ΠΜΣ «Οικονομική Επιστήμη»
ΕΚΠΑ Τμήμα Οικονομικών Επιστημών
Microeconomic Theory – Producer Theory January 2020
Instructor: Andreas Papandreou

Problem Set 3

1. Consider a duopoly with identical firms. The cost function for firm f is

$$C_0 + cq^f \ , f = 1,2$$

The inverse demand function is

$$\beta_0 - \beta q$$

where C_0, c, β_0 and β are all positive numbers and total output is given by

$$q = q^1 + q^2$$

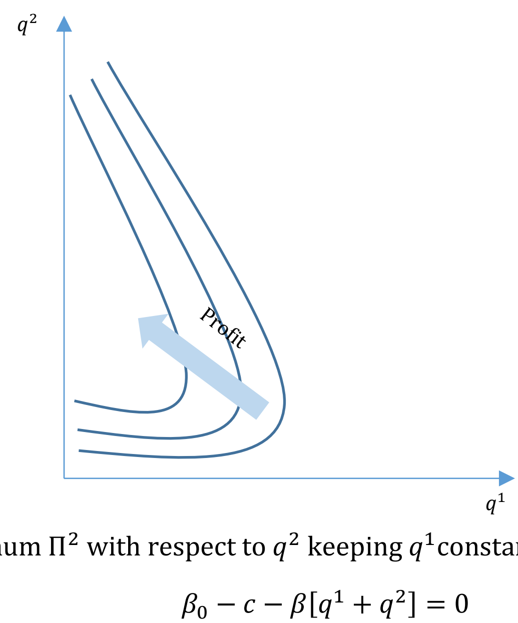
(a) *Find the isoprofit contour and the reaction function for firm 2.*

First we need to calculate the profit function of the firm 2. Thus we have,

$$\Pi^2 = pq^2 - [C_0 + cq^2] = \left[\beta_0 - \beta[q^1 + q^2]\right]q^2 - [C_0 + cq^2]$$

A typical isoprofit contour is given below for firm 2 by the locus of (q^1, q^2) satisfying

$$\left[\beta_0 - c - \beta[q^1 + q^2]\right]q^2 = constant$$

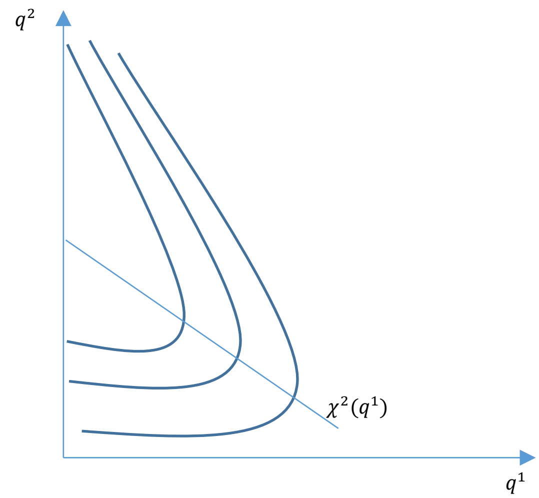


The FOC for maximum Π^2 with respect to q^2 keeping q^1constant yield

$$\beta_0 - c - \beta[q^1 + q^2] = 0$$

In this way we extract the Cournot reaction function for firm 2, that is

$$q^2 = \chi^2(q^1) = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1$$



That is a straight line. Note that this relationship holds wherever firm 2 can make positive profits.

(b) Find the Cournot-Nash equilibrium for the industry and illustrate it in q 1 ; q2 -space.

By symmetry we have

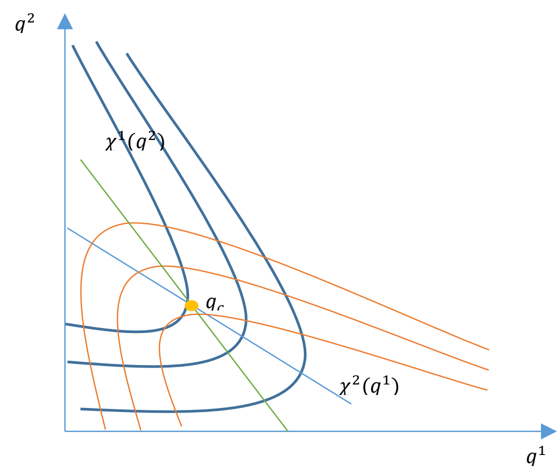
$$q^1 = \chi^1(q^2) = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^2$$

The the Cournot-Nash equilibrium is where $\chi^1(q^2) = \chi^2(q^1)$, that is

$$q^1 = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}\left[\frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1\right]$$

The solution yields

$$q^1 = q^2 = q_c = \frac{\beta_0 - c}{3\beta}$$



and by substituting in the inverse demand function, which is apparently a function of quantity that is $p(q) = \beta_0 - \beta q$ we have

$$p_c = \frac{2}{3}\beta_0 + \frac{1}{3}c$$

(c) Find the joint-profit maximising solution for the industry and illustrate it on the same diagram.

