Notes 23_10_2025

$$\int x^{\kappa}\,dx = \frac{x^{\kappa+1}}{\kappa+1} + c, \quad c \in \mathbb{R}$$

$\kappa \neq -1$

$$\sqrt[\kappa]{x^{\lambda}} = x^{\lambda/\kappa}$$

Π.χ. $\int \sqrt[3]{x^{2}}\,dx = \int x^{2/3}\,dx = \frac{3x^{5/3}}{5} + c$

$$\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx = \frac{x^{-3}}{-3} + c$$

$$\boxed{\int (\alpha x+\beta)^{\kappa}\,dx = \frac{1}{\alpha}\frac{(\alpha x+\beta)^{\kappa+1}}{\kappa+1} + c}$$

Π.χ. $\int (3x+2)^7\,dx = \frac{1}{3}\frac{(3x+2)^8}{8} + c$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$x \neq 0$

$$(\ln|x|)' = \frac{1}{x}, \quad x \neq 0$$

$$\boxed{\int \frac{1}{\alpha x+\beta}\,dx = \frac{1}{\alpha}\ln|\alpha x+\beta| + c}, \quad c \in \mathbb{R}$$

Π.χ. $\int \frac{1}{7x+4}\,dx = \frac{1}{7}\ln|7x+4| + c, \quad c \in \mathbb{R}$

$$\int e^x\,dx = e^x + c$$

$$\int e^{x} dx = e^{x} + c$$

$$\boxed{\int e^{\alpha x+\beta} dx = \frac{1}{\alpha} e^{\alpha x+\beta} + c}$$

π.χ. $\int e^{-x+4} dx = -e^{-x+4} + c$

$$\int \alpha^{x} dx = \frac{\alpha^{x}}{\ln\alpha} + c \qquad \left((\alpha^{x})' = \alpha^{x} \cdot \ln\alpha \right)$$

π.χ. $\int 3^{x} dx = \frac{3^{x}}{\ln 3} + c$

$$\boxed{\int \alpha^{\kappa x+\lambda} dx = \frac{1}{\kappa} \cdot \frac{\alpha^{\kappa x+\lambda}}{\ln\alpha} + c}$$

$f(x) = \alpha^{x}$

$0<\alpha<1 \Rightarrow f \downarrow$, $\alpha>1 \Rightarrow f \uparrow$

π.χ. $\int 3^{2x+4} dx = \frac{1}{2} \cdot \frac{3^{2x+4}}{\ln 3} + c$

ημίτονο

$$\int \sin x \, dx = -\cos x + c$$

$$\boxed{\int \sin(\alpha x+\beta) dx = -\frac{1}{\alpha}\cos(\alpha x+\beta) + c}$$

$$\int \cos x \, dx = \sin x + c \longrightarrow \boxed{\int \cos(\alpha x+\beta) dx = \frac{1}{\alpha}\sin(\alpha x+\beta) + c}$$

$$\int f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)\,dx =$$

$$\int f_1(x)dx \pm \int f_2(x)dx \pm \cdots \pm \int f_n(x)\,dx$$

$$\int \lambda \cdot f(x)\,dx = \lambda \cdot \int f(x)\,dx$$

Παραδείγματα:

▶Παράδειγμα 1.1. Να υπολογισθούν τα ολοκληρώματα:

i. $\int\left(2x^4 - 5\sin x + 7\sqrt{x}\right)dx$ ii. $\int \sqrt{x}\left(1+\sqrt[3]{x}\right)^2 dx$

iii. $\int \frac{\left(\sqrt{x}-1\right)^2}{x}dx$

i) $\int 2x^4 - 5\sin x + 7\sqrt{x}\,dx = 2\int x^4 dx - 5\cdot\int \sin x\,dx +$

$+ 7\int x^{1/2}dx = 2\frac{x^5}{5} + 5\cos x + 7\frac{x^{3/2}}{3/2} + c$

$= \frac{2x^5}{5} + 5\cos + \frac{14x^{3/2}}{3} + c$.

ii) $\int \sqrt{x}\left(1+\sqrt[3]{x}\right)^2 dx =$

$\int \boxed{\sqrt{x}}\left(1 + 2\sqrt[3]{x} + \boxed{\sqrt[3]{x^2}}\right)dx =$

$(\alpha \pm \beta)^2 = \alpha^2 \pm 2\alpha\beta + \beta^2$

$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$(\alpha-\beta)^3 = \alpha^3 - 3\alpha^2\beta + 3\alpha\beta^2 - \beta^3$

