S.O.S.

Ολοκλήρωση Ρητών συναρτήσεων

$$\int \frac{P(x)}{Q(x)}\,dx =$$

$\boxed{1}$ βαθμός$(P(x)) \geq$ βαθμός$(Q(x))$

$\frac{P(x)}{Q(x)}$ = καταχρηστική συνάρτηση

κάνουμε διαίρεση πολυωνύμων:

$$\begin{array}{l|l} P(x) & \underline{Q(x)} \\ & \pi(x) \\ \upsilon(x) & \end{array}$$

βαθμός$(\upsilon(x)) <$ βαθμός$(Q(x))$

Ταυτότητα Διαίρεσης:

$$\boxed{P(x) = \pi(x)\cdot Q(x) + \upsilon(x)}$$

$$= \int \frac{\pi(x)Q(x) + \upsilon(x)}{Q(x)}\,dx =$$

$$= \int \frac{\pi(x)\cdot \cancel{Q(x)}}{\cancel{Q(x)}} + \frac{\upsilon(x)}{Q(x)}\,dx =$$

$$= \int \pi(x)\,dx + \boxed{\int \frac{\upsilon(x)}{Q(x)}\,dx} \to \boxed{2}$$

βαθμός$(\upsilon(x)) <$ βαθμός$(Q(x))$

α) $\int \frac{A}{x-\alpha}\,dx = A\cdot \ln|x-\alpha| + c,$

$$\int \frac{A}{(x-\alpha)^{\kappa}}\,dx = A\int (x-\alpha)^{-\kappa}\,dx =$$

$\boxed{\kappa \neq 1}$

$$A\cdot \frac{(x-\alpha)^{-\kappa+1}}{-\kappa+1} + c$$

β) $\int \frac{Ax+B}{\alpha x^2+\beta x+\gamma}\,dx =$

$\alpha x^2+\beta x+\gamma \to \Delta>0 \to x_1, x_2$

$\to \Delta=0 \to x_1 =$ διπλή

$$\int \frac{Ax+B}{\alpha(x-x_1)(x-x_2)}\,dx$$

$\boxed{\alpha}$

$\to \underline{\Delta>0}$: $\alpha x^2+\beta x+\gamma = \alpha(x-x_1)(x-x_2)$

$\underline{\Delta=0}$: $\alpha x^2+\beta x+\gamma = \alpha\cdot(x-x_1)^2$

$\Delta<0$ (εκτός ύλης)

$\int \frac{}{\alpha(x-x_1)(x-x_2)}$

$\boxed{\acute{\eta}}$

$\Delta < 0$ (εντός ύλης)

$$\int \frac{Ax+B}{\alpha(x-x_1)^2}\,dx$$

Μέθοδος Απλών Κλασμάτων:

$$\frac{Ax+B}{\alpha(x-x_1)(x-x_2)} = \frac{\Gamma}{x-x_1} + \frac{\Delta}{x-x_2} \Rightarrow \dots \Rightarrow \Gamma, \Delta = ?$$

π.χ. $\int \frac{x+1}{x^2-3x+2}\,dx$

$x^2-3x+2 = 1(x-x_1)(x-x_2) = (x-1)(x-2)$

$$\frac{x+1}{x^2-3x+2} = \boxed{\frac{x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}} \Rightarrow$$

$$x+1 = A(x-2) + B(x-1)$$

Θέτω $x=2$, $3 = A\cdot 0 + B(2-1) \Rightarrow \boxed{3 = B}$

Θέτω $x=1$, $2 = A(1-2) + B\cdot 0 \Rightarrow \boxed{A=-2}$

$$\int \boxed{\frac{x+1}{x^2-3x+2}}\,dx = \int \frac{-2}{x-1} + \frac{3}{x-2}\,dx =$$

$$-2\cdot\int \frac{1}{x-1}\,dx + 3\int \frac{1}{x-2}\,dx =$$

$$= -2\ln|x-1| + 3\ln|x-2| + c, \quad c\in\mathbb{R}$$

π.χ. $\int \frac{7}{x^2-2x+1}dx =$ $\quad x^2-2x+1 = 1\cdot(x-1)^2$

$$= \int \frac{7}{(x-1)^2}dx = 7\int (x-1)^{-2}dx = 7\frac{(x-1)^{-2+1}}{-2+1} + c$$

$$= -7(x-1)^{-1} + c = \frac{-7}{x-1} + c.$$

$c \in \mathbb{R}$.

