Exercise 2.4 *Suppose a firm's production function has the* Cobb-Douglas *form*

$$q = z_1^{\alpha_1} z_2^{\alpha_2}$$

where z_1 and z_2 are inputs, q is output and α_1, α_2 are positive parameters.

1. *Draw the isoquants. Do they touch the axes?*
2. *What is the elasticity of substitution in this case?*
3. *Using the Lagrangean method find the cost-minimising values of the inputs and the cost function.*
4. *Under what circumstances will the production function exhibit (a) decreasing (b) constant (c) increasing returns to scale? Explain this using first the production function and then the cost function.*
5. *Find the conditional demand curve for input 1.*

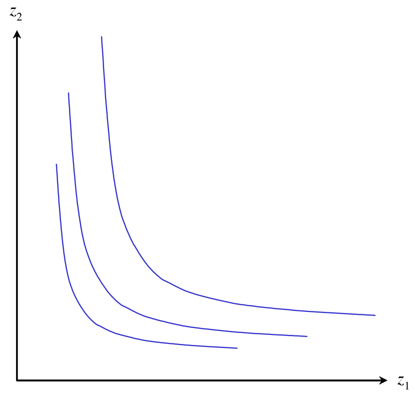

Figure 2.7: Isoquants: Cobb-Douglas

Outline Answer

1. The isoquants are illustrated in Figure 2.7. They do not touch the axes.
2. The elasticity of substitution is defined as

$$\sigma_{ij} := -\frac{\partial \log (z_j/z_i)}{\partial \log \left(\phi_j(\mathbf{z})/\phi_i(\mathbf{z})\right)}$$

which, in the two input case, becomes

$$\sigma = -\frac{\partial \log \left(\frac{z_1}{z_2}\right)}{\partial \log \left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)} \tag{2.1}$$

In case 1 we have $\phi(\mathbf{z}) = z_1^{\alpha_1} z_2^{\alpha_2}$ and so, by differentiation, we find:

$$\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})} = \frac{\alpha_1}{\alpha_2} / \frac{z_1}{z_2}$$

Taking logarithms we have

$$\log\left(\frac{z_1}{z_2}\right) = \log\left(\frac{\alpha_1}{\alpha_2}\right) - \log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)$$

or

$$u = \log\left(\frac{\alpha_1}{\alpha_2}\right) - v$$

where $u := \log(z_1/z_2)$ and $v := \log(\phi_1/\phi_2)$. Differentiating u with respect to v we have

$$\frac{\partial u}{\partial v} = -1. \tag{2.2}$$

So, using the definitions of u and v in equation (2.2) we have

$$\sigma = -\frac{\partial u}{\partial v} = 1.$$

3. This is a *Cobb-Douglas production function*. This will yield a unique interior solution; the Lagrangean is:

$$\mathcal{L}(\mathbf{z}, \lambda) = w_1 z_1 + w_2 z_2 + \lambda\left[q - z_1^{\alpha_1} z_2^{\alpha_2}\right] , \tag{2.3}$$

and the first-order conditions are:

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial z_1} = w_1 - \lambda \alpha_1 z_1^{\alpha_1 - 1} z_2^{\alpha_2} = 0 , \tag{2.4}$$

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial z_2} = w_2 - \lambda \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2 - 1} = 0 , \tag{2.5}$$

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial \lambda} = q - z_1^{\alpha_1} z_2^{\alpha_2} = 0 . \tag{2.6}$$

Using these conditions and rearranging we can get an expression for minimized cost in terms of and q:

$$w_1 z_1 + w_2 z_2 = \lambda \alpha_1 z_1^{\alpha_1} z_2^{\alpha_2} + \lambda \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2} = [\alpha_1 + \alpha_2] \lambda q.$$

