Άσκηση: $F = 2B^3(t) + t$, βρείτε dF

Λύση: $F(t, S) = 2S^3 + t$

$S = B(t)$

$dS = dB(t) = 0 \cdot dt + 1 \cdot dB(t)$

$\mu = 0, \quad \sigma = 1$

$\dfrac{\partial F}{\partial t} = 1, \quad \dfrac{\partial F}{\partial S} = 6S^2, \quad \dfrac{\partial^2 F}{\partial S^2} = 12S$

$$dF = \left[1 + 0 \ldots\ldots + \frac{1}{2} \cdot 1^2 \cdot 12\underline{S}\right] dt + 1 \cdot 6S^2 dB$$

$\Downarrow$

$$\boxed{dF = (1 + 6B(t))\,dt + 6B^2(t)\,dB(t)}$$

Στοχαστική Διαφορική Εξίσωση

Άσκηση: Αν $f = B^4(t)$, $B(t)$ κίνηση Brown, βρείτε df

Λύση: $f = f(t,S) = S^4$

όπου $S = B(t)$

$$dS = dB(t) = 0.dt + 1.dB(t)$$

$$\mu = 0 \quad , \quad \sigma = 1$$

$$\frac{\partial f}{\partial t} = 0 \quad , \quad \frac{\partial f}{\partial S} = 4S^3 \quad , \quad \frac{\partial^2 f}{\partial S^2} = 12S^2$$

$$df = \left[\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S^2}\right] dt + \sigma \frac{\partial f}{\partial S} dB$$

$$df = \left[0 + 0 + \frac{1}{2} \cdot 1 \cdot 12S^2\right] dt + 1.4S^3 dB$$

$$\boxed{dB^4(t) = 6B^2(t)\,dt + 4B^3(t)\,dB}$$

Άσκηση: Αν $f = e^{\mu t + \sigma B(t)}$, $\mu, \sigma \in \mathbb{R}$

βρείτε df.

Λύση: $f = f(t, S) = e^{S}$

$$S = \mu t + \sigma B(t)$$

$$dS = \underline{\mu} dt + \sigma dB(t) \qquad \mu = \mu \quad \sigma = \sigma$$

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial S} = e^{S}, \quad \frac{\partial^2 f}{\partial S^2} = e^{S}$$

$$df = \left[0 + \mu \cdot e^{S} + \frac{1}{2}\sigma^2 \cdot e^{S}\right] dt + \sigma e^{S} dB$$

$$df = \left[\mu \underline{\underline{e^{S}}} + \frac{1}{2}\sigma^2 \underline{\underline{e^{S}}}\right] dt + \sigma \underline{\underline{e^{S}}} dB \Rightarrow$$

$$\Rightarrow df = f\left[\left(\mu + \frac{1}{2}\sigma^2\right) dt + \sigma dB\right]$$

Άσκηση: η $S(t)$ ακολουθεί την

$$dS = \underline{\tilde{\mu} S}\, dt + \tilde{\sigma} S\, dB$$

Βρείτε το dF της $F = t + e^S$

Λύση: $F = F(t, S) = t + e^S$

$$\mu = \hat{\mu} S \quad , \quad \sigma = \tilde{\sigma} S$$

$$\frac{\partial F}{\partial t} = 1 \, , \quad \frac{\partial F}{\partial S} = e^S \, , \quad \frac{\partial^2 F}{\partial S^2} = e^S$$

$$dF = \left[1 + \hat{\mu} S e^S + \frac{1}{2} \tilde{\sigma}^2 S^2 e^S \right] dt +$$

$$+ \frac{1}{2} \tilde{\sigma} S e^S \, dB$$

Άσκηση: Ένα περουσιακό στοιχείο ακολουθεί την εξίσωση:

$$dW = \lambda(\hat{\mu} - W)\,dt + \hat{\sigma} W\,dB$$

βρείτε το διαφορικό της $F = \frac{1}{1-t} + e^{W} + t^{2}$.

Λύση: $$F(t,W) = \frac{1}{1-t} + e^{W} + t^{2}$$

$$\mu = \lambda(\hat{\mu} - W), \quad \sigma = \hat{\sigma} W$$

$$\frac{\partial F}{\partial t} = +\frac{1}{(1-t)^2} + 2t, \quad \frac{\partial F}{\partial W} = e^{W}, \quad \frac{\partial^2 F}{\partial W^2} = e^{W}$$

$$\boxed{dF = \left[\frac{1}{(1-t)^2} + 2t + \lambda(\hat{\mu} - W)\cdot e^{W} + \frac{1}{2}\hat{\sigma}^2 W^2 e^{W}\right]dt + \hat{\sigma} W e^{W}\cdot dB}$$

t	$B(t)$
5	1.5
5.01	1.7
5.02	2.7
5.03	-1
5.04	

Άσκηση: Αν $dS = rS\,dt + \sigma S\,dB$ (στοχαστικό επιτόκιο) βρείτε το διαφορικό της $S^* = Se^{-rt}$ (παρούσα αξία)

