# Using Technology to Support an Embodied Approach to Learning Concepts in Mathematics

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**David Tall**

Mathematics Education Research Centre
University of Warwick
CV4 7AL, UK
e-mail: david.tall@warwick.ac.uk

*In this presentation I shall explain what I mean by an 'embodied approach' to mathematics with particular applications to calculus. I shall contrast and compare it with two other modes: the 'proceptual' (manipulating symbols as process and concept) and the 'axiomatic' based on formal definitions and formal proof. Each of these has its own standard of 'truth'. I argue that the embodied mode, though it lacks mathematical proof when used alone, can provide a fundamental human basis for meaning in mathematics. I shall give examples of an embodied approach in mathematics using technology that makes explicit use of a visual and enactive interface.*

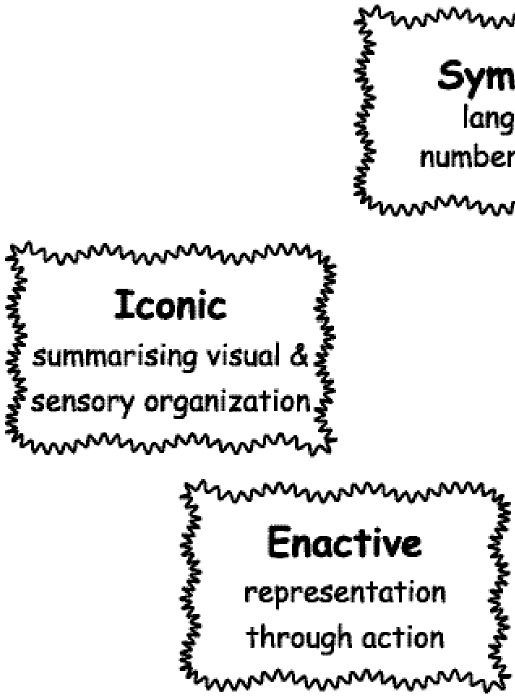
## Physical & mental tools and modes of representation

Man's use of mind is dependent upon his ability to develop and use "tools" or "instruments" or "technologies" that make it possible to express and amplify his powers. His very evolution as a species speaks to this point. It was consequent upon the development of bipedalism and the use of spontaneous pebble tools that man's brain and particularly his cortex developed. It was not a large-brained hominid that developed the technical-social life of the human; rather it was the tool-using, cooperative pattern that gradually changed man's morphology by favoring the survival of those who could link themselves with tool systems and disfavoring those who tried to do it on big jaws, heavy dentition, or superior weight. What evolved as a human nervous system was something, then, that required outside devices for expressing its potential.

(Bruner, *Education as Social Invention*, 1966, p. 25.)

What does it mean to translate experience into a model of the world. Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. […] There is a second system of representation that depends upon visual or other sensory organization and upon the use of summarizing images. […] We have come to talk about the first form of representation as **enactive**, the second is **iconic**. […] Finally, there is a representation in words or language. Its hallmark is that it is **symbolic** in nature.

(Bruner, 1966, pp. 10–11)




Bruner's three modes of representation

"… any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in recognizable form" (Bruner 1966. p. 44).

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Modern computer interfaces and Bruner's philosophy:

- Enactive interface,
- Icons as summarizing images to represent selectable options,
- Symbolism through keyboard input and internal processing.

‘Symbolism’ in mathematics requires further subcategories.

Bruner (1966, pp. 18, 19)

- “language in its natural form”
- the two “artificial languages of number and logic.”

To these categories we must add not just number, but algebraic and other functional symbolism (trigonometric, exponential and other functions in calculus) and the huge range of symbolism in axiomatic mathematics.

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The Reform movement in the calculus, for example the Harvard Calculus, focused on three representations: graphic, numeric and symbolic (or analytic):

> One of the guiding principles is the ‘Rule of Three,’ which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles. (Hughes Hallett 1991, p. 121)

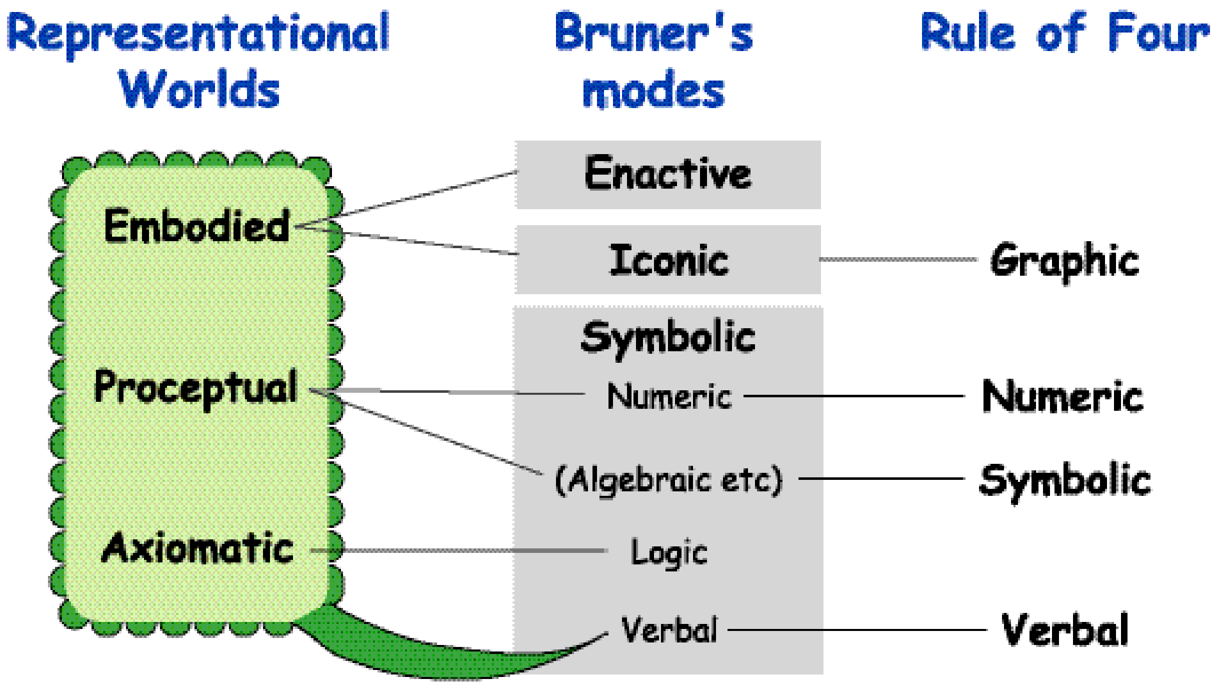
The ‘Rule of Three’ later became the ‘Rule of Four’, extending the representations to include the *verbal*.

Note:

- The enactive mode is completely omitted, presumably because it does not seem to be a central focus in the graphs and symbols of the calculus.
- The “verbal” mode was not seen as being important until late on in the development of the curriculum.
- Axiomatic formulations using logical deduction are not seen as part of calculus but of the later study of analysis.

My solution is to categorise representations into three different worlds of operation:

- **Embodied**: based on human perceptions and actions in a real-world context including but not limited to enactive and visual aspects.
- **Proceptual**: combining the role of symbols in arithmetic, algebra, symbolic calculus, based on the theory of these symbols acting dually as both process and concept (procept). (Tall *et al*, 2001, see below).
- **Axiomatic**: a formal approach starting from selected axioms and making logical deductions to prove theorems.




Three representational worlds

# Relationships with other theories

## Piaget

***sensori-motor / preconceptual / concrete operational / formal***

## SOLO taxonomy (Structure of Observed Learning Outcomes)

Biggs & Collis (1982)

***sensori-motor / ikonic / concrete-operational / formal / post-formal***

The SOLO taxonomy is intended to provide a template for assessment. Within each mode the assessment of concepts is performed according to how the student handles the particular concepts and whether this is:

- pre-structural (lacking knowledge of the assessed component)
- unistructural (focusing on a single aspect)
- multi-structural (focussing on several separate aspects)
- relational (relating different aspects together)
- extended abstract (seeing the concept from an overall viewpoint)

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*i) in each mode, increased sophistication occurs.*

*ii) As new modes develop, earlier modes may remain available.*

## Why the *Three* Worlds of Operation?

**(1) Embodied** : by taking enactive and visual/iconic *together* I focus on the physical senses of the individual that focuses on enaction, visual and spatial representations. It emphasises the role of the physical senses as a fundamental cognitive foundation of the calculus, almost absent from calculus reform movements.