$\int \sqrt{x}\,(1 + 2\sqrt[3]{x} + \sqrt[3]{x})\,dx =$

$\int x^{1/2} + 2x^{5/6} + x^{7/6}\,dx =$

$\dfrac{x^{3/2}}{3/2} + 2\cdot\dfrac{x^{11/6}}{11/6} + \dfrac{x^{13/6}}{13/6} + =$

$\dfrac{2x^{3/2}}{3} + \dfrac{12\cdot x^{11/6}}{11} + \dfrac{6x^{13/6}}{13} + C.$

$(\alpha-\beta)^3 = \alpha^3 - 3\alpha^2\beta + 3\alpha\beta^2 - \beta^3$

$\sqrt[k]{x^1}\cdot\sqrt[\lambda]{x^1} = x^{1/k}\cdot x^{1/\lambda}$

$= x^{\frac{1}{k}+\frac{1}{\lambda}} = x^{\frac{k+\lambda}{k\cdot\lambda}} =$

$= \sqrt[k\cdot\lambda]{x^{k+\lambda}}$

$\sqrt{x}\cdot\sqrt[3]{x} = x^{1/2}\cdot x^{1/3} =$

$= x^{\frac{1}{2}+\frac{1}{3}} = x^{5/6}$

$\sqrt{x}\cdot\sqrt[3]{x^2} =$

$x^{1/2}\cdot x^{2/3} =$

$x^{\frac{1}{2}+\frac{2}{3}} = x^{7/6}$

iii. $\displaystyle\int \frac{(\sqrt{x}-1)^2}{x}\,dx = \int \frac{x - 2\sqrt{x} + 1}{x}\,dx =$

$\boxed{(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2}$

$x^{1/2-1}$

$= \displaystyle\int \frac{x}{x} - \frac{2\sqrt{x}}{x} + \frac{1}{x}\,dx = \int 1\,dx - 2\int x^{-1/2}\,dx + \int \frac{1}{x}\,dx$

$= x - 2\,\dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \ln|x| + c = x - 4\cdot x^{1/2} + \ln|x| + c,$

$c \in \mathbb{R}$

Ολοκλήρωση κατά παράγοντες:

$$\boxed{\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\,g(x)\,dx}$$

- $\int p(x)\cdot \ln|x|\,dx$ $\quad p(x)=$ πολυώνυμο

$\int [P(x)]'\cdot \ln|x|\,dx$ $\qquad \int (x^2+4)\cdot \ln|x|\,dx$

$x^2+4 = \left(\frac{x^3}{3}+4x\right)'$

$f(x)$

π.χ. $\int (x^2+4)\cdot \ln|x|\,dx = \int \underbrace{\left(\frac{x^3}{3}+4x\right)'}_{g(x)}\cdot \boxed{\ln|x|}\,dx =$

$\left(\frac{x^3}{3}+4x\right)\cdot \ln|x| - \int \left(\frac{x^3}{3}+4x\right)\cdot \frac{1}{x}\,dx =$

$\left(\frac{x^3}{3}+4x\right)\ln|x| - \int \frac{1}{3}x^2+4\,dx =$

$\left(\frac{x^3}{3}+4x\right)\ln|x| - \frac{x^3}{9}+4x+c,\ c\in\mathbb{R}.$

σταθερό πολυώνυμο

π.χ. $\int \ln x\,dx = \int 1\cdot \ln x\,dx =$

π.χ. $\int \ln x \, dx = \int 1' \cdot \ln x \, dx =$

$\int (x)' \cdot \ln x \, dx = x \cdot \ln x - \int x \cdot \frac{1}{x} dx =$

$= x \ln x - x + c$

$$\boxed{\int \ln x \, dx = x \ln x - x + c}$$

- $\int e^x \cdot p(x) \, dx =$, $p(x) =$ πολυώνυμο

$= \int (e^x)' \cdot p(x) \, dx$

π.χ. $\int e^x (x^2 + 4x + 3) \, dx = \int (e^x)' \cdot (x^2 + 4x + 3) dx =$

$= e^x \cdot (x^2 + 4x + 3) - \int e^x \cdot (2x + 4) \, dx =$

$= e^x (x^2 + 4x + 3) - \left[\underbrace{\int (e^x)' (2x + 4) dx} \right] =$

$= e^x (x^2 + 4x + 3) - \left[e^x \cdot (2x + 4) - \int e^x (2x + 4)' dx \right] =$

$$= e^x(x^2+4x+3) - \left[e^x\cdot(2x+4) - \int e^x (2x+4)'\,dx\right] =$$

$$= e^x(x^2+4x+3) - e^x(2x+4) + 2\int e^x\,dx =$$

$$= e^x(x^2+2x-1) + 2e^x + c, \quad c\in\mathbb{R}.$$

- $\int e^x\cdot\sin x\,dx$ ή $\int e^x\cdot\cos x\,dx$

$$I = \int \boxed{e^x}\sin x\,dx = \int (e^x)'\cdot\sin x\,dx$$

$$e^x\cdot\sin x - \underbrace{\int \boxed{e^x}\cos x\,dx} = e^x\cdot\sin x - \int (e^x)'\cdot\cos x\,dx$$