Λύση:

$$F = F(t,S) = Se^{-rt}$$

$$\mu = rS \quad , \quad \sigma = \sigma S$$

$$\frac{\partial F}{\partial t} = -rSe^{-rt} \quad , \quad \frac{\partial F}{\partial S} = e^{-rt} \quad , \quad \frac{\partial^2 F}{\partial S^2} = 0$$

$$dS^* = \Big[-r\overset{S^*}{\underbrace{Se^{-rt}}} + r\overset{S^*}{\underbrace{Se^{-rt}}} + 0\Big]dt + \sigma\overset{S^*}{\underbrace{Se^{-rt}}}\,dB$$

$$dS^* = \underbrace{S^*[-r + r]}\,dt + \sigma S^*\,dB$$

$$\boxed{dS^* = \sigma S^*\,dB}$$

Άσκηση: Βρείτε $X(t)$ έτσι ώστε

$$dX(t) = X^3(t)\,dt - X^2(t)\,dB$$

Λύση: Έστω $X = F(t, S)$, $S = B(t)$

$\mu = 0$, $\sigma = 1$,

$$dX = \left[\frac{\partial F}{\partial t} + 0 + \frac{1}{2} 1^2 \cdot \frac{\partial^2 F}{\partial S^2}\right] dt + 1 \cdot \frac{\partial F}{\partial S} dB$$

$$\left.\begin{array}{l} dX = \left[\dfrac{\partial X}{\partial t} + \dfrac{1}{2} \dfrac{\partial^2 X}{\partial S^2}\right] dt + \dfrac{\partial X}{\partial S} dB \\ dX = X^3\,dt - X^2\,dB \end{array}\right\} \Rightarrow$$

(1) $$\frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial S^2} = X^3$$

, $X(t, S)$

(2) $$\frac{\partial X}{\partial S} = -X^2$$

(2) $\Rightarrow \dfrac{\partial X}{\partial S} = -X^2 \Rightarrow$

$$\int \frac{\partial X}{X^2} = -\int \partial S \Rightarrow -X^{-1} = -S + \varphi(t)$$

$$\Rightarrow X(t,S) = \frac{1}{S-\varphi(t)}$$

(1) $\dfrac{\partial X}{\partial t} + \dfrac{1}{2}\dfrac{\partial X^2}{\partial S^2} = X^3$ γιάνι σας

$$\Rightarrow \boxed{\varphi(t) = C}$$

$$\Rightarrow X(t,S) = \frac{1}{S-C} \Rightarrow$$

$$\Rightarrow \boxed{X(t,S) = \frac{1}{B(t)-C}}$$

Αν $X(0)=1 \xRightarrow{t=0} X(0) = \dfrac{1}{B(0)-C} \Rightarrow 1 = \dfrac{1}{-C}$

$C = -1$

Άσκηση: $\int_0^T e^{B(t)}\,dB(t)$

Λύση: $F = e^{B(t)} = e^{S}$, όπου $S = B(t)$

$\Longrightarrow \mu = 0, \quad \sigma = 1$

$$dF = \left[0 + 0 + \frac{1}{2}1^2 e^{S}\right]dt + 1\cdot e^{S}\,dB$$

$$\Downarrow$$

$$de^{B(t)} = \frac{1}{2}e^{B(t)}\,dt + \underline{e^{B(t)}\,dB}$$

$$\int_0^T de^{B(t)} = \frac{1}{2}\int_0^T e^{B(t)}\,dt + \int_0^T e^{B(t)}\,dB$$

$$e^{B(T)} - e^{\overset{=0}{B(0)}} = \frac{1}{2}\int_0^T e^{B(t)}\,dt + \int_0^T e^{B(t)}\,dB$$

$$\Rightarrow \boxed{\int_0^T e^{B(t)}\,dB = e^{B(T)} - 1 - \frac{1}{2}\int_0^T e^{B(t)}\,dt}$$

Άσκηση: Βρείτε $\int_0^T B(t)\,dB(t)$.

Λύση: Έστω $f = B^2(t)$

$f(t,S) = S^2$, $S = B(t) \Rightarrow$ $\mu = 0$, $\sigma = 1$

$$df = \left[0 + 0 + \frac{1}{2} 1^2 \cdot 2\right] dt + 1 \cdot 2S \cdot dB$$

$$\Downarrow$$

$$dB^2(t) = dt + 2B(t)\,dB \Rightarrow$$

$$\int_0^T dB^2(t) = \int_0^T dt + 2\int_0^T B(t)\,dB \Rightarrow$$

$$B^2(T) - B^2(0) = T - 0 + 2\int_0^T B(t)\,dB$$

$$\Rightarrow \boxed{\int_0^T B(t)\,dB = \frac{B^2(T) - T}{2}}$$

Άσκηση: $\int_0^T t\, B(t)\, dB$

Λύση: $F(t,S):\ \frac{\partial F}{\partial S} = tS \Rightarrow$

$\Rightarrow F = \frac{tS^2}{2}$, $F(t,S) = \frac{tS^2}{2}$, $S = B(t)$, $\mu = 0, \sigma = 1$

$$dF = \left[\frac{S^2}{2} + 0 + \frac{1}{2} 1^2 \cdot 2t\right] dt + 1 \cdot tS\, dB \Rightarrow$$

$$d\left[\frac{tB^2(t)}{2}\right] = \left[\frac{B^2(t)}{2} + t\right] dt + \underline{tB(t)\, dB}$$

$$\int_0^T d\left[t\,\frac{B^2(t)}{2}\right] = \int_0^T \left(\frac{B^2(t)}{2} + t\right) dt + \int_0^T tB(t)\, dB$$

$$\frac{TB^2(T)}{2} - 0 = \frac{1}{2}\int_0^T B^2(t)\, dt + \int_0^T t\, dt + I$$

$$\boxed{I = \frac{TB^2(T) - T^2/2 - \int_0^T B^2(t)\, dt}{2}}$$