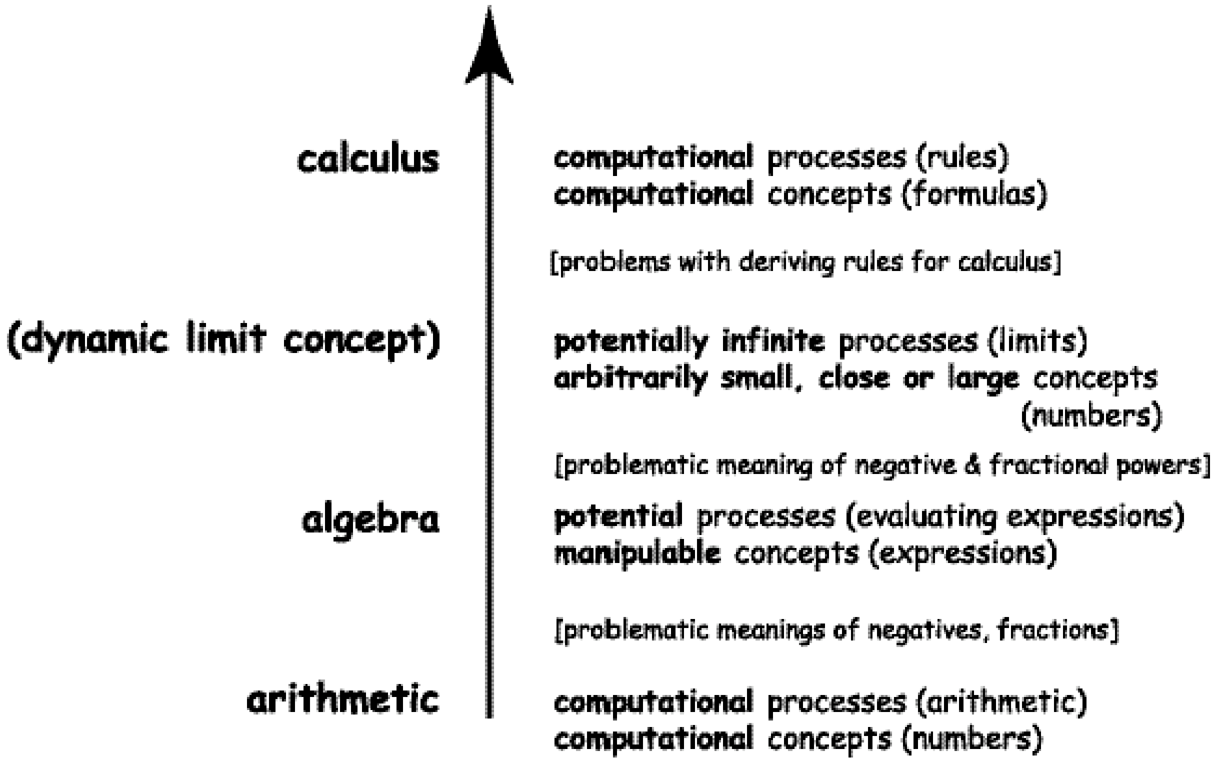
(2) **Proceptual** :

Why not use 'symbolic' ? … (many meanings).

Why not subdivide into numeric/algebraic or other?

(because the full development needs to be considered).

(… the subcategories can also be considered…)




Some different types of procept in mathematics

(3) **Axiomatic** :

Why *axiomatic* ? Why not just a *formal* presentation ?

(some 'formal' arguments are essentially proceptual)

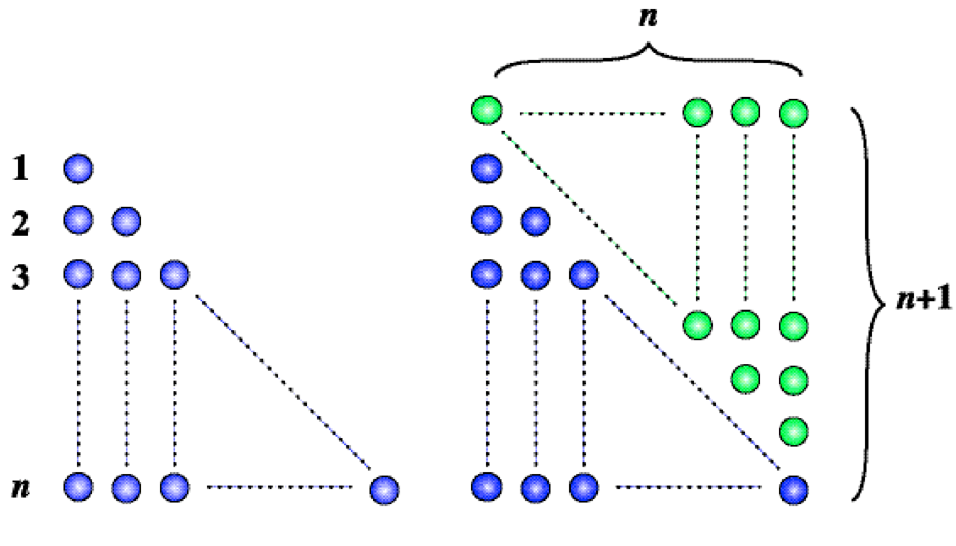
('axiomatic' means axioms for real numbers, formal definitions of limits, etc.)

## Different 'warrants for truth' in embodied, proceptual and axiomatic worlds.

Example: The sum of the first $n$ whole numbers is $\frac{1}{2}n(n+1)$.

*Proof 1: (embodied)*. Lay out rows of stones with 1 in the first, 2 in the second, and so on. Then take an equal layout of pebbles, turn it round and fit the two together. Visibly the two together make a rectangle size $n$ by $n+1$, giving $n(n+1)$ stones altogether, making $\frac{1}{2}n(n+1)$ in the original.

***The validity of this proof is in the visual picture***.




Embodied proof that the sum of the first $n$ whole numbers is $\frac{1}{2}n(n+1)$

*Proof 2: (proceptual)*. Write out the sum

$1+2+3+\ldots n$

backwards

as $n+\ \ldots\ +3+2+1$

and add the two together in order, pair by pair, to get

$(1+n) + (2+n-1) + \ldots + (n+1)$

to get $n$ lots of $n+1$, ie. $n(n+1)$, so, again, the original sum is half this, namely $\frac{1}{2}n(n+1)$. ***Validity by calculation.***

*Proof 3* : (axiomatic). By induction. ***Validity by proof.***

*(Is the third formal or axiomatic?* … Discuss.)

# THREE DIFFERENT WORLDS FOR CALCULUS AND THEIR WARRANTS FOR TRUTH

The **embodied world** is a world of sensory meaning. Its warrant for truth is that things behaves predictably in an expected way.

A (locally straight) graph *has* a slope, because you can *see* it.

It has an area underneath because you can *see* it.

What is necessary is experience of a range of alternatives to see graphs with corners or more wrinkled graphs where the slope varies wildly.

The **proceptual world** is a world where calculations can be made (both arithmetic and algebraic). A graph has a slope (derivative) or an area (integral) because you can *calculate* it.

The **axiomatic world** is a world where explicit axioms are assumed to hold and definitions are given formally in terms of quantified set-theoretic statements. A function has derivative or integral because you can *prove* it.

# AN EMBODIED APPROACH TO THE CALCULUS

*What it is not:*

It is *not* and approach that begins with formal ideas of limits, but with embodied ideas of graphical representations of functions.

It is *not* an approach based ***only*** on real-world applications. Each real world application involving say length, area, velocity, acceleration, density, weight etc has specific sensory perceptions that are *in addition* to the ideas of the calculus and therefore may cloud the issue. The important focus is on the embodied properties of the graph of a function.