Ασκήσεις:

► Άσκηση 6.1. (Βλέπε ->Λυμένες ασκήσεις Κεφαλαίου XII)

Να υπολογισθεί το ολοκλήρωμα:

$$\int \frac{x+1}{2x^2-10x+12}dx$$

βαθμός $(x+1)$ < βαθμός $(2x^2-10x+12)$

$$2x^2-10x+12 = 2(x^2-5x+6) = 2(x-2)(x-3)$$

$$\frac{x+1}{2x^2-10x+12} = \boxed{\frac{x+1 \cdot \cancel{2(x-2)(x-3)}}{\cancel{2(x-2)(x-3)}} = \frac{A \cdot \cancel{2(x-2)(x-3)}}{\cancel{x-2}} + \frac{B \cdot \cancel{2(x-2)(x-3)}}{\cancel{x-3}}}$$

$$x+1 = 2A(x-3) + 2B(x-2)$$

Θέτω $x=2$: $3 = -2A \Rightarrow \boxed{A = \frac{-3}{2}}$

$x=3$: $4 = 2B \Rightarrow \boxed{B=2}$

$$\int \boxed{\frac{x+1}{2x^2-10x+12}}dx = \int \frac{-3/2}{x-2} + \frac{2}{x-3}\,dx =$$

$$-\frac{3}{2}\int\frac{1}{x-2}dx+2\int\frac{1}{x-3}dx=$$

$$-\frac{3}{2}\ln|x-2|+2\ln|x-3|+c\ ,\ c\in\mathbb{R}.$$

▶ Άσκηση 6.2. Να υπολογισθεί το ολοκλήρωμα:

$$\int\frac{x+1}{x^3+x^2-6x}dx$$

βαθμός$(x+1)$ < βαθμός(x^3+x^2-6x)

$$x^3+x^2-6x=x(x^2+x-6)=x(x+3)(x-2)$$

$$\boxed{x_1=-3\ ,\ x_2=2}\quad (x-0)$$

$$\frac{x+1}{x^3+x^2-6x}=\boxed{\frac{x+1}{x(x+3)(x-2)}=\frac{A}{x}+\frac{B}{x+3}+\frac{\Gamma}{x-2}}$$

$$x+1=A(x+3)(x-2)+B\cdot x(x-2)+\Gamma\cdot x\cdot(x+3)$$

Θέτω:

$x=0:\ 1=-6A\Rightarrow\boxed{A=-\frac{1}{6}}$

$x=2:\ 3=10\Gamma\Rightarrow\boxed{\Gamma=\frac{3}{10}}$

$x=-3:\ -2=15B\Rightarrow\boxed{B=\frac{-2}{15}}$

$$\int\frac{x+1}{x^3+x^2-6x}dx=\int\frac{-1/6}{x}+\frac{-2/15}{x+3}+\frac{3/10}{x-2}dx$$

$$-\frac{1}{6}\int\frac{1}{x}dx-\frac{2}{15}\int\frac{1}{x+3}dx+\frac{3}{10}\int\frac{1}{x-2}dx=$$

$$-\frac{1}{6}\int\frac{1}{x}dx-\frac{2}{15}\int\frac{1}{x+3}dx+\frac{3}{10}\int\frac{1}{x-2}dx=$$

$$-\frac{1}{6}\ln|x|-\frac{2}{15}\ln|x+3|+\frac{3}{10}\ln|x-2|+c,\ c\in\mathbb{R}$$

