## Duopoly

### MICROECONOMICS

*Principles and Analysis*

Frank Cowell

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**Prerequisites**

Almost essential
- Monopoly

Useful, but optional
- Game Theory: Strategy and Equilibrium

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Overview...

How the basic elements of the firm and of game theory are used.
Basic ingredients

- Two firms:
  - Issue of entry is not considered.
  - But monopoly could be a special limiting case.

- Profit maximisation.

- Quantities or prices?
  - There’s nothing within the model to determine which “weapon” is used.
  - It’s determined *a priori*.
  - Highlights artificiality of the approach.

- Simple market situation:
  - There is a known demand curve.
  - Single, homogeneous product.
Reaction

- We deal with “competition amongst the few”.
- Each actor has to take into account what others do.
- A simple way to do this: *the reaction function*.
- Based on the idea of “best response”.
  - We can extend this idea…
  - In the case where more than one possible reaction to a particular action.
  - It is then known as a reaction *correspondence*.
- We will see how this works:
  - Where reaction is in terms of prices.
  - Where reaction is in terms of quantities.
Overview...

Introduction to a simple simultaneous move price-setting problem.
Competing by price

- There is a market for a single, homogeneous good.
- Firms announce prices.
- Each firm does not know the other’s announcement when making its own.
- Total output is determined by demand.
  - Determinate market demand curve
  - Known to the firms.
- Division of output amongst the firms determined by market “rules.”
- Let’s take a specific model with a clear-cut solution…
Bertrand – basic set-up

- Two firms can potentially supply the market.
- Each firm: zero fixed cost, constant marginal cost $c$.
- If one firm alone supplied the market it would charge monopoly price $p_M > c$.
- If both firms are present they announce prices.
- The outcome of these announcements:
  - If $p^1 < p^2$ firm 1 captures the whole market.
  - If $p^1 > p^2$ firm 2 captures the whole market.
  - If $p^1 = p^2$ the firms supply equal amounts to the market.
- What will be the equilibrium price?
Bertrand – best response?

- Consider firm 1’s response to firm 2
  - If firm 2 foolishly sets a price $p^2$ above $p_M$ then it sells zero output.
    - Firm 1 can safely set monopoly price $p_M$.
  - If firm 2 sets $p^2$ above $c$ but less than or equal to $p_M$ then firm 1 can “undercut” and capture the market.
    - Firm 1 sets $p^1 = p^2 - \delta$, where $\delta > 0$.
    - Firm 1’s profit always increases if $\delta$ is made smaller…
    - …but to capture the market the discount $\delta$ must be positive!
    - So strictly speaking there’s no best response for firm 1.
  - If firm 2 sets price equal to $c$ then firm 1 cannot undercut
    - Firm 1 also sets price equal to $c$.
  - If firm 2 sets a price below $c$ it would make a loss.
    - Firm 1 would be crazy to match this price.
    - If firm 1 sets $p^1 = c$ at least it won’t make a loss.
- Let’s look at the diagram…
Bertrand model – equilibrium

- Marginal cost for each firm
- Monopoly price level
- Firm 1's reaction function
- Firm 2's reaction function
- Bertrand equilibrium
Bertrand – assessment

- Using “natural tools” – prices.
- Yields a remarkable conclusion.
- Mimics the outcome of perfect competition
  - Price = MC.
- But it is based on a special case.
- Neglects some important practical features
  - Fixed costs.
  - Product diversity
  - Capacity constraints.
- Outcome of price-competition models usually very sensitive to these.
Overview...

The link with monopoly and an introduction to two simple “competitive” paradigms.

- Duopoly
  - Background
  - Price competition
  - Quantity competition
  - Assessment

- Collusion
- The Cournot model
- Leader-Follower
quantity models

- Now take output quantity as the firms’ choice variable.
- Price is determined by the market once total quantity is known:
  - An auctioneer?
- Three important possibilities:
  1. Collusion:
     - Competition is an illusion.
     - Monopoly by another name.
     - But a useful reference point for other cases
  2. Simultaneous-move competing in quantities:
     - Complementary approach to the Bertrand-price model.
  3. Leader-follower (sequential) competing in quantities.
Collusion – basic set-up

- Two firms agree to maximise joint profits.
- This is what they can make by acting as though they were a single firm.
  - Essentially a monopoly with two plants.
- They also agree on a rule for dividing the profits.
  - Could be (but need not be) equal shares.
- In principle these two issues are separate.
The profit frontier

- To show what is possible for the firms…
- …draw the *profit frontier*.
- Show the possible combination of profits for the two firms
  - given demand conditions
  - given cost function
- Start with the case where cash transfers between the firms are not possible
Frontier – non-transferable profits