*What it is:*

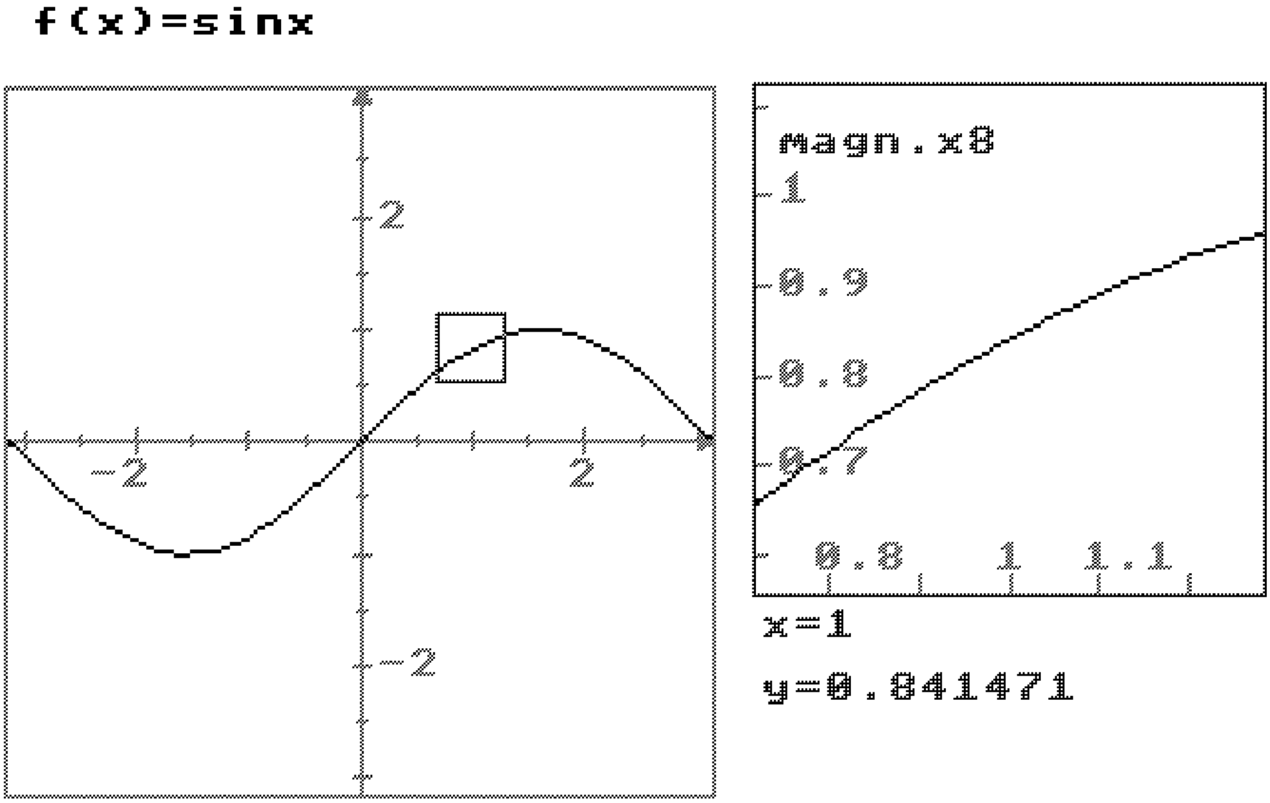
It is a study of functions involving variables that are *numbers*, not quantities with dimensions. (So that the slope is a variable *number*, the area is a *number* and slopes or areas themselves have graphs that are numerical quantities.)

It is at its best when the embodied is linked to proceptual calculations (numeric and algebraic).

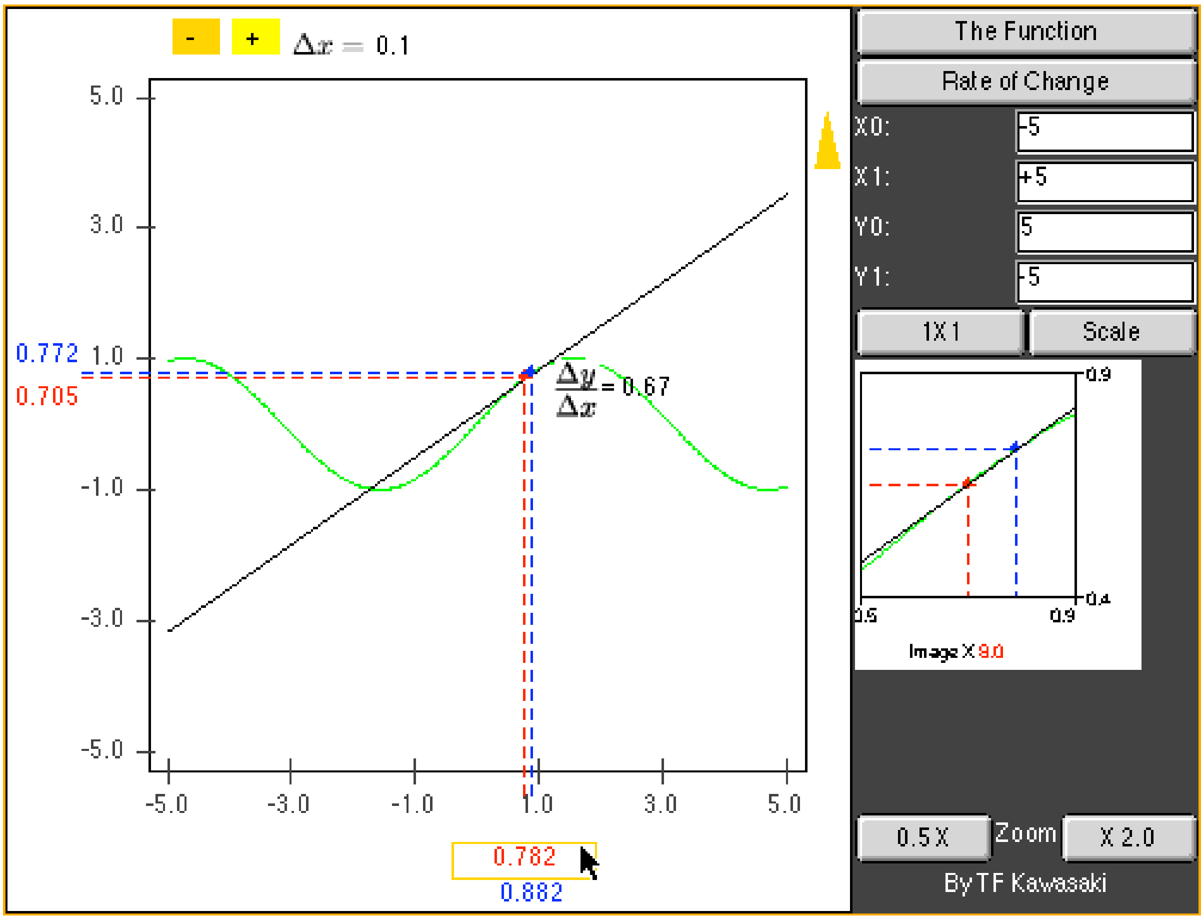
Where appropriate, it can be used to motivate ideas that later can be turned into axiomatic proofs.

# COMPUTER ENVIRONMENTS FOR COGNITIVE DEVELOPMENT

- a *generic organiser* is an environment (or microworld) which enables the learner to manipulate *examples* and (if possible) *non-examples* of a specific mathematical concept or a related system of concepts. (Tall, 1989).
- a *cognitive root* (Tall,1989) is a cognitive unit which is (potentially) meaningful to the student at the time, yet contain the seeds of cognitive expansion to formal definitions and later theoretical development. (Usually embodied!)