We can then eliminate λ:

$$\left.\begin{array}{l} w_1 - \lambda \alpha_1 \frac{q}{z_1} = 0 \\ w_2 - \lambda \alpha_2 \frac{q}{z_2} = 0 \end{array}\right\}$$

which implies

$$\left.\begin{array}{l} z_1^* = \frac{\alpha_1}{w_1} \lambda q \\ z_2^* = \frac{\alpha_2}{w_2} \lambda q \end{array}\right\}. \tag{2.7}$$

Substituting the values of z_1^* and z_2^* back in the production function we have

$$\left[\frac{\alpha_1}{w_1} \lambda q\right]^{\alpha_1} \left[\frac{\alpha_2}{w_2} \lambda q\right]^{\alpha_2} = q$$

which implies

$$\lambda q = \left[q \left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2} \right]^{\frac{1}{\alpha_1+\alpha_2}} \tag{2.8}$$

So, using (2.7) and (2.8), the corresponding cost function is

$$\begin{aligned} C(\mathbf{w}, q) &= w_1 z_1^* + w_2 z_2^* \\ &= [\alpha_1 + \alpha_2] \left[q \left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2} \right]^{\frac{1}{\alpha_1+\alpha_2}}. \end{aligned}$$

4. Using the production functions we have, for any $t > 0$:

$$\phi(t\mathbf{z}) = [tz_1]^{\alpha_1} [tz_2]^{\alpha_2} = t^{\alpha_1+\alpha_2}\phi(\mathbf{z}).$$

Therefore we have DRTS/CRTS/IRTS according as $\alpha_1 + \alpha_2 \lesseqqgtr 1$. If we look at average cost as a function of q we find that AC is increasing/constant/decreasing in q according as $\alpha_1 + \alpha_2 \lesseqqgtr 1$.

5. Using (2.7) and (2.8) conditional demand functions are

$$H^1(\mathbf{w}, q) = \left[q \left[\frac{\alpha_1 w_2}{\alpha_2 w_1}\right]^{\alpha_2} \right]^{\frac{1}{\alpha_1+\alpha_2}}$$

$$H^2(\mathbf{w}, q) = \left[q \left[\frac{\alpha_2 w_1}{\alpha_1 w_2}\right]^{\alpha_1} \right]^{\frac{1}{\alpha_1+\alpha_2}}$$

and are smooth with respect to input prices.

Exercise 2.5 *Suppose a firm's production function has the* Leontief *form*

$$q = \min\left\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\right\}$$

where the notation is the same as in Exercise 2.4.

1. *Draw the isoquants.*
2. *For a given level of output identify the cost-minimising input combination(s) on the diagram.*
3. *Hence write down the cost function in this case. Why would the Lagrangean method of Exercise 2.4 be inappropriate here?*
4. *What is the conditional input demand curve for input 1?*
5. *Repeat parts 1-4 for each of the two production functions*

$$\begin{aligned} q &= \alpha_1 z_1 + \alpha_2 z_2 \\ q &= \alpha_1 z_1^2 + \alpha_2 z_2^2 \end{aligned}$$

   Explain carefully how the solution to the cost-minimisation problem differs in these two cases.

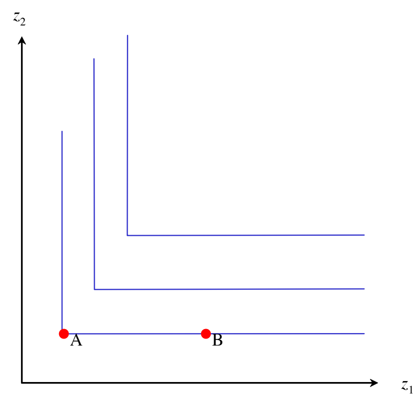

Figure 2.8: Isoquants: Leontief

Outline Answer

1. The Isoquants are illustrated in Figure 2.8 – the so-called *Leontief* case,
2. If all prices are positive, we have a unique cost-minimising solution at A: to see this, draw any straight line with positive finite slope through A and take this as an isocost line; if we considered any other point B on the isoquant through A then an isocost line through B (same slope as the one through A) must lie above the one you have just drawn.