Writing $q \ = q^1 + q^2, 2$, the two firms' joint profits are given by

$$\Pi^{12} = pq - [2C_0 + cq] = [\beta_0 - \beta q]q - [2C_0 + cq]$$

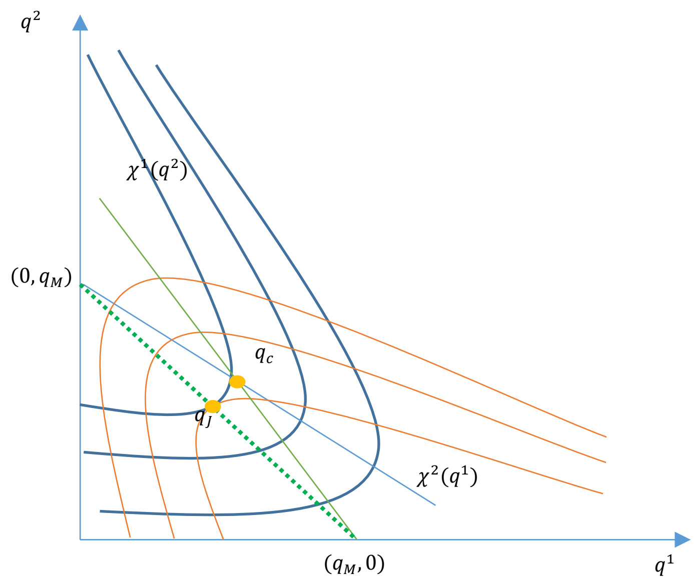
FOC yields

$$\beta_0 - c - 2\beta q = 0$$

which gives the collusive monopoly solution as

$$q_M = \frac{\beta_0 - c}{2\beta}$$

while the corresponding monopoly price, extracted in a same manner as on part (a), should be $p_M = \frac{1}{2}[\beta_0 - c]$. However, the break-down into outputs q^1 and q^2 is in principle is undefined.



The points $(0, q_M)$and $(0, q_M)$ are the endpoints of the two reaction functions (each indicates the amount that one firm would produce if it knew that the other was producing zero). The solution lies somewhere on the line joining these two points. In particular the symmetric joint-profit maximising outcome $(q_J;\ q_J)$ lies exactly at the midpoint where the isoprofit contour of firm 1 is tangent to the isoprofit contour of firm 2.

(d) If firm 1 acts as leader and firm 2 as a follower find the Stackelberg solution.

If firm 1 is the leader and firm 2 is the follower then firm 1 can predict firm 2ís output using the reaction function $q^2 = \chi^2(q^1) = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1$, and build this into its optimisation problem. The leaderís proÖts are therefore given as

$$\Pi_S^1 = pq^1 - [C_0 + cq^1] = [\beta_0 - \beta[q^1 + \chi^2(q^1)]]q^1 - [C_0 + cq^1]$$
$$= \left[\beta_0 - \beta\left[q^1 + \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1\right]\right]q^1 - [C_0 + cq^1]$$
$$= \frac{1}{2}[\beta_0 - c - \beta q^1] - q^1 - C_0$$

Again using FOC for the leader, we have,

$$\frac{1}{2}[\beta_0 - c] - \beta q^1 = 0$$

that yields the leader's output,

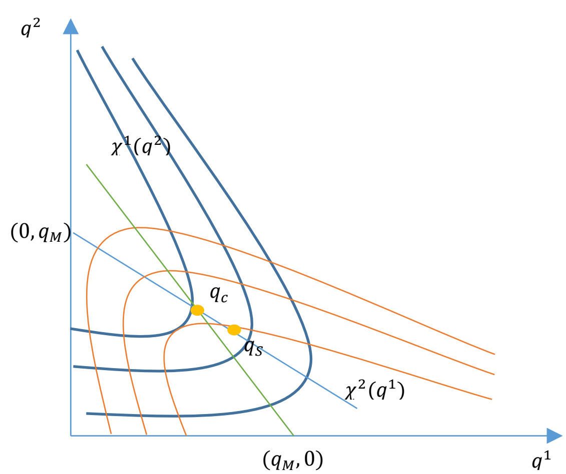
$$\mathrm{q}_S^1 = \frac{\beta_0 - c}{2\beta}$$