▶ **Άσκηση 6.5.** Να υπολογισθεί το ολοκλήρωμα:

$$\int\frac{2x^3-8x^2+9x+1}{x^2-4x+4}dx$$

βαθμός$(2x^3-8x^2+9x+1)>$ βαθμός (x^2-4x+4)

$$\begin{array}{r|l} \cancel{2x^3}-\cancel{8x^2}+9x+1 & x^2-4x+4 \\ \hline \ominus\ \cancel{2x^3}-\cancel{8x^2}+8x & 2x \\ \hline x+1 & \end{array}$$

Ταυτότητα διαίρεσης: $\Delta(x)=\pi(x)\cdot\delta(x)+\upsilon(x)$

$$\boxed{2x^3-8x^2+9x+1=2x\cdot(x^2-4x+4)+x+1}$$

$$\int\frac{2x^3-8x^2+9x+1}{x^2-4x+4}dx=\int\frac{2x(x^2-4x+4)+x+1}{x^2-4x+4}dx$$

$$= \int \frac{2x\,\cancel{(x^2-4x+4)}}{\cancel{x^2-4x+4}}\,dx + \int \frac{x+1}{x^2-4x+4}\,dx$$

$$= \underbrace{\boxed{\int 2x\,dx}}_{I_1} + \underbrace{\boxed{\int \frac{x+1}{x^2-4x+4}\,dx}}_{I_2}$$

$$I_1 = \int 2x\,dx = x^2 + c_1$$

$$I_2 = \int \frac{x+1}{x^2-4x+4}\,dx = \boxed{\int \frac{x+1}{(x-2)^2}\,dx}$$

Θέτω $u = x-2 \Rightarrow x = u+2$, $\quad \dfrac{du}{dx} = (x-2)' \Rightarrow \dfrac{du}{dx} = 1$

$du = dx$

$$I_2 = \int \frac{u+3}{u^2}\,du = \int \frac{1}{u}\,du + \int \frac{3}{u^2}\,du =$$

κανόνας δύναμης

$$= \ln|u| + 3\left(-\frac{1}{u}\right) + c_2$$

$$= \ln|x-2| - \frac{3}{x-2} + C_2$$

$$I_1 + I_2 = x^2 + \ln|x-2| - \frac{3}{x-2} + C, \; C\in\mathbb{R}$$

$$\int \frac{x+1}{(x-2)^2}\,dx \longrightarrow \text{B' ΤΡΟΠΟΣ}$$

$$\frac{x+1}{(x-2)^2} = \frac{A}{(x-2)^1} + \frac{B}{(x-2)^2}$$

π.χ.

$$\frac{x+1}{(x-2)^4} = \frac{A}{(x-2)^1} + \frac{B}{(x-2)^2} + \frac{\Gamma}{(x-2)^3} + \frac{\Delta}{(x-2)^4}$$

$$\frac{x+1}{\cancel{(x-2)^2}}\,(x-2)^2 = \frac{A}{\cancel{x-2}}\,(x-2)^2 + \frac{B}{\cancel{(x-2)^2}}\,(x-2)^2$$

$$\boxed{x+1 = A(x-2) + B}$$

Θέτω $x=2$: $3 = B \Rightarrow \boxed{B=3}$

$x=0$: $1 = A(-2) + B$

$1 = -2A + 3 \Rightarrow 2A = 2 \Rightarrow \boxed{A=1}$

$$\int \frac{x+1}{(x-2)^2}\,dx = \int \frac{A}{x-2}\,dx + \int \frac{B}{(x-2)^2}\,dx =$$

$$\int \frac{1}{\quad}\,dx + \int \frac{3}{\quad}\,dx =$$

$$\int \frac{1}{x-2}\,dx + \int \frac{3}{(x-2)^2}\,dx =$$

$$= \ln|x-2| + 3\,\frac{(x-2)^{-2+1}}{-2+1} + c$$

$$= \ln|x-2| - \frac{3}{x-2} + c\,.$$

Π.χ.