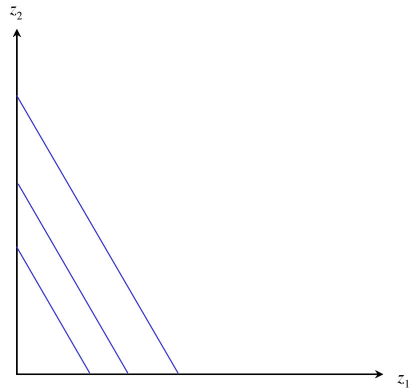

Figure 2.9: Isoquants: linear

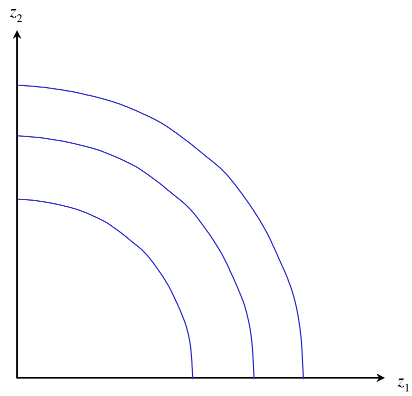

Figure 2.10: Isoquants: non-convex to origin

3. The coordinates of the corner A are $(\alpha_1 q, \alpha_2 q)$ and, given $\mathbf{w}$, this immediately yields the minimised cost.

$$C(\mathbf{w}, q) = w_1\alpha_1 q + w_2\alpha_2 q.$$

The methods in Exercise 2.4 since the Lagrangean is not differentiable at the corner.

4. Conditional demand is constant if all prices are positive

$$\begin{aligned} H^1(\mathbf{w}, q) &= \alpha_1 q \\ H^2(\mathbf{w}, q) &= \alpha_2 q. \end{aligned}$$

5. Given the linear case

$$q = \alpha_1 z_1 + \alpha_2 z_2$$

- Isoquants are as in Figure 2.9.
- It is obvious that the solution will be either at the corner $(q/\alpha_1, 0)$ if $w_1/w_2 < \alpha_1/\alpha_2$ or at the corner $(0, q/\alpha_2)$ if $w_1/w_2 > \alpha_1/\alpha_2$, or otherwise anywhere on the isoquant
- This immediately shows us that minimised cost must be.

$$C(\mathbf{w}, q) = q \min\left\{\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}\right\}$$

- So conditional demand can be multivalued:

$$H^1(\mathbf{w}, q) = \begin{cases} \frac{q}{\alpha_1} & \text{if } \frac{w_1}{w_2} < \frac{\alpha_1}{\alpha_2} \\ z_1^* \in \left[0, \frac{q}{\alpha_1}\right] & \text{if } \frac{w_1}{w_2} = \frac{\alpha_1}{\alpha_2} \\ 0 & \text{if } \frac{w_1}{w_2} > \frac{\alpha_1}{\alpha_2} \end{cases}$$

$$H^2(\mathbf{w}, q) = \begin{cases} 0 & \text{if } \frac{w_1}{w_2} < \frac{\alpha_1}{\alpha_2} \\ z_2^* \in \left[0, \frac{q}{\alpha_2}\right] & \text{if } \frac{w_1}{w_2} = \frac{\alpha_1}{\alpha_2} \\ \frac{q}{\alpha_2} & \text{if } \frac{w_1}{w_2} > \frac{\alpha_1}{\alpha_2} \end{cases}$$