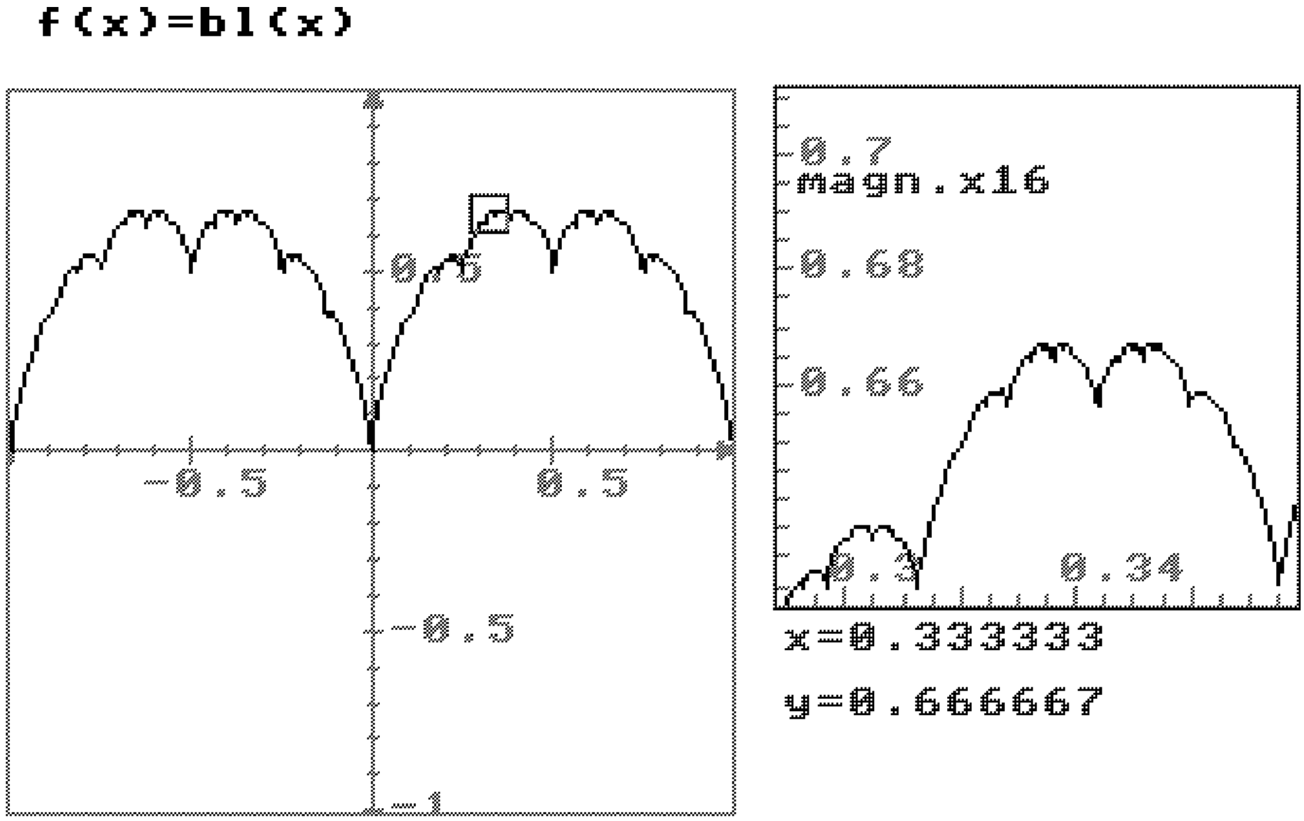

*Local straightness* is a cognitive root for differentiation. The program *Magnify* is a generic organizer for it.




Dragging the view-point along the graph to see the changing slope of a locally straight graph

Teresinha Kawasaki – Visual Calculus

A generic organiser for 'local straightness' as a cognitive root for differentiability and the enactive change in the slope as the variable 'rate of change'.



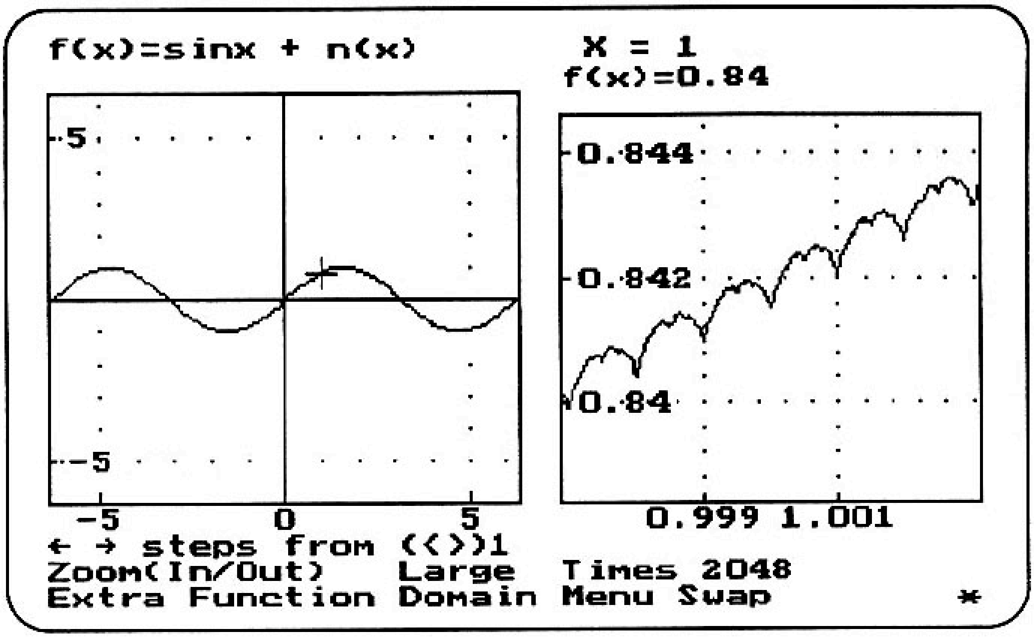

A graph which nowhere looks straight

It is the sum of saw-teeth

s($x$)= min($d$($x$), 1– $d$($x$)), where $d(x) = x$–INT$x$,

$s_n(x)=s(2^{n-1}x)/ 2^{n-1}$.

$bl(x) = s_1(x)+ s_2(x)+ s_3(x)+ \ldots$



A ‘smooth-looking curve’ that magnifies ‘rough’.

$n(x)=bl(1000x)/1000$

$\sin x$ is differentiable everywhere

$\sin x + n(x)$ is differentiable nowhere!

# EMBODIED LOCAL STRAIGHTNESS

Using an enactive interface.

**Differentiation**

a) zoom in under enactive control to sense the lessening curvature and establish local straightness by sensing it ‘happen’.

b) drag a magnification window along a locally straight graph to see its changing slope.

c) explore ‘corners’ (with different left and right slopes) and more general ‘wrinkled’ curves to sense that not all graphs are locally straight.

d) use software to draw the slope function to establish visual relationship between a locally straight function and its slope.

eg slope of $x^2$ is $2x$, of $x^2$ is $3x^2$, and relate to symbolic calculation.

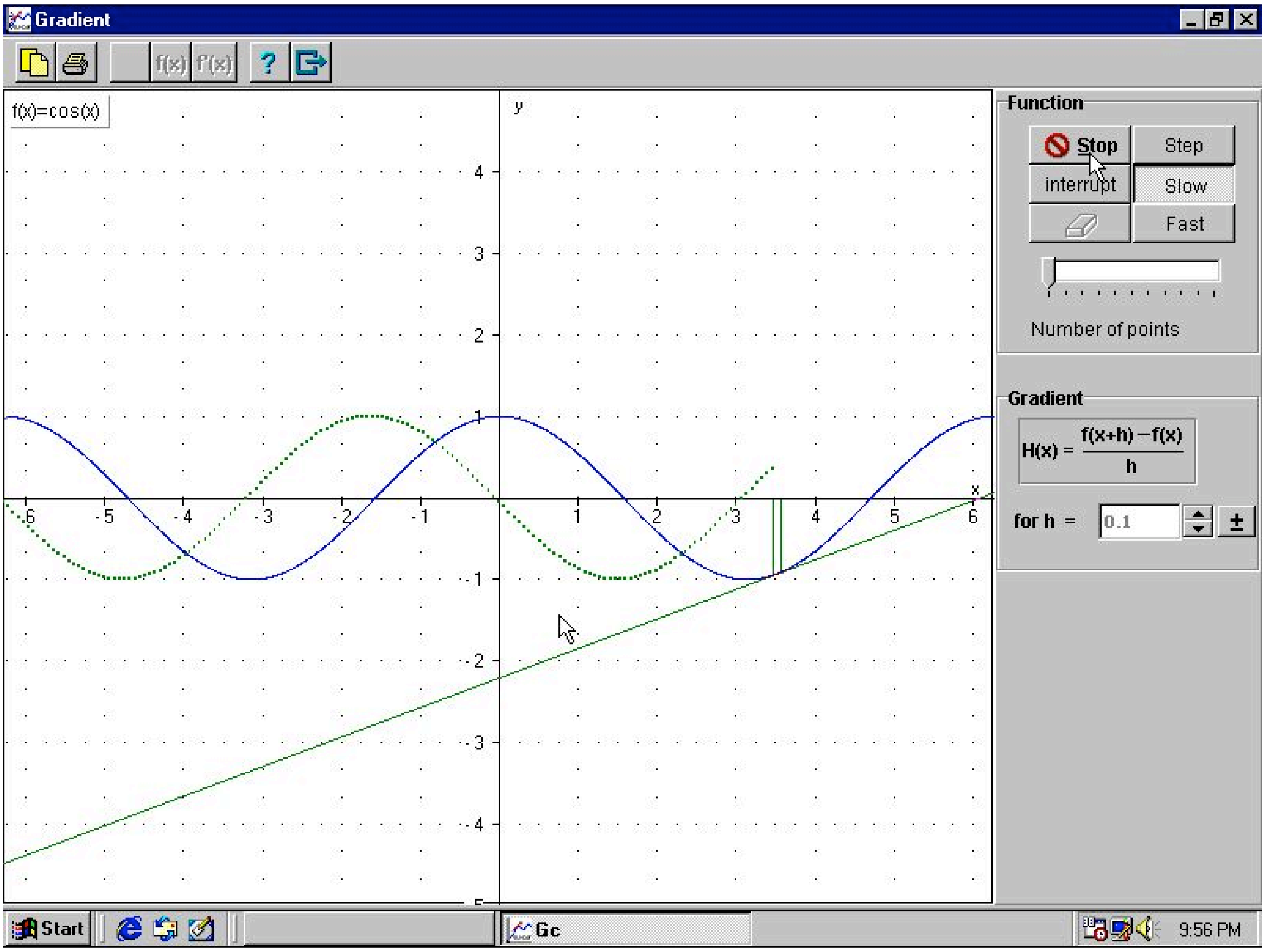
e) Slope functions of of sinx, cosx, and ‘explain’ the minus sine in the derivative of cosx is –sinx…

f) Explore $2^x$, $3^x$ and vary the parameter $k$ in $k^x$ to find a value of $k$ such that the slope of $k^x$ is again $k^x$.