And by replacing this quantity on the reaction function used, that is $\chi^2(q^1) = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1$we obtain

$$\mathrm{q}_S^2 = \frac{\beta_0 - c}{4\beta}$$

again the corresponding price is

$$\mathrm{q}_S^1 = \frac{1}{4}\beta_0 + \frac{3}{4}c$$



(e) *Draw the set of payoff possibilities and plot the payoffs for cases 2-4 and for the case where there is a monopoly.*

2. You are given the following payoffs associated with two pure strategies of each of two players (a, b) *in a simultaneous move game.*

	Player b		
Player a		s_1^b	s_2^b
	s_1^a	3,5	10,0
	s_2^a	6,2	6,4

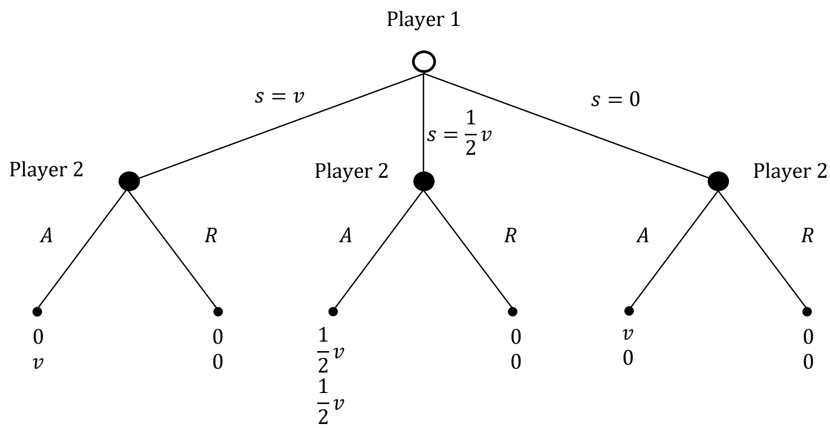
(a) *Are there any dominant strategies in this game? Explain.*
(b) *Are there any Nash equilibria in this game? Explain.*
(c) *How would you describe this game? Can you think of any real world examples?*
(d) *Find the mixed-strategy equilibrium.*
(e) *Show the mixed-strategy equilibrium in the space of probabilities. Explain.*
(f) *Show an extensive form of this simultaneous move game. Explain.*

**The key for this exercise is on this directory:
https://eclass.uoa.gr/modules/document/index.php?course=ECON258&openDir=/54ce0fc5DNfu

3. Consider a sequential-move bargaining game between Player 1 (proposer) and Player 2 (responder). Player 1 makes a take-it-or-leave-it offer to Player 2, specifying an amount $s = \left\{0, \frac{1}{2}v, v\right\}$ out of an initial surplus v, i.e., no share of the pie, half of the pie, or all of the pie. If Player 2 accepts such a distribution Player 2 receives the offer $+s$, while Player 1 keeps the remaining surplus $v - s$. If Player 2 rejects, both players get a zero payoff.

(a) Describe the strategy space for every player.

The best way of finding strategies spaces, is to use the extended form of the game given. Thus we construct the extended form of the given bargaining game which is



Strategy set for player 1 is

$$S_1 = \left\{0, \frac{1}{2}v, v\right\}$$

while the strategy set for player 2 is

$$S_2 = \{\text{AAA, AAR, ARR, RRR, RRA, RAA, ARA, RAR}\}$$

For every triplet, the first component specifies player 2's response upon observing that Player 1 makes an offer $s = v$ (in the left-hand side of the game tree in the figure above), the second component is his response to an offer $s = \frac{1}{2}v$, and the third component describes player 2's response to an offer $s = 0$ (in the right-hand side of the game tree)

(b) Provide the normal-form representation of this bargaining game.

