] - v_2L[v_1] = v_1\lambda v_2 - v_2\lambda v_1 = \lambda(v_1v_2 - v_2v_1) = 0.$$

It follows that $Q(x) = p(x)(v_2v_1' - v_1v_2')$ is a constant function. But because v_1 and v_2 both satisfy the boundary conditions, we have $Q(a) = Q(b) = 0$. Therefore, $Q(x) \equiv 0$ and

$$W(v_1, v_2) = v_1v_2' - v_2v_1' = 0.$$

We conclude that v_1 and v_2 are linearly dependent. □

Note that this result only applies to *regular* Sturm-Liouville problems; for periodic problems, recall that most eigenvalues have multiplicity 2.

Countably Infinite Eigenvalues

The following essential result characterizes the behavior of the entire set of eigenvalues of Sturm-Liouville problems.

Proposition 6 *The set of eigenvalues of a regular Sturm-Liouville problem is countably infinite, and is a monotonically increasing sequence*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$$

with $\lim_{n\to\infty} \lambda_n = \infty$*. The same is true for a periodic Sturm-Liouville problem, except that the sequence is monotonically nondecreasing.*

The difference in behavior of the eigenvalues between the regular and periodic problems is due to the fact that the eigenvalues of a regular problem are simple, whereas for the periodic case they can have multiplicity 2.

The following result follows from the preceding proposition, as well as earlier results pertaining to the eigenfunctions. Recall that $E_r(a, b)$ is the space of piecewise continuous functions on $[a, b]$ with inner product $\langle , \rangle_r$, where $r(x)$ is the weight function from (1).

Corollary 1 *A regular or periodic Sturm-Liouville problem admits an orthonormal sequence of real-valued eigenfunctions in* $E_r(a, b)$*. Furthermore, the sequence of eigenvalues is not bounded above, but is bounded below.*

Completeness

The eigenfunctions of a Sturm-Liouville problem can be used to describe piecewise continuous functions, which is very useful for solving time-dependent PDE for which separation of variables yields a Sturm-Liouville problem.

Proposition 7 *The orthonormal set* $\{v_n\}_{n=0}^{\infty}$ *of eigenfunctions of a regular or periodic Sturm-Liouville problem is a basis for* $E_r(a,b)$*; that is,* $E_r(a,b)$ *is complete.*

The expansion of a function $v \in E_r(a,b)$ in the orthonormal basis of eigenfunctions, given by

$$v = \sum_{n=0}^{\infty} a_n v_n, \quad a_n = \langle v_n, v\rangle_r = \int_a^b \overline{v_n(x)} v(x) r(x)\, dx,$$

is called an *eigenfunction expansion* of v.

The eigenfunction expansion has these essential properties.

Proposition 8 *Let* $\{v_n\}_{n=0}^{\infty}$ *be an orthonormal set of eigenfunctions of a regular or periodic Sturm-Liouville problem.*

1. *If* f *is continuous and piecewise differentiable on* $[a,b]$ *and satisfies the boundary conditions of the Sturm-Liouville problem, then the eigenfunction expansion of* f *converges uniformly to* f *on* $[a,b]$.

2. *If* f *is piecewise differentiable on* $[a,b]$*, then for* $x \in (a,b)$ *the eigenfunction expansion of* f *converges to* $[f(x_+) + f(x_-)]/2$*, where* $f(x_+)$ *and* $f(x_-)$ *are the left- and right-hand limits of* f *at* x.