## Embodied Local Straightness & Mathematical Local Linearity

‘Local straightness’ is a primitive human perception of the visual aspects of a graph. It has global implications as the individual looks *along* the graph and sees the changes in gradient, so that the gradient of the whole graph is seen *as a global entity*.

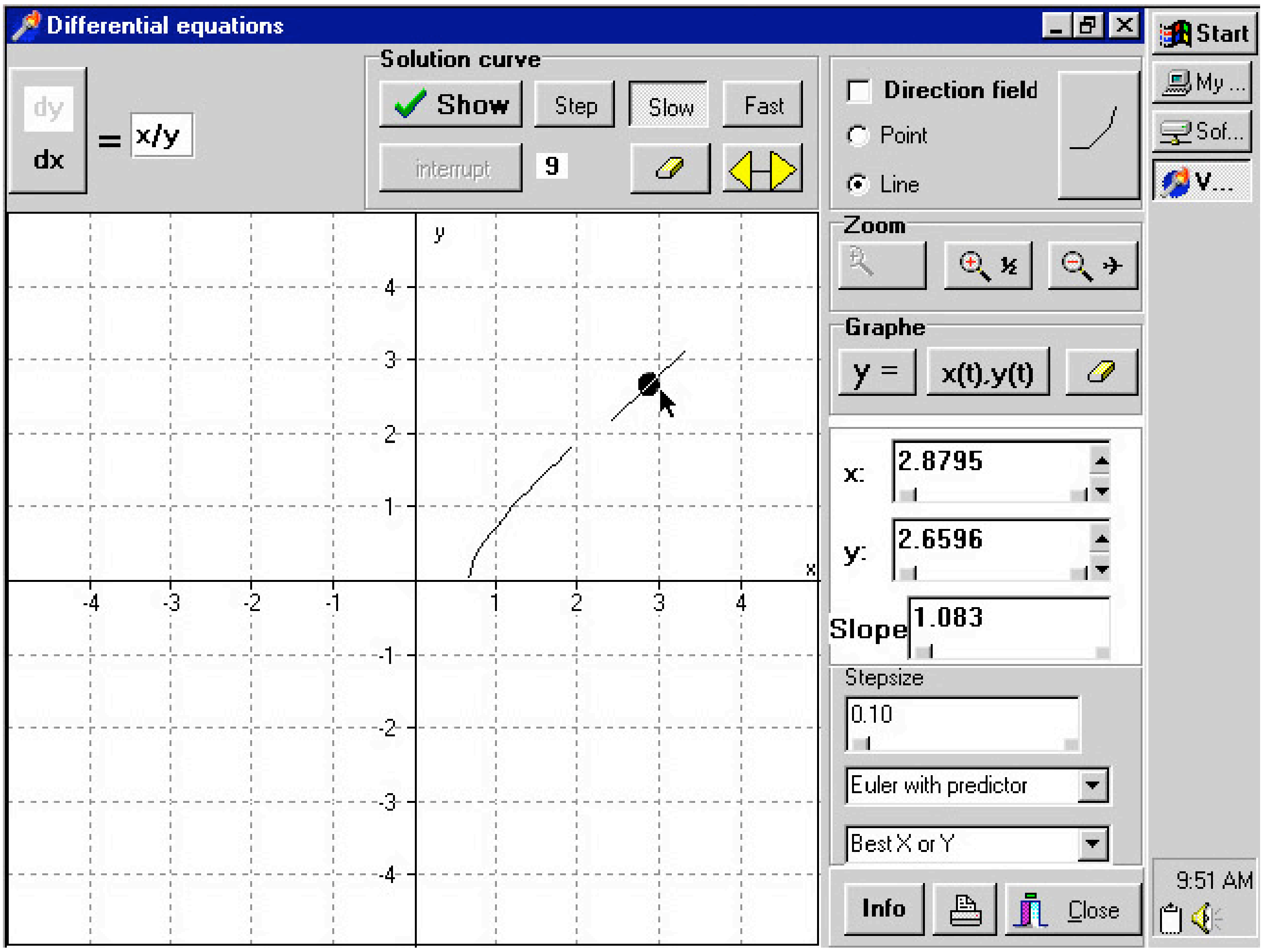
Local linearity is a *symbolic linear approximation* to the slope *at a single point* on the graph, having a linear *function* approximating the graph at that point. It is a *mathematical* formulation of gradient, taken first as a limit at a point $x$, and only then varying $x$ to get the formal derivative. Local straightness *remains at an embodied level* and links readily to the global view.




The gradient of cos*x* (drawn with Blokland et al (2000))

- an ‘embodied approach’.
- it can be linked directly to numeric and graphic derivatives, as required.
- it fits exactly with the notion of local straightness.
- it uses enactive software to build up the concept in an embodied form.