Using the three strategies for Player 1 and the eight available strategies for Player 2, the 3 x 8 matrix below represents the normal-form representation of this game

Player 2

Player 1		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
	$s = v$	$0, v$	$0, v$	$0, v$	$0, v$	0,0	0,0	0,0	0,0
	$s = \frac{1}{2}v$	$\frac{1}{2}v, \frac{1}{2}v$	$\frac{1}{2}v, \frac{1}{2}v$	0,0	0,0	$\frac{1}{2}v, \frac{1}{2}v$	$\frac{1}{2}v, \frac{1}{2}v$	0,0	0,0
	$s = 0$	$v, 0$	0,0	$v, 0$	0,0	$v, 0$	0,0	$v, 0$	0,0

(c) Does any player have strictly dominated pure strategies?

No player has any strictly dominated pure strategy since

For player 1, we find that $s = \frac{1}{2}v$ yields a weakly (not strictly) higher payoff than $s = v$, that is $u_1\left(s = \frac{1}{2}v, s_2\right) \geq u_1(s = v, s_2)$ for all strategies of player 2, $s_2 \in S_2$ (i.e., some columns in the above matrix), which is satisfied with strict equality for some strategies of player 2, such as ARR, RRR or RRA.

Similarly, $s = 0$ yields a weakly (but not strictly) higher payoff than $s = v$. That is, $u_1(s = 0, s_2) \geq u_1(s = v, s_2)$ for all $s2 \in S2$, with strict equality for some $s2 \in S2$, such as ARR and RRR.

Finally, $s = \frac{1}{2}v$ yields a higher payoff than $s = 0$ against some strategies of player 2, such as AAR, but a lower payoff against other strategies, such as AAA and RAR. Hence, there is no weakly dominated strategy for Player 1.

The same holds for player 2, which is extracted in the same manner for player 1.

(d) Does any player have strictly dominated mixed strategies?

Once we have shown that there is no strictly dominated pure strategy, we focus on the existence of strictly dominated mixed strategies. We know that player 1 is never going to mix assigning a strictly positive probability to his pure strategy $s = v$ (i.e., offering the whole pie to Player 2) given that it will reduce for sure his expected payoff, for any strategy with which player 2 responds. Indeed, since such strategy yields a strictly lower (or equal) payoff than other of his available strategies, such as $s = 0$ or $s = \frac{1}{2}v$.

If Player 1 mixes between $s = 0$ and $s = \frac{1}{2}v$, we can see that he is going to obtain a mixed strategy σ_1 that yields an expected utility, $u_1(\sigma_1, s_1)$, which exceeds his utility from selecting the pure strategy $s = v$. That is,

$$u_1(\sigma_1, s_1) \geq u_1(s = v, s_2)\ \forall\, s_2 \in S_2$$

with strict equality for s2 = ARR and s2 = RRR, but strict inequality (yielding a strictly higher expected payoff) for all other strategies of player 2. We can visually check this result in the above normal-form representation by noticing that $s = v$, in the top row, yields a zero payoff for any strategy of player 2. However, a linear combination of strategies $s = \frac{1}{2}v$ and $s = 0$, in the middle and bottom rows, yields a positive expected payoff for columns AAA, AAR, RRA, RAA, ARA and RAR; since all of them contain at least one positive payoff for player 1 in the middle or bottom row. However, in the remaining columns (ARR and RRR), player 1's payoff is zero both in the middle and bottom row, thus implying that his expected payoff, zero, coincides with his payoff from playing the pure strategy $s = v$ in the top row. A similar argument applies to Player 2. Therefore, there is no strictly dominated mixed strategy.

4. Consider the following payoff matrix depicting a simultaneous-move game between players 1 and 2.

		Player 2		
		d	*e*	*f*
Player 1	*A*	8 ,0	8 , 1	8 , 8
	B	8 ,0	6 , 8	9 , 1
	C	6 , 8	9 , 1	8 , 1

(a) *What equilibrium prediction can you find using Iterated Deletion of Strictly Dominated Strategies (IDSDS)?*

Player 1 has no strictly dominated strategies. Indeed, while strategy A performs better than C against d, it yields the same payoff when player 2 selects f, and performs strictly worse when player 2 selects column e. Similarly, player 2 has no strictly dominated strategies. In particular, f performs better than d and e when player 1 chooses row A, but yields a weakly lower payoff when player 1 selects row C.