Example 5 We compute the expansion of $f(x) = 1$ in the orthonormal basis $\{\sqrt{2/L}\sin(n\pi x/L)\}_{n=1}^{\infty}$, which are eigenfunctions of the Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < L, \quad v(0) = v(L) = 0.$$

We have

$$f(x) = 1 = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where

$$\begin{aligned}
b_n &= \left\langle \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}, 1 \right\rangle \\
&= \sqrt{\frac{2}{L}} \int_0^L \sin\frac{n\pi x}{L}\, dx \\
&= -\sqrt{\frac{2}{L}} \frac{L}{n\pi} \cos\frac{n\pi x}{L}\bigg|_0^L \\
&= -\frac{\sqrt{2L}}{n\pi}[(-1)^n - 1] \\
&= \frac{\sqrt{2L}}{n\pi}[1 - (-1)^n] \\
&= \begin{cases} 0 & n \text{ even} \\ \frac{2\sqrt{2L}}{n\pi} & n \text{ odd} \end{cases}.
\end{aligned}$$

This yields

$$f(x) = \sqrt{\frac{2}{L}} \sum_{k=1}^{\infty} \frac{2\sqrt{2L}}{(2k-1)\pi} \sin \frac{(2k-1)\pi x}{L} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin \frac{(2k-1)\pi x}{L}.$$

This series expansion converges to $f(x) = 1$ on $(0, L)$, but it does not converge uniformly on $[0, L]$, because the boundary conditions are not satisfied. Due to this compatibility between $f(x)$ and the boundary condition, truncated expansions exhibit oscillations at $x = 0$ and $x = L$ characteristic of Gibbs' phenomenon. □

Example 6 We compute the expansion of $f(x) = x$ in the orthonormal basis $\{\sqrt{1/L}\} \cup \{\sqrt{2/L}\cos(n\pi x/L)\}_{n=1}^{\infty}$, which are eigenvalues of the Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < L, \quad v'(0) = v'(L) = 0.$$

We have

$$f(x) = x = \sqrt{\frac{1}{L}} a_0 + \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where

$$a_0 = \left\langle \sqrt{\frac{1}{L}}, x \right\rangle = \sqrt{\frac{1}{L}} \int_0^L x \, dx = \sqrt{\frac{1}{L}} \frac{L^2}{2} = \frac{L^{3/2}}{2},$$

and, for $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \left\langle \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}, x \right\rangle \\
&= \sqrt{\frac{2}{L}} \int_0^L x \cos \frac{n\pi x}{L} \, dx \\
&= \sqrt{\frac{2}{L}} \left[\frac{L}{n\pi} x \sin \frac{n\pi x}{L} \Big|_0^L - \frac{L}{n\pi} \int_0^L \sin \frac{n\pi x}{L} \, dx \right] \\
&= \sqrt{\frac{2}{L}} \frac{L^2}{(n\pi)^2} \cos \frac{n\pi x}{L} \Big|_0^L \\
&= \sqrt{\frac{2}{L}} \frac{L^2}{(n\pi)^2} [(-1)^n - 1] \\
&= \begin{cases} 0 & n \text{ even} \\ -\frac{2L\sqrt{2L}}{(n\pi)^2} & n \text{ odd} \end{cases}.
\end{aligned}$$

We conclude that

$$f(x) = \sqrt{\frac{1}{L}} \frac{L^{3/2}}{2} - \sqrt{\frac{2}{L}} \sum_{k=1}^{\infty} \frac{2L\sqrt{2L}}{((2k-1)\pi)^2} \cos \frac{(2k-1)\pi x}{L} = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{L}.$$

This expansion converges uniformly to x on $[0, \pi]$, even though it does not satisfy the boundary conditions. Truncated expansions do not exhibit Gibbs' phenomenon. □

Rayleigh Quotients

We now develop a useful technique for estimating eigenvalues, which is very useful for numerical computation.

Definition 1 *The* **principal eigenvalue***, also known as the* **ground state energy***, of a Sturm-Liouville problem is the minimal eigenvalue λ_0. The* **principal eigenfunction** *is the eigenfunction corresponding to the principal eigenvalue.*

Definition 2 *Let L be the differential operator from (1). The expression*

$$R(u) = \frac{\langle u, L[u] \rangle}{\langle u, u \rangle_r} = \frac{\int_a^b u L[u]\, dx}{\int_a^b u^2 r\, dx}$$

is called the **Rayleigh quotient** *of u.*

Proposition 9 *The principal eigenvalue λ_0 of a regular Sturm-Liouville problem (1), (2) satisfies the* **variational principle***, known as the* **Rayleigh-Ritz formula:**

$$\lambda_0 = \inf_{u \in V, u \neq 0} R(u),$$

where V is the space of all twice continuously differentiable functions on $[a, b]$ that satisfy the boundary conditions (2).