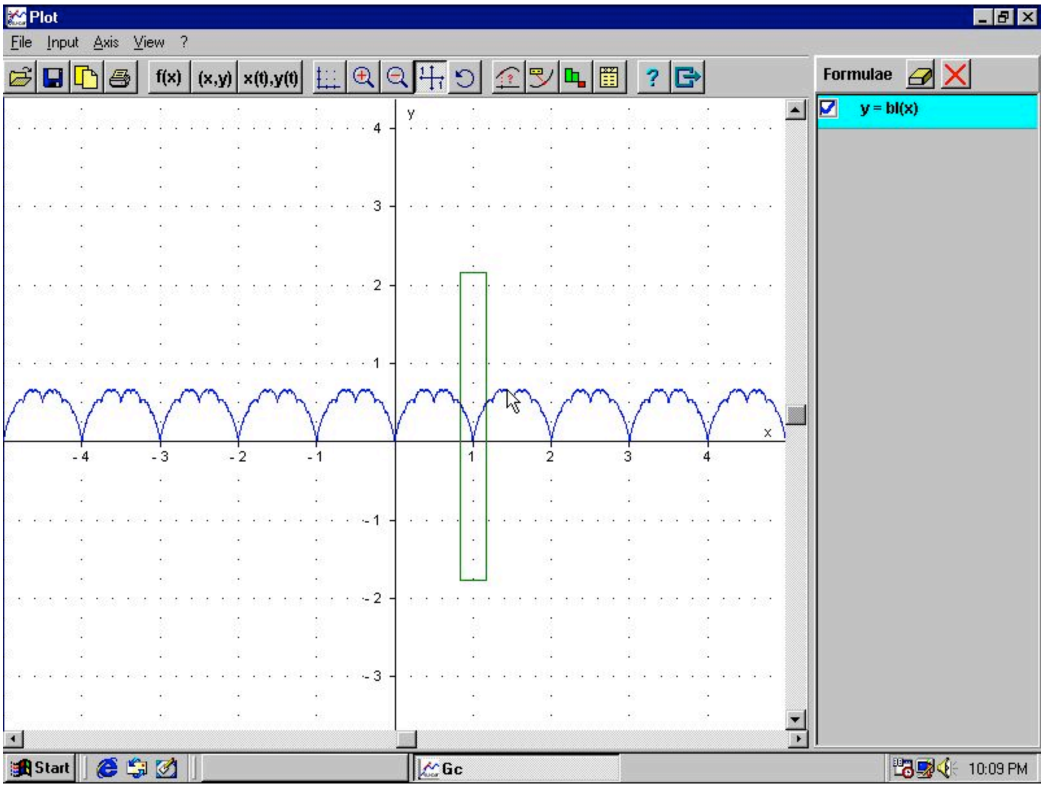
# LOCAL LINEARITY AND THE SOLUTION OF DIFFERENTIAL EQUATIONS

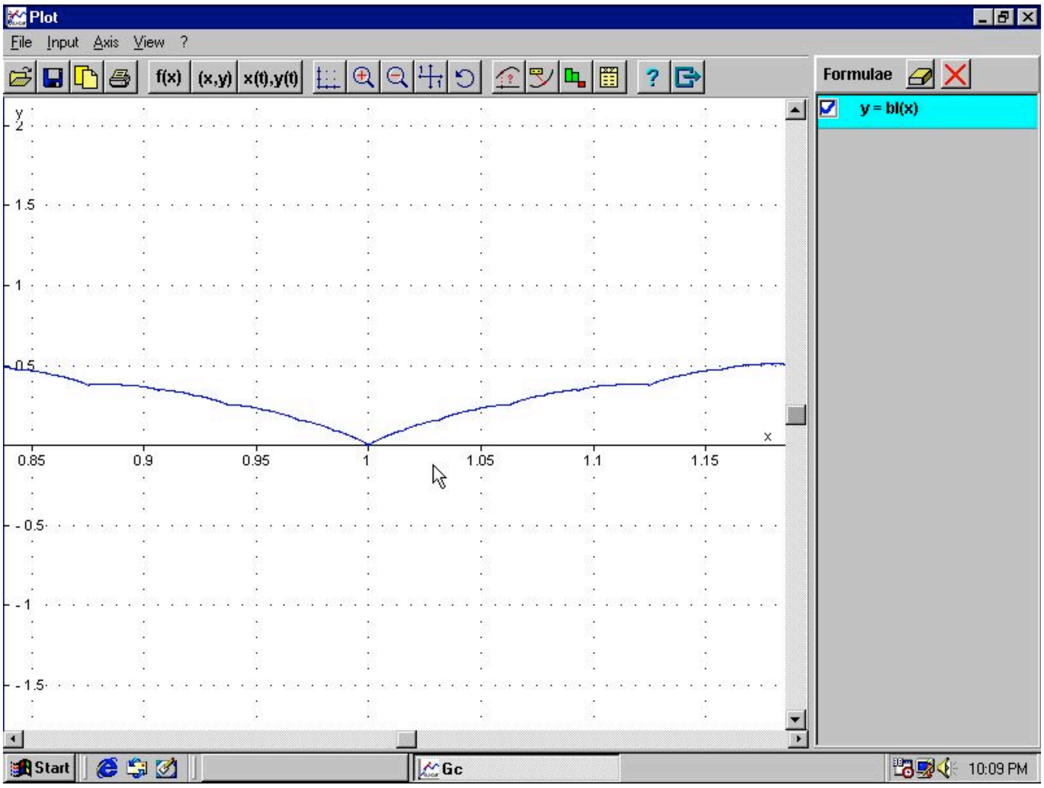



A generic organiser to build a solution of a first order differential equation by hand,
(Blokland *et al*, (2000)).

# CONTINUITY

The blancmange graph and a rectangle to be stretched to fill the screen:

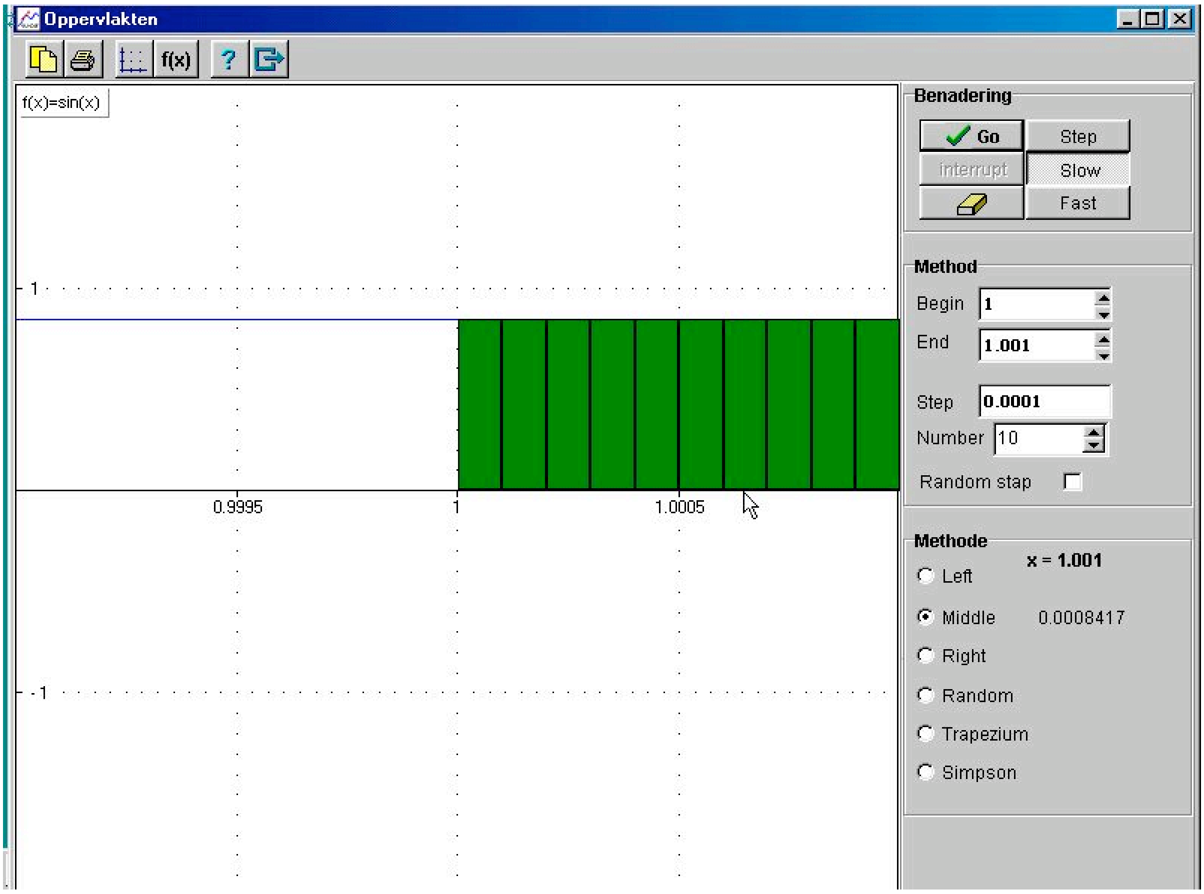


***Embodied definition***: A real function is continuous if it can be pulled flat.

Draw the graph with pixels height 2□, imagine $(a, f(a))$ in the middle of a pixel. Find an interval a–□ to $a$+□ in which the graph lies inside the pixel height $f(a)$±□.

Example: f($x$)= sin$x$ pulled flat from .999 to 1.001:



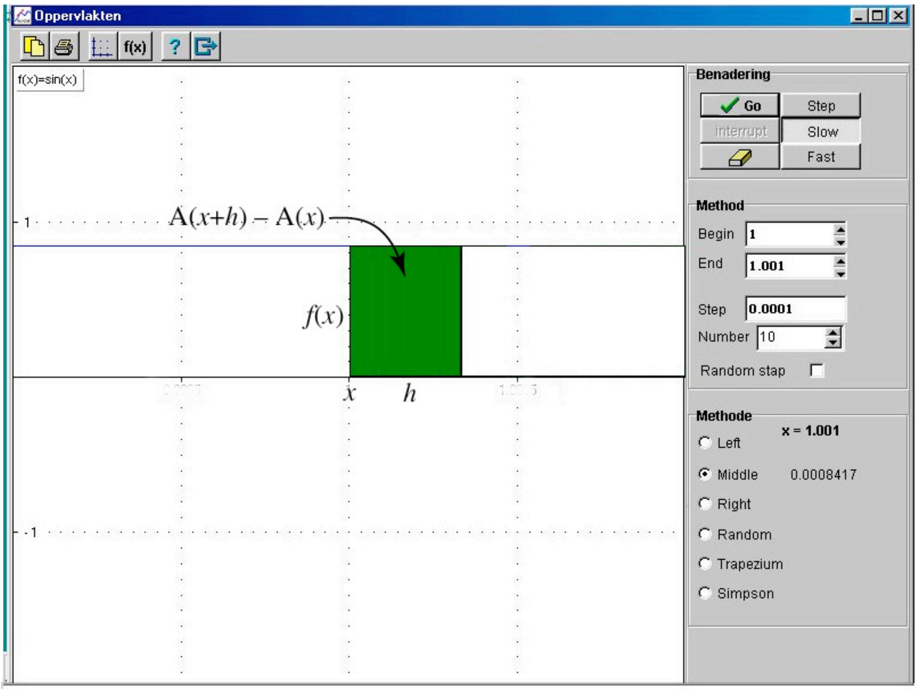

Area under sin$x$ from 1 to 1.001 stretched horizontally

This is the ***Fundamental Theorem of Calculus*** embodied. (Think about it!)

## Formalizing the embodiment of the fundamental theorem.

Let A($x$) be the area under a continuous graph over a closed interval [$a$,$b$] from $a$ to a variable point $x$.

Here, ‘continuous’ means ‘may be stretched horizontally to “look flat”’. This means that, given an $\varepsilon > 0$, and a drawing in which the value $(x_0, f(x_0))$ lies in the centre of a practical line of thickness $f(x_0) \pm \varepsilon$, then a value $\delta > 0$ can be found so that the graph over the interval from $x_0 - \delta$ to $x_0 + \delta$ lies completely within the practical line.




