Θέμα 1ον

$$y' = \frac{3x^2-2y-y^3}{2x+3xy^2}$$

$$\Updownarrow$$

$$\underbrace{(2x+3xy^2)}_{Q}dy + \underbrace{(-3x^2+2y+y^3)}_{P}dx = 0$$

$$\frac{\partial P}{\partial y} = 2+3y^2 \quad , \quad \frac{\partial Q}{\partial x} = 2+3y^2$$

άρα υπάρχει λύση.

Ζητάμε Φ:

$$\frac{\partial \phi}{\partial x} = -3x^2+2y+y^3 \quad , \quad \frac{\partial \phi}{\partial y} = 2x+3xy^2$$

$$\Downarrow$$

$$\Phi = \int(-3x^2+2y+y^3)dx = -\frac{3x^3}{3} + 2yx + y^3x + G(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = \frac{\partial(-x^3+2yx+y^3x+G(y))}{\partial y} = 2x+3xy^2$$

$$\Rightarrow 2x + 3y^2x + G'(y) = 2x+3xy^2$$

$$\Rightarrow G'(y) = 0 \Rightarrow G(y) = K$$

$$\Rightarrow \Phi = -x^3 + 2xy + xy^3 + K$$

άρα η λύση $y(x)$ της Δ.Ε. δίνεται πεπλεγμένα από την σχέση:

$$\boxed{-x^3 + 2xy(x) + xy^3(x) + K = C}$$

$\Leftarrow$

Θέμα $2^{ον}$: Με D-τελεστές έχουμε:

$$(D-3D^0)x(t) - 6D^0 y(t) = t^2$$

$$D x(t) + (D-3D^0) y(t) = e^t$$

$$\Downarrow$$

$$\Delta = \begin{vmatrix} D-3D^0 & -6D^0 \\ D & D-3D^0 \end{vmatrix} = D^2 + 9D^0$$

$$\Downarrow$$

$$\Delta_x = \begin{vmatrix} t^2 & -6D^0 \\ e^t & D-3D^0 \end{vmatrix} = 2t - 3t^2 + 6e^t$$

$$\Delta_y = \begin{vmatrix} D-3D^0 & t^2 \\ D & e^t \end{vmatrix} = e^t - 3e^t - 2t =$$

$$= -2e^t - 2t$$

$$x(t) = \frac{\Delta_x}{\Delta} \Rightarrow x'' + 9x = \cancel{e^t - 3e^t - 2t}\ 6e^t - 3t^2 + 2t \quad (\mathrm{I})$$

$$y(t) = \frac{\Delta_y}{\Delta} \Rightarrow y'' + 9y = -2e^t - 2t \quad (\mathrm{II})$$

Χαρ. Εξίσωση $= \lambda^2+9 \Rightarrow \lambda_1 = 3i, \ \lambda_2 = -3i$

$\Rightarrow \alpha = 0, \ \beta = 3 \Rightarrow$

$$\boxed{\begin{aligned} x_{om}(t) &= c_1 \text{ημ}3t + c_2 \text{συν}3t \\ y_{om}(t) &= \tilde{c}_1 \text{ημ}3t + \tilde{c}_2 \text{συν}3t \end{aligned}}$$

$x_m(t) = Ae^t + Bt^2 + \Gamma t + \Delta \Rightarrow$

$x_m''(t) = Ae^t + 2B \Rightarrow$

$Ae^t + 2B + 9(Ae^t + Bt^2 + \Gamma t + \Delta) = 6e^t - 3t^2 + 2t$

$$\Rightarrow \left.\begin{aligned} 10A &= 6 \\ 9B &= -3 \\ 9\Gamma &= 2 \\ 2B + 9\Delta &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 3/5 \\ B &= -1/3 \\ \Gamma &= 2/9 \end{aligned}, \quad \Delta = \frac{2}{27}$$

$x_m(t) = \frac{3}{5}e^t - \frac{t^2}{3} + \frac{2}{9}t + \frac{2}{27} \Rightarrow$

$$\boxed{x(t) = c_1 \text{ημ}3t + c_2 \text{συν}3t + \frac{3}{5}e^t - \frac{t^2}{3} + \frac{2}{9}t + \frac{2}{27}}$$

$$\left.\begin{aligned} Y_m(t) &= Ae^t + Bt + \Gamma \\ Y_m''(t) &= Ae^t \end{aligned}\right\} \Rightarrow$$

$$Ae^t + 9(Ae^t + Bt + \Gamma) = -2e^t - 2t \Rightarrow$$

$$\left.\begin{aligned} 10A &= -2 \\ 9B &= -2 \\ 9\Gamma &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\frac{1}{5} \\ B &= -\frac{2}{9} \\ \Gamma &= 0 \end{aligned} \Rightarrow$$

$$Y_m(t) = -\frac{e^t}{5} - \frac{2}{9}t \Rightarrow$$

$$\Rightarrow \boxed{Y(t) = \tilde{C}_1 \,\text{ημ}\, 3t + \tilde{C}_2 \,\text{συν}\, 3t - \frac{e^t}{5} - \frac{2}{9}t}$$