(b) *Can you identify one or more Nash equilibria in pure strategies?*

Best responses are

$BR_1(d) = \{A, B\}, BR_1(e) = \{C\}, BR_1(f) = \{B\}$,for player 1

$BR_2(A) = \{f\}, BR_2(B) = \{e\}, BR_1(C) = \{d\}$, for player 2

The matrix below provides the Best Response payoffs which are underlined.

		Player 2		
		d	*e*	*f*
Player 1	*A*	$\underline{8}$,0	8 , 1	8 , $\underline{8}$
	B	$\underline{8}$,0	6 , $\underline{8}$	$\underline{9}$, 1
	C	6 , $\underline{8}$	$\underline{9}$, 1	8 , 1

The fact that there is no cell with both payoffs being underlined indicates that no strategy profile (that is, no cell in the matrix) has both players choosing a mutual best response to each other's strategies. As a consequence, there is no Nash equilibrium in pure strategies.

(c) *Can you find a mixed strategy Nash equilibrium?*

Once we have shown that there is no strictly dominated pure strategy, we focus on the existence of strictly dominated mixed strategies.

Let p_1 be the probability that player 1 chooses A, p_2 be the probability that he chooses B, and thus $1 - p1 - p2$ the probability he chooses C: Similarly, let q_1 be the probability that player 2 chooses d and q_2 be the probability that he chooses e; and thus $1 - q1 - q2$

represents the probability he chooses f. The conditions for a mixed strategy to exist for player 1 are (keep in mind that in order to find expected utility we multiply the payoff of the current player times the probability of the opposite player to choose the current strategy)

$$EU_1(A) = EU_1(B) = EU_1(C)$$

$$8q_1 + 8q_2 + 8(1 - q_1 - q_2) = 8q_1 + 6q_2 + 9(1 - q_1 - q_2) = 6q_1 + 9q_2 + 8(1 - q_1 - q_2)$$

$$\Leftrightarrow\ 8 = 9 - q_1 - 3q_2 = 8 - 2q_1 + q_2 \Leftrightarrow \begin{cases} q_1 = \dfrac{1}{7} \\ q_2 = \dfrac{2}{7} \end{cases}$$

Similarly for player 2 to use a mixed strategy we need,

$$EU_2(d) = EU_2(e) = EU_2(f)$$

$$0p_1 + 0p_2 + 8(1 - p_1 - p_2) = 1p_1 + 8p_2 + 0(1 - p_1 - p_2) = 8p_1 + 1p_2 + 0(1 - p_1 - p_2)$$

$$\Leftrightarrow\ 8 - 8p_1 - 8p_2 = p_1 + 8p_2 = 8p_1 + p_2 \Leftrightarrow \begin{cases} p_1 = \dfrac{8}{25} \\ p_2 = \dfrac{8}{25} \end{cases}$$

Therefore, the mixed strategy Nash equilibrium of this game is

$$\left(\underbrace{\frac{8}{25}A \frac{8}{25}B \frac{9}{25}C}_{Player\ 1},\ \underbrace{\frac{1}{7}d \frac{2}{7}e \frac{4}{7}f}_{Player\ 2} \right)$$

5. Consider the following simultaneous-move game with payoff matrix

Player 2

		A	B
Player 1	A	$u_1(A,A), u_2(A,A)$	$u_1(A,B), u_2(A,B)$
	B	$u_1(B,A), u_2(B,A)$	$u_1(B,B), u_2(B,B)$

In addition, assume that $u_i(A,A) = uj\ (A,A)$ *and* $u_i(B,B) = u_j\ (B,B)$*, and that* $u_i(A,B) = u_j\ (B,A)$ *for every player* $i = \{1,2\}$ *and* $j \neq i$*. Find players' strictly dominated strategies and the Nash equilibria of the game (allowing for both pure and mixed strategies) in the following settings. Interpret and relate your results to common games.*

(a) $u_i(A,A) > u_i(B,A)$ *and* $u_i(A,B) > u_i(B,B)$ *for every player i.*

Every player i finds B to be strictly dominated by A since $u_i(A,A) > u_i(B,A)$ when his opponent (player $j \neq i$) chooses A, and $u_i(A,B) > u_i(B,B)$ when his opponent chooses B. That is, regardless of the strategy selected by player i's opponent, strategy A yields player i a strictly higher payoff than strategy B. Hence (A, A) is the only strategy profile surviving IDSDS *(Iterated Deletion of Strictly Dominated Strategies)*, and thus becomes the unique NE *(Nash Equilibrium)* of the game, which involves the use of strictly dominant strategies by both players. The game has no msNE *(mixed strategy Nash Equilibrium)* since every player only assign a positive probability to his undominated strategy, A. Therefore, this game resembles a standard Prisonner's Dilemma game.