- Suppose profits can’t be transferred between firms
- Take case of identical firms
  - Constant returns to scale
  - Decreasing returns to scale in each firm (1): MC always rising
  - Decreasing returns to scale in each firm (2): capacity constraints
- Increasing returns to scale in each firm (fixed cost and constant marginal cost)
Frontier – transferable profits

Increasing returns to scale (without transfers)
Now suppose firms can make “side-payments”
So profits can be transferred between firms
Profits if everything were produced by firm 1
Profits if everything were produced by firm 2
The profit frontier if transfers are possible
Joint-profit maximisation with equal profit shares

Cash transfers “convexify” the set of attainable profits.
Collusion – simple model

- Take the special case of the “linear” model where marginal costs are identical: $c^1 = c^2 = c$.

- Will both firms produce a positive output?
  - If unlimited output is possible then only one firm needs to incur the fixed cost…
  - …in other words a true monopoly.
  - But if there are capacity constraints then both firms may need to produce.
  - Both firms incur fixed costs.

- We examine both cases – capacity constraints first.
Collusion: capacity constraints

- If both firms are active total profit is
  \[ [a - bq] q - [C^1_0 + C^2_0 + cq] \]

- Maximising this, we get the FOC:
  \[ a - 2bq - c = 0. \]

- Which gives equilibrium quantity and price:
  \[ q = \frac{a - c}{2b}; \quad p = \frac{a + c}{2}. \]

- So maximised profits are:
  \[ \Pi_M = \frac{[a-c]^2}{4b} - [C^1_0 + C^2_0]. \]

- Now assume the firms are identical: \( C^1_0 = C^2_0 = C_0 \).

- Given equal division of profits each firm’s payoff is
  \[ \Pi_j = \frac{[a-c]^2}{8b} - C_0. \]
Collusion: no capacity constraints

- With no capacity limits and constant marginal costs…
- …there seems to be no reason for both firms to be active.
- Only need to incur one lot of fixed costs $C_0$.
  - $C_0$ is the smaller of the two firms’ fixed costs.
  - Previous analysis only needs slight tweaking.
- Modify formula for $\Pi_J$ by replacing $C_0$ with $\frac{1}{2}C_0$.
- But is the division of the profits still implementable?
Overview...

Simultaneous move “competition” in quantities

Duopoly

Background

Price competition

Quantity competition

- Collusion
- The Cournot model
- Leader-Follower

Assessment
Cournot – basic set-up

- Two firms.
  - Assumed to be profit-maximisers
  - Each is fully described by its cost function.

- Price of output determined by demand.
  - Determinate market demand curve
  - Known to both firms.

- Each chooses the quantity of output.
  - Single homogeneous output.
  - Neither firm *knows* the other’s decision when making its own.

- Each firm makes an *assumption* about the other’s decision
  - Firm 1 assumes firm 2’s output to be given number.
  - Likewise for firm 2.

- How do we find an equilibrium?
Cournot – model setup

- Two firms labelled $f = 1, 2$
- Firm $f$ produces output $q^f$.
- So total output is:
  - $q = q^1 + q^2$
- Market price is given by:
  - $p = p(q)$
- Firm $f$ has cost function $C^f(\cdot)$.
- So profit for firm $f$ is:
  - $p(q)q^f - C^f(q^f)$
- Each firm’s profit depends on the other firm’s output
  - (because $p$ depends on total $q$).
Cournot – firm’s maximisation

- Firm 1’s problem is to choose $q^1$ so as to maximise
  \[ \Pi^1(q^1; q^2) := p(q^1 + q^2)q^1 - C^1(q^1) \]
- Differentiate $\Pi^1$ to find FOC:
  \[ \frac{\partial \Pi^1(q^1; q^2)}{\partial q^1} = p \frac{q(q^1 + q^2)}{q^1} + p(q^1 + q^2) - C_q^1(q^1) \]
- For an interior solution this is zero.
- Solving, we find $q^1$ as a function of $q^2$.
- This gives us 1’s reaction function, $\chi^1$:
  \[ q^1 = \chi^1(q^2) \]
- Let’s look at it graphically…
Cournot – the reaction function

- Firm 1’s iso-profit curves
- Assuming 2’s output constant at $q_0$ ...
- ...firm 1 maximises profit
- If 2’s output were constant at a higher level
- 2’s output at a yet higher level
- The reaction function