Then (for $-\delta < h < \delta$), the area $A(x+h)-A(x)$ lies between $(f(x)-\varepsilon)h$ and $(f(x)+\varepsilon)h$, so (for $h \neq 0$),

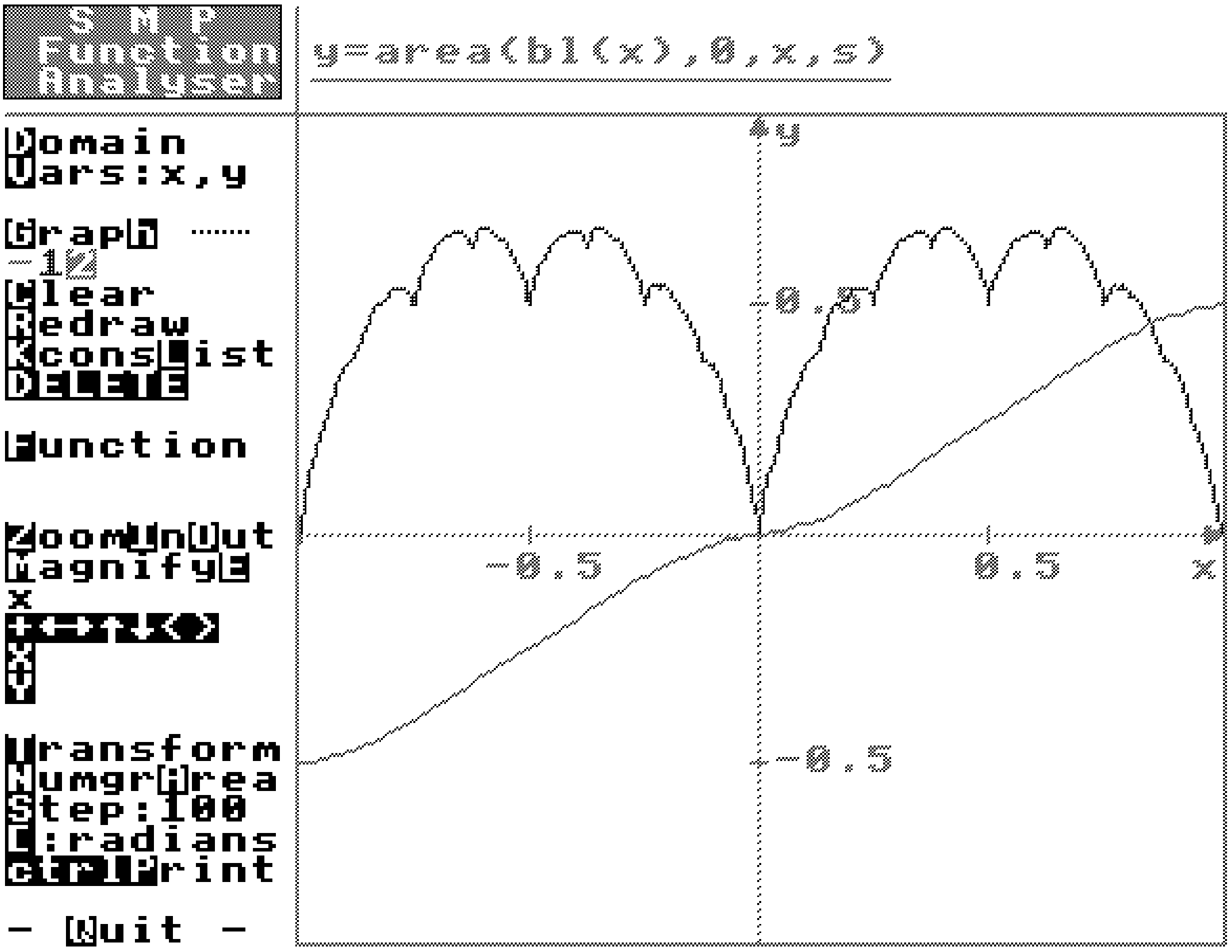
$$\frac{A(x+h) - A(x)}{h} \text{ lies between } f(x)-\varepsilon \text{ and } f(x)+\varepsilon$$

and so, for $|h| < \delta$,

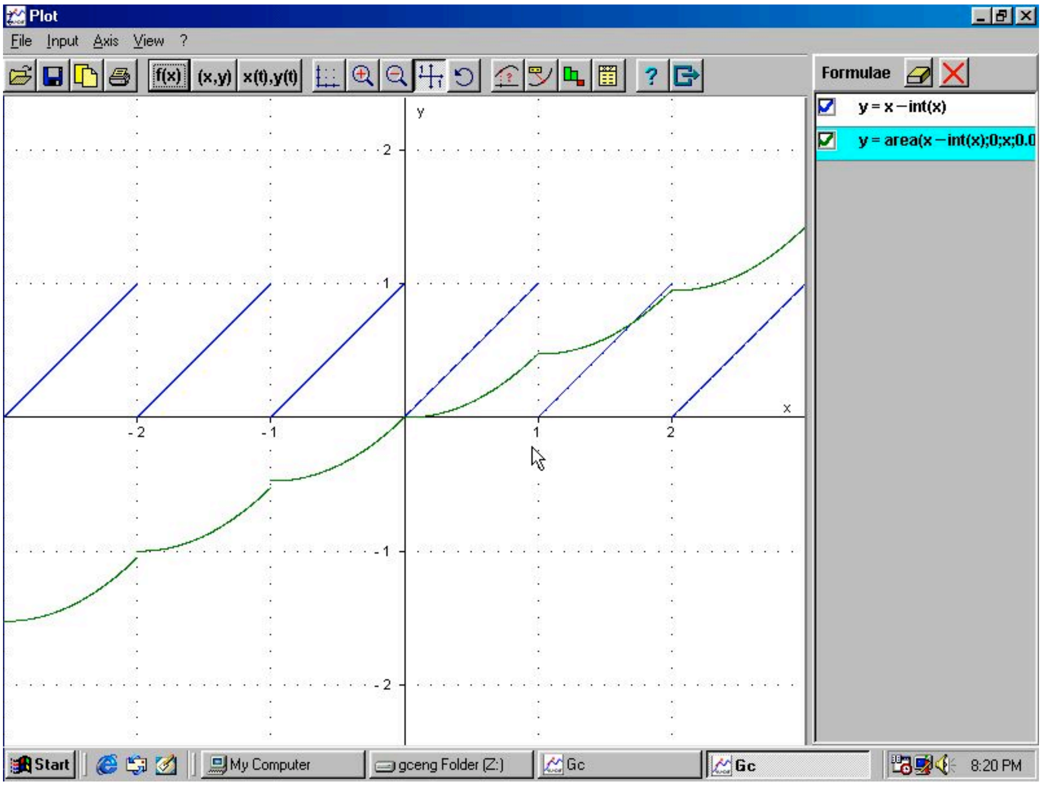
$$\left|\frac{A(x+h) - A(x)}{h}\right| < \varepsilon$$

*ie the embodied idea implies the axiomatic definition.*

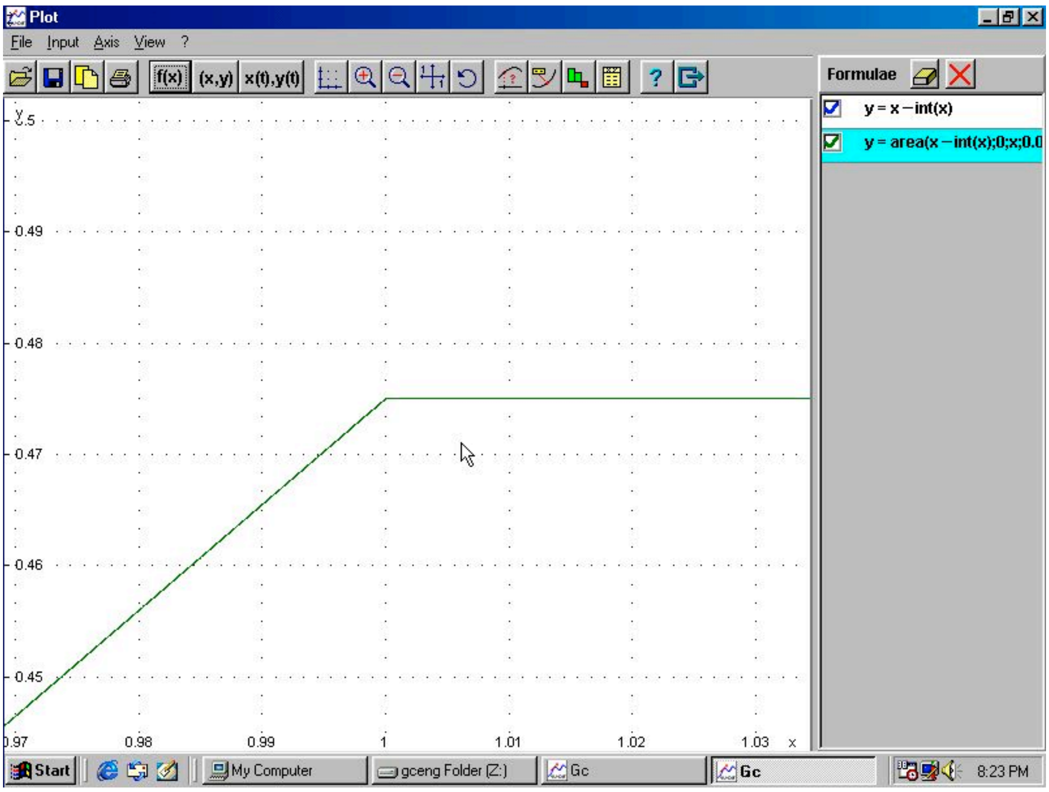
# EMBODIED AREA AND FORMAL RIEMANN INTEGRATION




The area function under the blancmange and the derivative of the area (from Tall, 1991b)
The embodied notions of 'area' and 'area-so-far' as cognitive roots can support Riemann and even Lebesgue integration.