Θέμα $3^{ον}$: Το F είναι:

$F = ty' - (y')^2$, η εξίσωση Euler είναι:

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow 0 - \frac{d}{dt}\left(t - 2y'\right) = 0$$

$$\Rightarrow 1 - 2y'' = 0 \Rightarrow \boxed{2y'' = 1}$$

$$\left.\begin{array}{l} \Rightarrow y(t) = \frac{t^2}{4} + c_1 t + c_2 \\ y(0) = 0 \Rightarrow c_2 = 0 \\ y(4) = 3 \Rightarrow 3 = \frac{16}{4} + 4c_1 \Rightarrow c_1 = -1/4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{y^*(t) = \frac{t^2}{4} - \frac{t}{4}}$$

Συνθήκη δευτέρας τάξεως =

Φτιάχνουμε τόν πίνακα:

$$\begin{pmatrix} F_{y'y'} & F_{yy'} \\ F_{y'y} & F_{yy} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

οι ιδιοτιμές είναι

$-2 < 0 \quad , \quad 0 \leq 0$

άρα αρνητικά ημιορισμένος,

άρα έχουμε __μέγιστο__

Θέμα $4^{ον}$: χαρ. εξίσωση:

$$\lambda^2 - 4\lambda + 16 = 0 \Rightarrow$$

$$\Rightarrow \lambda_1 = 2 - 2\sqrt{3}\,i, \quad \lambda_2 = 2 + 2\sqrt{3}\,i$$

$$\Rightarrow \alpha = 2, \quad \beta = 2\sqrt{3}$$

$$r = \sqrt{4 + 4\cdot 3} = \sqrt{16} = 4, \quad \eta\mu\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{3}} \Rightarrow$$

$$\Rightarrow \boxed{Y_n^{\text{ομ}} = c_1 4^n \,\text{συν}\left(n\cdot\frac{\pi}{3} + c_2\right)}$$

$$Y_n^* = A \Rightarrow A - 4A + 16A = 26 \Rightarrow 13A = 26 \Rightarrow \boxed{A = 2}$$

$$Y_n^* = 2 \Rightarrow$$

$$\Rightarrow \boxed{Y_n = c_1 4^n \,\text{συν}\left(n\cdot\frac{\pi}{3} + c_2\right) + 2}$$

(β) $y_{n+2} + y_{n+1} - 2y_n = n$, χαρ. Εξίσωση

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -2 \end{matrix} \Rightarrow$$

$$Y_n^{om} = C_1 \cdot 1^n + C_2 (-2)^n = C_1 + C_2 (-2)^n$$

$$y_n^* = An^2 + Bn + \Gamma \Longrightarrow$$

$$A(n+2)^2 + B(n+2) + \Gamma + A(n+1)^2 + B(n+1) + \Gamma -$$

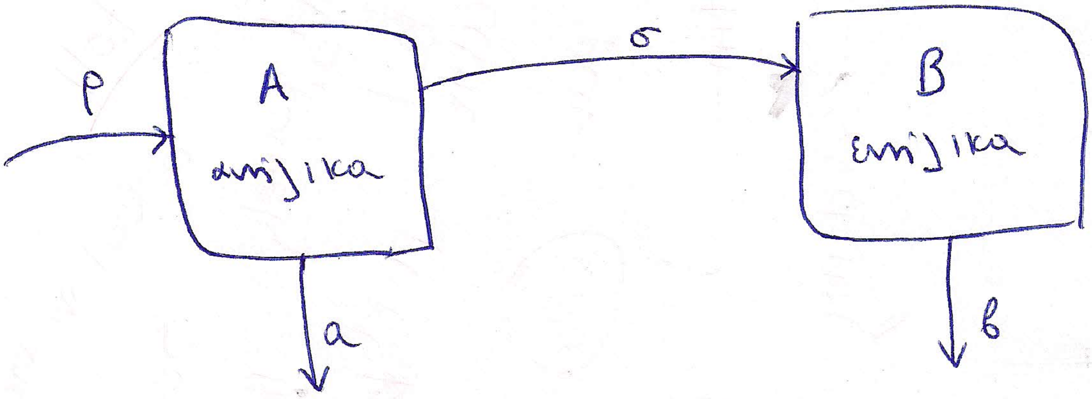
$$-2\left[An^2 + Bn + \Gamma\right] = n \Longrightarrow$$