Firm 1’s choice given that 2 chooses output $q_0$
Cournot – solving the model

- \( \chi^1(\cdot) \) encapsulates profit-maximisation by firm 1.
- Gives firm’s reaction 1 to a fixed output level of the competitor firm:
  - \( q^1 = \chi^1(q^2) \)
- Of course firm 2’s problem is solved in the same way.
- We get \( q^2 \) as a function of \( q^1 \):
  - \( q^2 = \chi^2(q^1) \)
- Treat the above as a pair of simultaneous equations.
- Solution is a pair of numbers \( (q^1_C, q^2_C) \).
  - So we have \( q^1_C = \chi^1(\chi^2(q^1_C)) \) for firm 1…
  - … and \( q^2_C = \chi^2(\chi^1(q^2_C)) \) for firm 2.
- This gives the *Cournot-Nash equilibrium* outputs.
Cournot-Nash equilibrium (1)

- Firm 2's Iso-profit curves
- If 1's output is $q_0 \ldots$
- ...firm 2 maximises profit
- Repeat at higher levels of 1's output
- Firm 2's reaction function
- Combine with firm 's reaction function
- "Consistent conjectures"
Cournot-Nash equilibrium (2)

- Firm 1’s Iso-profit curves
- Firm 2’s Iso-profit curves
- Firm 1’s reaction function
- Firm 2’s reaction function
- Cournot-Nash equilibrium
- Outputs with higher profits for both firms
- Joint profit-maximising solution
The Cournot-Nash equilibrium

- Why “Cournot-Nash”?  
- It is the general form of Cournot’s (1838) solution.

- But it also is the Nash equilibrium of a simple quantity game:
  - The players are the two firms.
  - Moves are simultaneous.
  - Strategies are actions – the choice of output levels.
  - The functions give the best-response of each firm to the other’s strategy (action).

- To see more, take a simplified example…
Cournot – a “linear” example

- Take the case where the inverse demand function is:
  \[ p = \beta_0 - \beta q \]
- And the cost function for \( f \) is given by:
  \[ C^f(q^f) = C_0^f + c^f q^f \]
- So profits for firm \( f \) are:
  \[ [\beta_0 - \beta q] q^f - [C_0^f + c^f q^f] \]
- Suppose firm 1’s profits are \( \Pi \).
- Then, rearranging, the iso-profit curve for firm 1 is:
  \[ q^2 = \frac{\beta_0 - c^1}{\beta} - q^1 - \frac{C_0^1 + \Pi}{\beta q^1} \]
Cournot – solving the linear example

- Firm 1’s profits are given by
  - $\Pi^1(q^1; q^2) = [\beta_0 - \beta q] q^1 - [C_0^1 + c^1 q^1]$
- So, choose $q^1$ so as to maximise this.
- Differentiating we get:
  \[
  \frac{\partial \Pi^1(q^1; q^2)}{\partial q^1} = -2\beta q^1 + \beta_0 - \beta q^2 - c^1
  \]
- FOC for an interior solution ($q^1 > 0$) sets this equal to zero.
- Doing this and rearranging, we get the reaction function:
  \[
  q^1 = \max \left\{ \frac{\beta_0 - c^1}{2\beta} - \frac{1}{2} q^2, 0 \right\}
  \]
The reaction function again

\[ \chi^1(\cdot) \]

\[ \Pi^1(q^1; q^2) = \text{const} \]

- Firm 1’s iso-profit curves
- Firm 1 maximises profit, given \( q^2 \).
- The reaction function
Finding Cournot-Nash equilibrium

- Assume output of both firm 1 and firm 2 is positive.
- Reaction functions of the firms, \( \chi^1(\cdot), \chi^2(\cdot) \) are given by:
  \[
  q^1 = \frac{a - c^1}{2b} - \frac{1}{2}q^2; \quad q^2 = \frac{a - c^2}{2b} - \frac{1}{2}q^1.
  \]
- Substitute from \( \chi^2 \) into \( \chi^1 \):
  \[
  q^1_C = \frac{a - c^1}{2b} - \frac{1}{2} \left[ \frac{a - c^2}{2b} - \frac{1}{2}q^1_C \right].
  \]
- Solving this we get the Cournot-Nash output for firm 1:
  \[
  q^1_C = \frac{a + c^2 - 2c^1}{3b}.
  \]
- By symmetry get the Cournot-Nash output for firm 2:
  \[
  q^2_C = \frac{a + c^1 - 2c^2}{3b}.
  \]
Cournot – identical firms

- Take the case where the firms are identical.
  - This is useful but very special.
- Use the previous formula for the Cournot-Nash outputs.
  \[ q_C^1 = \frac{a + c^2 - 2c^1}{3b} ; q_C^2 = \frac{a + c^1 - 2c^2}{3b} . \]
- Put \( c^1 = c^2 = c \). Then we find \( q_C^1 = q_C^2 = q_C \) where
  \[ q_C = \frac{a - c}{3b} . \]
- From the demand curve the price in this case is \( \frac{1}{3}[a+2c] \)
- Profits are
  \[ \Pi_C = \frac{(a - c)^2}{9b} - C_0 . \]
Symmetric Cournot

- A case with identical firms
- Firm 1’s reaction to firm 2
- Firm 2’s reaction to firm 1
- The Cournot-Nash equilibrium
Cournot – assessment

- Cournot-Nash outcome straightforward.
  - Usually have continuous reaction functions.