Proof: Using the orthonormality and completeness of the eigenfunctions, as well as the monotonicity of the eigenvalues, we obtain

$$\begin{aligned}
R(u) &= \frac{\langle u, L[u] \rangle}{\langle u, u \rangle_r} \\
&= \frac{\left\langle \sum_{m=0}^{\infty} a_m v_m, L\left[\sum_{n=0}^{\infty} a_n v_n\right] \right\rangle}{\left\langle \sum_{m=0}^{\infty} a_m v_m, \sum_{n=0}^{\infty} a_n v_n \right\rangle_r} \\
&= \frac{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \overline{a_m} a_n \langle v_m, L[v_n] \rangle}{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \overline{a_m} a_n \langle v_m, v_n \rangle_r} \\
&= \frac{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \overline{a_m} a_n \langle v_m, \lambda_n r v_n \rangle}{\sum_{n=0}^{\infty} |a_n|^2} \\
&= \frac{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \overline{a_m} a_n \lambda_n \langle v_m, v_n \rangle_r}{\sum_{n=0}^{\infty} |a_n|^2} \\
&= \frac{\sum_{n=0}^{\infty} |a_n|^2 \lambda_n}{\sum_{n=0}^{\infty} |a_n|^2} \\
&\geq \frac{\sum_{n=0}^{\infty} |a_n|^2 \lambda_0}{\sum_{n=0}^{\infty} |a_n|^2} \\
&\geq \lambda_0 \frac{\sum_{n=0}^{\infty} |a_n|^2}{\sum_{n=0}^{\infty} |a_n|^2} \\
&\geq \lambda_0.
\end{aligned}$$

If we choose $u = v_0$, then $R(u) = \lambda_0$. This proves the result. It is important to note that we have used the fact that u is twice continuously differentiable to conclude that its eigenfunction expansion converges uniformly on $[a, b]$, which allows term-by-term integration and differentiation. $\square$

Using integration by parts, the Rayleigh-Ritz formula can be rewritten as follows:

$$\begin{aligned}\lambda_0 &= \inf_{u\in V, u\neq 0} R(u) \\ &= \inf_{u\in V, u\neq 0} \frac{\int_a^b -u(p(x)u')' - uq(x)u\,dx}{\int_a^b u^2 r\,dx} \\ &= \inf_{u\in V, u\neq 0} \frac{\int_a^b p(x)(u')^2 - q(x)u^2\,dx - puu'|_a^b}{\int_a^b u^2 r\,dx}.\end{aligned}$$

This leads to the following result.

Corollary 2 *If $q \leq 0$ and $puu'|_a^b \leq 0$ for $u \in V$, then the eigenvalues of the Sturm-Liouville problem are nonnegative. In particular, the eigenvalues are nonnegative for the Dirichlet, Neumann and periodic Sturm-Liouville problems.*

Example 7 Consider the Sturm-Liouville problem

$$v'' + \lambda v = 0, \quad 0 < x < 1, \quad v(0) = v(1) = 0.$$

The principal eigenfunction is $v_0(x) = \sin \pi x$, with corresponding eigenvalue $\lambda_0 = \pi^2$. We can estimate this eigenvalue using a *test function* $u(x) = x - x^2$, which, like $\sin \pi x$, has roots at $x = 0, 1$ and is concave down on $(0, 1)$. We have

$$R(u) = \frac{-\int_0^1 (x - x^2)(-2)\,dx}{\int_0^1 (x - x^2)^2\,dx} = 10 \geq \pi^2 \approx 9.87.$$

That is, the Rayleigh quotient yields a upper bound of the principal eigenvalue. $\square$

Zeros of Eigenfunctions

Proposition 10 *The nth eigenfunction v_n of a regular Sturm-Liouville problem has exactly n roots on the interval (a, b).*