$$\int \frac{x+1}{(x-1)(x-2)^2(x-3)}\,dx$$

$$\frac{x+1}{(x-1)\boxed{(x-2)^2}(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{\Gamma}{(x-2)^2} + \frac{\Delta}{x-3}$$

$$x+1 = A(x-2)^2(x-3) + B(x-1)(x-2)(x-3) + \Gamma(x-1)(x-3) + \boxed{\Delta(x-1)(x-2)^2}$$

Θέτω $x=1$: $\quad 2 = -2A \Rightarrow \boxed{A=-1}$

$x=2$: $\quad 3 = -\Gamma \Rightarrow \boxed{\Gamma=-3}$

$x=3$: $\quad 4 = 2\Delta \Rightarrow \boxed{\Delta = 2}$

$\longrightarrow \boxed{x=0}$: $\quad 1 = -12A - 6B + 3\Gamma - 4\Delta$

$$1 = 12 - 6B - 9 - 8 \Rightarrow$$

$$-6B = 6 \Rightarrow \boxed{B = -1}$$

$$\int \frac{x+1}{(x-1)(x-2)^2(x-3)}dx = \int \frac{A}{x-1}dx + \int \frac{B}{x-2}dx + \int \frac{\Gamma}{(x-2)^2}dx$$

$$+ \int \frac{\Delta}{x-3}dx$$

$$= -\int \frac{1}{x-1}dx - \int \frac{1}{x-2}dx - 3\int \frac{1}{(x-2)^2}dx + 2\int \frac{1}{x-3}dx$$

$$= -\ln|x-1| - \ln|x-2| - 3\frac{(x-2)^{-2+1}}{-2+1} + 2\ln|x-3| + c$$

$$= -\ln|x-1| - \ln|x-2| + \frac{3}{x-2} + \boxed{2}\ln|x-3| + c, \quad c \in \mathbb{R}$$

$$= \ln\frac{(x-3)^2}{|x-1|\,|x-2|} + \frac{3}{x-2} + c, \quad c \in \mathbb{R}$$

$$\boxed{\begin{array}{l} \ln\frac{\alpha}{\beta} = \ln\alpha - \ln\beta \\ \ln \alpha\cdot\beta = \ln\alpha + \ln\beta \end{array}}$$

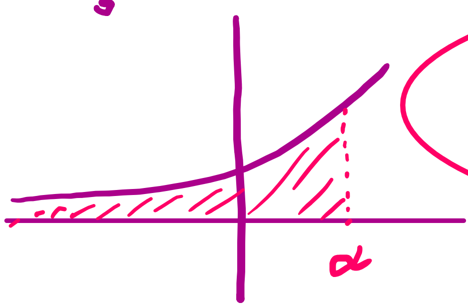
Γενικευμένο Ολοκλήρωμα.

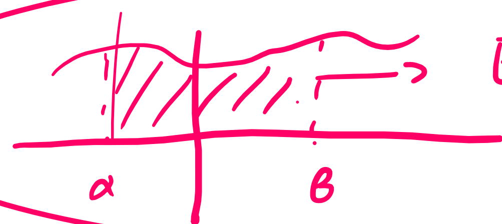
Γενικευμένα Ολοκληρώματα.

$$\int_{\alpha}^{b} f(x)\,dx \qquad f: [\alpha,\beta] \to \mathbb{R}$$

f συνεχής

$$\boxed{\int_{-\infty}^{\alpha} f(x)\,dx}\ , \quad \boxed{\int_{\alpha}^{+\infty} f(x)\,dx}\ , \quad \boxed{\int_{-\infty}^{+\infty} f(x)\,dx}$$