(b) $u_i(A,A) > u_i(B,A)$ *and* $u_i(A,B) < u_i(B,B)$ *for every player i.*

In this setting, every player i prefers to respond selecting the same strategy as his opponent, i.e., choose A when his opponent selects A, but B otherwise. Hence, there is no strictly dominated (nor strictly dominant) strategy for either player. In other words, if we rely on IDSDS *(Iterated Deletion of Strictly Dominated Strategies)*, the entire matrix would be our most precise equilibrium prediction, i.e., four strategy profiles, (A,A); (A,B); (B,B) and (B,A), survive IDSDS *(Iterated Deletion of Strictly Dominated Strategies)*. However, when we apply NE *(Nash Equilibrium)*, we can easily show that a more precise equilibrium prediction emerges. In particular, we can underline best response payoffs as follows

		Player 2	
		A	B
Player 1	A	$\underline{u_1(A,A)}, \underline{u_2(A,A)}$	$u_1(A,B), u_2(A,B)$
	B	$u_1(B,A), u_2(B,A)$	$\underline{u_1(B,B)}, \underline{u_2(B,B)}$

There are two psNE *(pure strategy Nash Equilibrium)* since players are playing mutual best responses: (A,A) and (B,B). Hence, this is a coordination game similar to the Battle of the Sexes game. In this way we have to find the msNE *(mixed strategy Nash Equilibrium)* of the game given. Let p denote the probability that player 1 chooses A and q be the probability that player 2 does. Hence, the value of p that makes player 2 indifferent between A and B is

$$EU_2(A) = EU_2(B)$$

$$pu_2(A,A) + (1-p)u_2(B,A) = pu_2(A,B) + (1-p)u_2(B,B)$$

$$p = \frac{u_2(B,B) - u_2(B,A)}{[u_2(A,A) - u_2(A,B)] + [u_2(B,B) - u_2(B,A)]}$$

The numerator is positive since $u_2(B,B) > u_2(B,A)$ by definition. In addition, the denominator is larger than the numerator given that $u_2(A,A) - u_2(A,B) > 0$ and $u_2(B,B) - u_2(B,A) > 0$,) ultimately implying that the probability we found satisfies $p \in (0,1)$. Similarly, the value of q that makes player 1 indifferent between choosing A and B is

$$EU_1(A) = EU_1(B)$$

$$qu_1(A,A) + (1-q)u_1(B,A) = qu_1(A,B) + (1-q)u_1(B,B)$$

$$q = \frac{u_1(B,B) - u_1(B,A)}{[u_1(A,A) - u_1(A,B)] + [u_1(B,B) - u_1(B,A)]}$$

The numerator is positive since $u_1(B,B) > u_1(B,A)$ by definition. In addition, the denominator is larger than the numerator given that $u_1(A,A) - u_1(A,B) > 0$ and $u_1(B,B) -$

$u_1(B,A) > 0,$) ultimately implying that the probability we found satisfies $q \in (0,1)$. In this way the msNE *(mixed strategy Nash Equilibrium)* of the above game is,

$$\{(pA,(1-p)B),(qA,(1-q)B)\}$$

(c) $u_i(A,A) < u_i(B,A)$ *and* $u_i(A,B) < u_i(B,B)$ *for every player i.*

This case is symmetric to the game analyzed in part (a), but has strategy B (A) as strictly dominant (dominated) for both players. Hence, (B,B) is the unique strategy profile surviving IDSDS, and therefore becomes the unique psNE *(pure strategy Nash Equilibrium)*, which has both players using dominant strategies. Hence, the game also resembles a Prisonner's Dilemma.

(d) $u_i(A,A) < u_i(B,A)$ *and* $u_i(A,B) > u_i(B,B)$ *for every player i.*

This game is symmetric to that in part (b). Both probabilities and Equilibrium yield the same result. (You should try it on your own for practice! Only the best response actually changes, which is actually the opposite of part (b)).

However, since the payoff structure of this game is different, the values of ratios p and q would not coincide with those of part (b). Hence, we should not generally expect players to randomize with the same probabilities in a Coordination and in an Anti-coordination game, n game such as the Game of Chicken where, in any psNE *(pure strategy Nash Equilibrium)*, players choose opposite strategies.