- Case 3 is a test to see if you are awake: the isoquants are not convex to the origin: an experiment with a straight-edge to simulate an isocost line will show that it is almost like case 2 – the solution will be either at the corner $(\sqrt{q/\alpha_1}, 0)$ if $w_1/w_2 < \sqrt{\alpha_1/\alpha_2}$ or at the corner $(0, \sqrt{q/\alpha_2})$ if $w_1/w_2 > \sqrt{\alpha_1/\alpha_2}$ (but nowhere else). So the cost function is :

$$C(\mathbf{w}, q) = \min\left\{w_1\sqrt{\frac{q}{\alpha_1}}, w_2\sqrt{q/\alpha_2}\right\}.$$

The conditional demand function is similar to, but slightly different from, the previous case:

$$H^1(\mathbf{w}, q) = \begin{cases} \frac{q}{\alpha_1} & \text{if } \frac{w_1}{w_2} < \sqrt{\frac{\alpha_1}{\alpha_2}} \\ z_1^* \in \left\{0, \frac{q}{\alpha_1}\right\} & \text{if } \frac{w_1}{w_2} = \sqrt{\frac{\alpha_1}{\alpha_2}} \\ 0 & \text{if } \frac{w_1}{w_2} > \sqrt{\frac{\alpha_1}{\alpha_2}} \end{cases}$$

$$H^2(\mathbf{w}, q) = \begin{cases} 0 & \text{if } \frac{w_1}{w_2} < \sqrt{\frac{\alpha_1}{\alpha_2}} \\ z_2^* \in \left\{0, \frac{q}{\alpha_2}\right\} & \text{if } \frac{w_1}{w_2} = \sqrt{\frac{\alpha_1}{\alpha_2}} \\ \frac{q}{\alpha_2} & \text{if } \frac{w_1}{w_2} > \sqrt{\frac{\alpha_1}{\alpha_2}} \end{cases}$$

Note the discontinuity exactly at $w_1/w_2 = \sqrt{\alpha_1/\alpha_2}$

Exercise 2.6 *Assume the production function*

$$\phi(\mathbf{z}) = \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}}$$

where z_i is the quantity of input i and $\alpha_i \geq 0$, $-\infty < \beta \leq 1$ are parameters. This is an example of the CES (Constant Elasticity of Substitution) production function.

1. *Show that the elasticity of substitution is $\frac{1}{1-\beta}$.*
2. *Explain what happens to the form of the production function and the elasticity of substitution in each of the following three cases: $\beta \to -\infty$, $\beta \to 0$, $\beta \to 1$.*
3. *Relate your answer to the answers to Exercises 2.4 and 2.5.*

Outline Answer

1. Differentiating the production function

$$\phi(\mathbf{z}) := \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}}$$

it is clear that the marginal product of input i is

$$\phi_i(\mathbf{z}) := \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}-1} \alpha_i z_i^{\beta-1} \tag{2.9}$$

Therefore the MRTS is

$$\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})} = \frac{\alpha_1}{\alpha_2}\left[\frac{z_1}{z_2}\right]^{\beta-1} \tag{2.10}$$

which implies

$$\log\left(\frac{z_1}{z_2}\right) = \frac{1}{1-\beta}\log\frac{\alpha_1}{\alpha_2} - \frac{1}{1-\beta}\log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right).$$

Therefore

$$\sigma = -\frac{\partial \log\left(\frac{z_1}{z_2}\right)}{\partial \log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)} = \frac{1}{1-\beta}$$

2. Clearly $\beta \to -\infty$ yields $\sigma = 0$ ($\phi(z) = \min\{\alpha_1 z_1, \alpha_2 z_2\}$), $\beta \to 0$ yields $\sigma = 1$ ($\phi(z) = z_1^{\alpha_1} z_2^{\alpha_2}$), $\beta \to 1$ yields $\sigma = \infty$ ($\phi(z) = \alpha_1 z_1 + \alpha_2 z_2$).
3. The case $\beta \to -\infty$ corresponds to that in part 1 of Exercise 2.5; $\beta \to 0$. corresponds to that in Exercise 2.4; $\beta \to 1$. corresponds to that in part 5 of Exercise 2.5 .