- Apparently “suboptimal” from the selfish point of view of the firms.
  - Could get higher profits for all firms by collusion.

- Unsatisfactory aspect is that price emerges as a “by-product.”
  - Contrast with Bertrand model.

- Absence of time in the model may be unsatisfactory.
Overview...

Duopoly

Background

Price competition

Quantity competition

Assessment

Sequential “competition” in quantities

• Collusion
• The Cournot model
• Leader-Follower
Leader-Follower – basic set-up

- Two firms choose the quantity of output.
  - Single homogeneous output.
- Both firms know the market demand curve.
- But firm 1 is able to choose first.
  - It announces an output level.
- Firm 2 then moves, knowing the announced output of firm 1.
- Firm 1 knows the reaction function of firm 2.
- So it can use firm 2’s reaction as a “menu” for choosing its own output…
Leader-follower – model

- Firm 1 (the leader) knows firm 2’s reaction.
  - If firm 1 produces \( q^1 \) then firm 2 produces \( \chi^2(q^1) \).
- Firm 1 uses \( \chi^2 \) as a feasibility constraint for its own action.
- Building in this constraint, firm 1’s profits are given by
  \[
p(q^1 + \chi^2(q^1)) q^1 - C^1(q^1)
  \]
- In the “linear” case firm 2’s reaction function is
  \[
  q^2 = \frac{a - c^2}{2b} - \frac{1}{2}q^1.
  \]
- So firm 1’s profits are
  \[
  \left[ a - b \left[ q^1 + \frac{[a - c^2]}{2b} - \frac{1}{2}q^1 \right] \right] q^1 - [C_0^1 + c^1q^1] \]
Solving the leader-follower model

- Simplifying the expression for firm 1’s profits we have:
  \[ \frac{1}{2} \left[ a + c^2 - b q^1 \right] q^1 - [C_0^1 + c^1 q^1] \]
- The FOC for maximising this is:
  \[ \frac{1}{2} \left[ a + c^2 \right] - b q^1 - c^1 = 0 \]
- Solving for \( q^1 \) we get:
  \[ q^1_{S} = \frac{a + c^2 - 2c^1}{2b} \]
- Using 2’s reaction function to find \( q^2 \) we get:
  \[ q^2_{S} = \frac{a + 2c^1 - 3c^2}{4b} \]
Leader-follower – identical firms

- Again assume that the firms have the same cost function.
- Take the previous expressions for the Leader-Follower outputs:
  \[ q_s^1 = \frac{a + c^2 - 2c^1}{2b} ; \quad q_s^2 = \frac{a + 2c^1 - 3c^2}{4b} . \]
- Put \( c^1 = c^2 = c \); then we get the following outputs:
  \[ q_s^1 = \frac{a - c}{2b} ; \quad q_s^2 = \frac{a - c}{4b} . \]
- Using the demand curve, market price is \( \frac{1}{4} [a + 3c] \).
- So profits are:
  \[ \Pi_s^1 = \frac{[a - c]^2}{8b} - C_0 ; \quad \Pi_s^2 = \frac{[a - c]^2}{16b} - C_0 . \]
Leader-Follower

- Firm 1’s Iso-profit curves
- Firm 2’s reaction to firm 1
- Firm 1 takes this as an opportunity set...
- ...and maximises profit here
- Firm 2 follows suit

- Leader has higher output (and follower less) than in Cournot-Nash
- “S” stands for von Stackelberg
Overview...

How the simple price- and quantity-models compare.
Comparing the models

- The price-competition model may seem more “natural”
- But the outcome \( (p = MC) \) is surely at variance with everyday experience.
- To evaluate the quantity-based models we need to:
  - Compare the quantity outcomes of the three versions
  - Compare the profits attained in each case.
Output under different regimes

- Reaction curves for the two firms.
- Joint-profit maximisation with equal outputs
- Cournot-Nash equilibrium
- Leader-follower (Stackelberg) equilibrium
Profits under different regimes

- Joint-profit maximisation with equal shares
- Attainable set with transferable profits
- Profits at Cournot-Nash equilibrium
- Profits in leader-follower (Stackelberg) equilibrium

Cournot and leader-follower models yield profit levels inside the frontier.
What next?

- Introduce the possibility of entry.
- General models of oligopoly.
- Dynamic versions of Cournot competition