$$= e^x\cdot\sin x - \left[e^x\cdot\cos x + \int e^x\cdot\sin x\,dx\right] \Rightarrow$$

$$I = e^x\cdot\sin x - e^x\cdot\cos x - I \Rightarrow$$

$$2I = e^x\cdot\sin x - e^x\cdot\cos x \Rightarrow$$

$$I = \frac{e^x(\sin x - \cos x)}{} + c$$

$$I = \frac{e^x(\sin x - \cos x)}{2} + c$$

Μέθοδος της αντικατάστασης :

$$\int f(\boxed{\overset{u}{g(x)}}) \cdot \boxed{g'(x)\,dx} = \int f(u)\,du$$

Θέτω $u = g(x) \Rightarrow$

$$\boxed{\frac{du}{dx} = g'(x) \Rightarrow}$$

$$\boxed{du = g'(x) \cdot dx}$$

$$\int (\underbrace{x^2+3x+7}_{g(x)})^2 \cdot (\underbrace{2x+3}_{g'(x)})\,dx$$

$$f'(x) = \frac{df}{dx}$$

▶ **Παράδειγμα 3.1.** Να υπολογισθούν τα ολοκληρώματα:

i. $\int 3x^2(x^3+5)^7 dx$ **ii.** $\int (x^3+2x+6)^7(3x^2+2)\,dx$

i) $\int \underbrace{3x^2}_{g'(x)} \cdot (\underbrace{x^3+5}_{g(x)})^7 dx =$

Θέτω $u = x^3+5 \Rightarrow \frac{du}{dx} = (x^3+5)' \Rightarrow$

$\frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2 \cdot dx$

$$\frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2\,dx$$

$$\int u^7\,du = \frac{u^8}{8} + c =$$

$$= \frac{(x^3+5)^8}{8} + c, \quad c \in \mathbb{R}$$

$$\int \underbrace{(x^3 + 2x + 6)}_{g(x)}{}^7\,\underbrace{(3x^2 + 2)}_{g'(x)}\,dx$$

Θέτω $u = x^3 + 2x + 6$

$$\frac{du}{dx} = 3x^2 + 2 \Rightarrow \boxed{du = (3x^2+2)\,dx}$$

$$\int u^7\,du = \frac{u^8}{8} + c =$$

$$= \frac{(x^3+2x+6)^8}{8} + c, \quad c \in \mathbb{R}.$$

▶ Παράδειγμα 3.3. Να υπολογισθεί το ολοκλήρωμα: $\int (ln\,x)^3 \frac{1}{x}\,dx$

▶ **Παράδειγμα 3.3.** Να υπολογισθεί το ολοκλήρωμα: $\int (ln\,x)^3 \frac{1}{x} dx$

$$\int (\ln x)^3 \cdot \frac{1}{x} dx$$

$\ln x \to g(x)$, $\frac{1}{x} \to g'(x)$

$$\int \frac{(\ln x)^3}{x} dx$$

Θέτω $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} \cdot dx$

$$\int u^3 \cdot du = \frac{u^4}{4} + c = \frac{(\ln x)^4}{4} + c,\ c \in \mathbb{R}$$

▶ **Παράδειγμα 3.4.** Να υπολογισθούν τα ολοκληρώματα:

i. $\int sin^4 x\, cos\, x\ dx$ ii. $\int \sqrt{sin\, x}\, cos\, x\ dx$

$$\int \frac{\sin x}{x} dx$$

ii) $\int \sqrt{\sin x} \cdot \cos x\, dx =$

Θέτω $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow \boxed{du = \cos x \cdot dx}$

$$= \int \sqrt{u} \cdot du = \frac{u^{3/2}}{3/2} + c =$$

$$= \frac{2\sqrt{\sin^3 x}}{3} + c, \quad c \in \mathbb{R}$$

Να υπολογισθεί το ολοκλήρωμα:

$$\int e^{\sin x} \sin x \cos x \, dx =$$

Θέτω $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x\, dx$

$$= \int e^u \cdot u \cdot du = \int u \cdot e^u du =$$

$$= \int u \cdot (e^u)' du = u \cdot e^u - \int (u)' \cdot e^u du =$$

$$u \cdot e^u - \int e^u du = u \cdot e^u - e^u = (u-1)e^u =$$

$$= (\sin x - 1)e^{\sin x} + c, \quad c \in \mathbb{R}.$$

Να υπολογισθούν τα παρακάτω ολοκληρώματα. :