$$\Rightarrow 6An + 5A + 3B = n \Rightarrow$$

$$\begin{matrix} 6A = 1 \\ 5A + 3B = 0 \end{matrix} \Rightarrow \begin{matrix} A = 1/6 \\ B = -\frac{5}{18} \end{matrix} \Rightarrow y_n^* = \frac{n^2}{6} - \frac{5}{18}n + \Gamma$$

$$\Rightarrow \boxed{Y_n = C_1 + C_2(-2)^n + \frac{n^2}{6} - \frac{5}{18}n + \Gamma}$$

Θέμα 5ον: Σχηματογραφικά έχουμε:



Αν A_n, B_n οι αντίστοιχοι πληθυσμοί έχουμε:

$$\left.\begin{aligned} A_{n+1} &= A_n - \sigma A_n - \alpha A_n + \rho B_n \\ B_{n+1} &= B_n - \beta B_n + \sigma A_n \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 1-\sigma-\alpha & \rho \\ \sigma & 1-\beta \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

$$\Downarrow$$

$$\vec{X}_{n+1} = \mathcal{A} \vec{X}_n$$ ομογενές σύστημα

Θέτοντας $p=\frac{2}{15}$, $\alpha=\frac{1}{15}$, $b=\frac{7}{15}$, $\sigma=\frac{4}{15}$ έχουμε:

$$A=\begin{pmatrix} \frac{2}{3} & \frac{2}{15} \\ \frac{4}{15} & \frac{8}{15} \end{pmatrix} \Rightarrow$$

$$\Rightarrow |A-\lambda I|=0 \Leftrightarrow \left(\frac{2}{3}-\lambda\right)\left(\frac{8}{15}-\lambda\right)-\frac{4}{15}\cdot\frac{2}{15}=0$$

$$\Rightarrow \frac{2}{3}\cdot\frac{8}{15}-\frac{2}{3}\lambda-\frac{8}{15}\lambda+\lambda^2-\frac{4}{15}\cdot\frac{2}{15}=0 \xrightarrow{\text{πολ. } 15^2}$$

$$15^{\cancel{2}}\cdot\frac{2}{3}\cdot\frac{8}{\cancel{15}}-15^2\frac{2}{3}\lambda-\frac{8}{15}15^2\lambda+15^2\lambda^2-\frac{4}{15}\cdot\frac{2}{15}\cdot15^2=0 \Rightarrow$$

$$\Rightarrow 80-150\lambda-120\lambda+15^2\lambda^2-8=0 \Rightarrow$$

$$\Rightarrow 15^2\lambda^2-270\lambda+72=0 \Rightarrow \boxed{\left(\lambda^2-\frac{6}{5}\lambda+\frac{8}{25}\right)=0}$$

$$\Rightarrow \boxed{\lambda_1=\frac{4}{5}, \quad \lambda_2=\frac{2}{5}}$$

Ιδιοδιανύσματα για $\lambda_1=\frac{4}{5}$

$$\begin{pmatrix} \frac{2}{3}-\frac{4}{5} & \frac{2}{15} \\ \frac{4}{15} & \frac{8}{15}-\frac{4}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{2}{15} & \frac{2}{15} \\ \frac{4}{15} & -\frac{4}{15} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$

$$\Rightarrow -x+y=0 \Rightarrow y=x \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

για $\lambda_2 = \frac{2}{5}$ το ιδιοδιάνυσμα είναι $\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$$\Downarrow$$

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = c_1 \left(\frac{4}{5}\right)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left(\frac{2}{5}\right)^n \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

Για $n=0$ έχουμε:

$$\left.\begin{array}{l} A_0 = c_1 - \frac{c_2}{2} \\ B_0 = c_1 + c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 - \frac{c_2}{2} = 1 \\ c_1 + c_2 = 2 \end{array} \Rightarrow \begin{array}{l} c_1 = \frac{4}{3} \\ c_2 = \frac{2}{3} \end{array}$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} A_n = \frac{4}{3} \cdot \left(\frac{4}{5}\right)^n + \frac{2}{3} \cdot \left(\frac{2}{5}\right)^n \cdot \left(-\frac{1}{2}\right) \\ B_n = \frac{4}{3} \left(\frac{4}{5}\right)^n + \frac{2}{3} \left(\frac{2}{5}\right)^n \end{array}}$$