The area function for the discontinuous function
*x–int*(*x*) calculated from 0.




The area function magnified.

# INTEGRATING HIGHLY DISCONTINUOUS FUNCTIONS

such as *f(x)=x for x rational, f(x)=*1*–x for x irrational*.

**Idea**: if $(x_n)$ is a sequence of *rationals* $x_n = a_n/b_n$ tending to the real number $x$, then if $x$ is rational, the sequence $(x_n)$ is ultimately constant and equal to $x$ otherwise the denominators $b_n$ grow without limit.

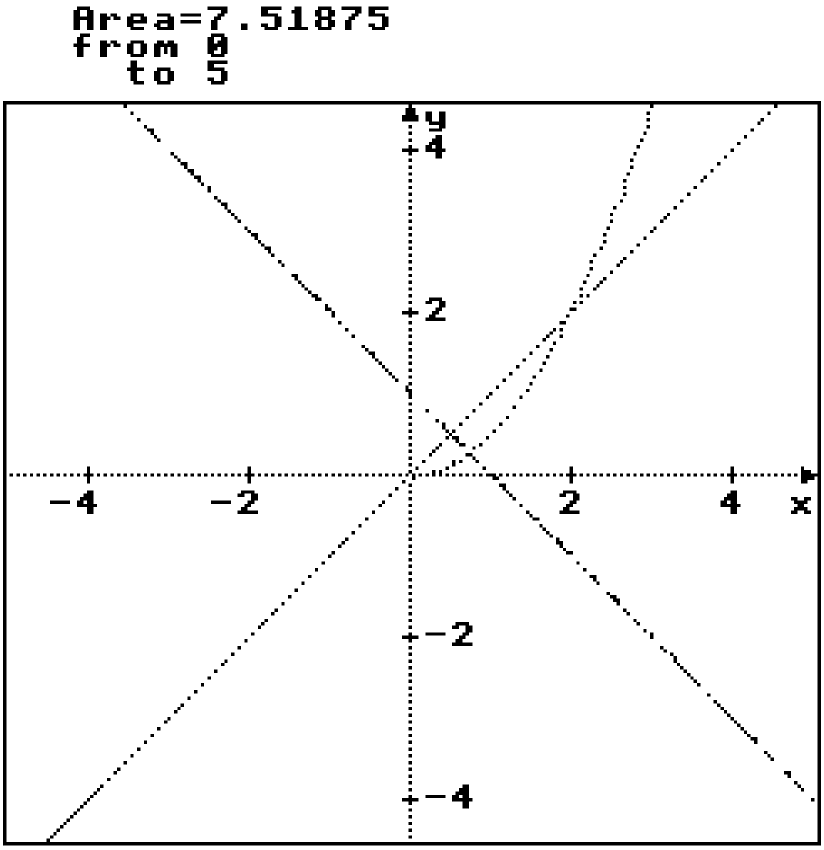
**Definition**: $x$ is ***(ε,N)-rational*** if the sequence of rationals is computed by the continued fraction method and, as soon as $|x - a_n/b_n| < \varepsilon$, then $b_n \le N$.

*Code (for Luiz!)*

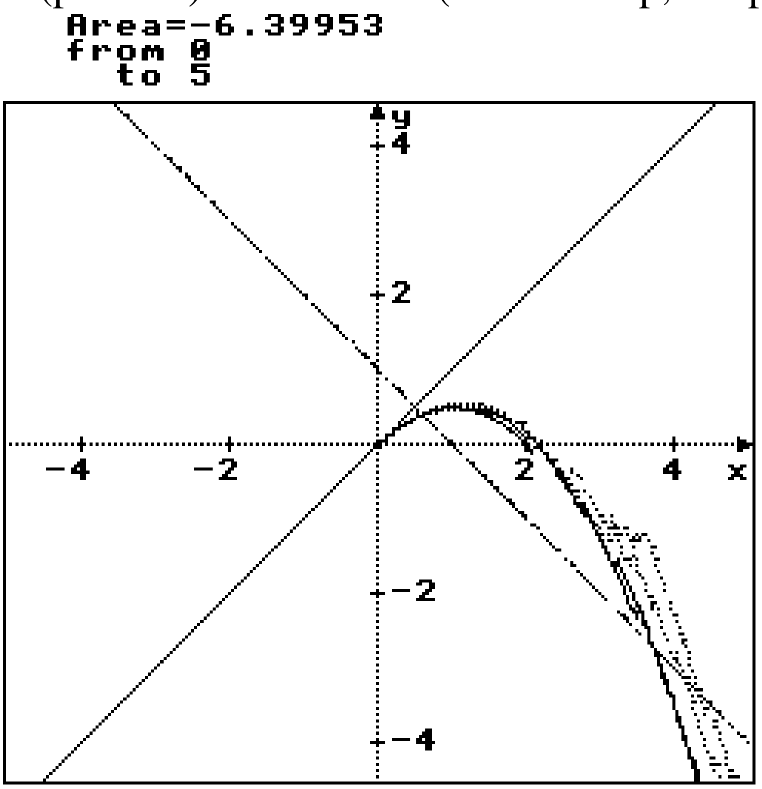
Define a function *rational(x,e,k)* which returns *true* or *false* depending on whether $x$ is an $(e,K)$-rational or not:

```
DEFINITION rational(x,e,K)
r=x: a1=0: b1=1: a2=1: b2=0
REPEAT
     n=INT(r): d=r–n: a=n*a2+a1: b=n*b2+b1
     IF d<>0 THEN
r=1/d: a1=a2: b1=b2: a2=a: b2=b
UNTIL ABS(a/b–x)<e
IF b<K THEN return TRUE ELSE return FALSE
```

Try, say $\varepsilon=10^{-8}$, $N = 10^4$ for single precision arithmetic or $\varepsilon=10^{-16}$, $N = 10^8$ for double precision.




The (pseudo-) rational area (rational step, midpt)




The (pseudo-) irrational area (random step)

## Reflections

Embodied calculus hasn't happened widely yet. Why not?

***Programming numerical algorithms: (pre-1980)***

***Graphics: (early 1980s), eg using graph-plotting programs***

***Enactive control (1984) allowing interactive exploration (eg Cabri)***

***Computer algebra systems (early 80s, generally available in late 80s)***

***Personal portable tools (eg TI-92, PDAs, portables, iBooks with wireless etc)***

***Multi-media (1990s)***

***The World Wide Web (1990s)***

Constant innovation caused new ideas to be implemented. Mathematicians naturally wanted the latest and "best" tools.

Computer algebra systems give symbolic power while the power of an enactive interface was still to be fully understood. An embodied approach combines a good human interface into symbolic power. Those focusing mainly on the symbolic rather than the embodied have a great deal of mental reconstruction to do to begin to even understand the power of building on embodied enaction.

**Epilogue**

- The human mind does not always do mathematics logically, but is guided by a concept image that can be both helpful and also deceptive.
- Symbolism is more precise and safer than visualisation, but cognitive development of symbols in arithmetic, algebra and calculus have many potential cognitive pitfalls.
- *Local straightness* provides an embodied foundation for the calculus.
- The *local slope of the graph* as rate of change is an embodied foundation for the slope function (derivative).
- *Finding a graph given its slope* is an embodied foundation for differential equations.
- *Local flatness* is a cognitive foundation for continuity and the fundamental theorem of calculus

- An embodied approach can be used to link directly to

  ☆ proceptual symbolism in calculus

  and to

  ☆ axiomatic formulation in analysis.