25. $\int ln(1-x)dx, \quad \int ln(1-x^2)dx$

$$\ln(\alpha \cdot \beta) = \ln \alpha + \ln \beta$$
$$\ln\left(\frac{\alpha}{\beta}\right) = \ln \alpha - \ln \beta$$
$$\ln \alpha^{\kappa} = \kappa \cdot \ln \alpha$$

$$\int \ln(1-x)\, dx =$$

$u = 1-x \Rightarrow \frac{du}{dx} = -1 \Rightarrow$

$$u = 1-x \quad \frac{du}{dx} = -1$$

$$\boxed{du = -1 \cdot dx} \Rightarrow -du = dx$$

(Βλέπε προηγούμενες σημειώσεις)

$$\int \ln(u)\cdot(-1)\,du = -\int \ln u\,du =$$

$$= -\left(u\cdot\ln u - u\right) + c =$$

$$= -(1-x)\ln(1-x) + (1-x) + c,\quad c\in\mathbb{R}.$$

$$\int \ln(1-x^2)dx = \int \ln\Big[\underbrace{(1-x)(1+x)}_{\ln \alpha\cdot\beta = \ln\alpha + \ln\beta}\Big]dx =$$

$$= \int \ln(1-x) + \ln(1+x)\,dx =$$

$$\boxed{\int \ln(1-x)dx} + \int \ln(1+x)\,dx =$$

Βλέπε προηγούμενη άσκηση

Θέτω $u = 1+x \Rightarrow$

$$\frac{du}{dx} = 1 \Rightarrow \boxed{du = dx}$$

$$\int \ln u\,du =$$

$$= u\ln u - u + c =$$

$$= (1+x)\ln(1+x) - (1+x) + c,\quad c\in\mathbb{R}$$

κ.λ.π.

$$\int_1^e (x^2+x)\cdot \ln x \, dx$$

$$\boxed{\int_\alpha^\beta f(x)\cdot g'(x)dx = \Big[f(x)\cdot g(x)\Big]_\alpha^\beta - \int_a^\beta f'(x)\cdot g(x)dx}$$

$$= \int_1^e \left(\frac{x^3}{3}+\frac{x^2}{2}\right)'\cdot \ln x \, dx = \left[\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\ln x\right]_1^e -$$

$$\int_1^e \left(\frac{x^3}{3}+\frac{x^2}{2}\right)\cdot \frac{1}{x}dx =$$

$$= \left[\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\cdot \ln x\right]_1^e - \int_1^e \frac{x^2}{3}+\frac{x}{2}dx =$$

$\ln 1 = 0$

$$= \left(\frac{e^3}{3}+\frac{e^2}{2}\right) - 0 - \left[\frac{x^3}{9}+\frac{x^2}{4}\right]_1^e =$$

$$= \frac{e^3}{3}+\frac{e^2}{2} - \left(\frac{e^3}{9}+\frac{e^2}{4}-\frac{1}{9}-\frac{1}{4}\right) = \dots$$

- $$\int_1^e \frac{(\ln x)^2}{x} dx = \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\int_1 \frac{(\ln x)}{x}\,dx = \int_0 u\,du = \left[\frac{\ }{3}\right]_0 = \frac{\ }{3}$$

Θέτω $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow \boxed{du = \frac{1}{x}dx}$

Για $x = 1 \rightsquigarrow u = \ln 1 = 0$

Για $x = e \rightsquigarrow u = \ln e = 1$

$$\int \frac{P(x)}{Q(x)}\,dx$$ → Ολοκλήρωμα ρητής πολυωνυμικής συνάρτησης

βαθμός του $P(x)$

$$P(x) = \alpha_\nu x^{\boxed{\nu}} + \alpha_{\nu-1} x^{\nu-1} + \dots + \alpha_1 \cdot x + \alpha_0$$

$$Q(x) = \beta_\mu \cdot x^{\boxed{\mu}} + \beta_{\mu-1} \cdot x^{\mu-1} + \dots + \beta_1 \cdot x + \beta_0$$

1) Αν ο βαθμός του αριθμητή είναι μεγαλύτερος ή ίσος με τον βαθμό του παρανομαστή τότε κάνουμε διαίρεση του $P(x)$ με το $Q(x)$

$$\begin{array}{l|l} P(x) & \underline{Q(x)} \\ \vdots & \Pi(x) \\ \upsilon(x) & \end{array}$$

βαθμός $(\upsilon(x)) <$ βαθμός $(Q(x))$

Ταυτότητα διαίρεσης: $\boxed{P(x) = \pi(x)\cdot Q(x) + \upsilon(x)}$