$$E = \int_{\alpha}^{\beta} f(x)\,dx$$

$$1.\bar{9} = 1.999\ldots = 2$$

$$x = 1.999\ldots \Rightarrow$$

$$10x = 19.999\ldots \Rightarrow$$

$$10x = 18 + \boxed{1.999\ldots}$$

$$10x = 18 + x$$

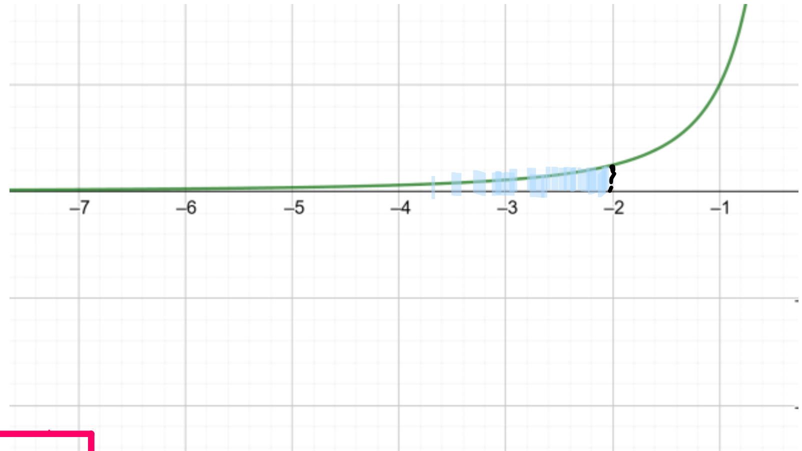
$$10x - x = 18 \Rightarrow 9x = 18 \Rightarrow \boxed{x = 2}$$

<u>π.χ.</u>

$$\int^{-2} \frac{1}{\ldots}\,dx =$$

π.χ.

$$\int_{-\infty} \frac{1}{x^2}\,dx =$$



$$\lim_{k\to-\infty} \boxed{\int_{k}^{-2} \frac{1}{x^2}\,dx} = \lim_{k\to-\infty} \left[\frac{x^{-2+1}}{-2+1}\right]_{k}^{-2} =$$

$$\lim_{k\to-\infty} \left[-\frac{1}{x}\right]_{k}^{-2} = \lim_{k\to-\infty} \left(\frac{1}{2} + \frac{1}{k}\right)$$

$$= \frac{1}{2} + 0 = \frac{1}{2}.$$

Άρα, το γενικευμένο ολοκλήρωμα συγκλίνει στο $\boxed{1/2}$

π.χ.

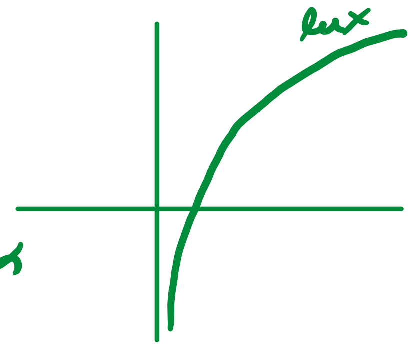
$\int^{+\infty}$ $\int^{k}$

π.χ.

$$\int_2^{+\infty} \frac{1}{x}\,dx = \lim_{k\to+\infty}\int_2^{k} \frac{1}{x}\,dx =$$

$$= \lim_{k\to+\infty}\Big[\ln|x|\Big]_2^{k} = \lim_{k\to+\infty}\left(\ln k - \ln 2\right)$$

$$= +\infty - \ln 2 = \boxed{+\infty}$$



Άρα, το γενικευμένο ολοκλήρωμα αποκλίνει.

$$I = \int_{-\infty}^{+\infty} f(x)\,dx = \overbrace{\int_{-\infty}^{\alpha} f(x)\,dx}^{=I_1} + \overbrace{\int_{\alpha}^{+\infty} f(x)}^{=I_2}$$

$$\boxed{\int_\alpha^\beta f(x)\,dx = \int_\alpha^\gamma f(x)\,dx + \int_\gamma^\beta f(x)\,dx}$$

Για να συγκλίνει το I θα πρέπει να συγκλίνει <u>και</u> το I_1 <u>και</u> το I_2