The Multi-Output Firm

MICROECONOMICS
Principles and Analysis
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Introduction

- This presentation focuses on analysis of firm producing more than one good
  - modelling issues
  - production function
  - profit maximisation

- For the single-output firm, some things are obvious:
  - the direction of production
  - returns to scale
  - marginal products

- But what of multi-product processes?

- Some rethinking required...?
  - nature of inputs and outputs?
  - tradeoffs between outputs?
  - counterpart to cost function?
Overview...

A fundamental concept

The Multi-Output Firm

Net outputs

Production possibilities

Profit maximisation
Multi-product firm: issues

■ “Direction” of production
  ◆ Need a more general notation

■ Ambiguity of some commodities
  ◆ Is paper an input or an output?

■ Aggregation over processes
  ◆ How do we add firm 1’s inputs and firm 2’s outputs?
Net output

- Net output, written as $q_i$,
  - if **positive** denotes the amount of good $i$ produced as output
  - if **negative** denotes the amount of good $i$ used up as output

- Key concept
  - treat outputs and inputs symmetrically
  - offers a representation that is consistent

- Provides consistency
  - in aggregation
  - in “direction” of production
Approaches to outputs and inputs

- A standard “accounting” approach
- An approach using “net outputs”
- How the two are related
- A simple sign convention

<table>
<thead>
<tr>
<th>NET OUTPUTS</th>
<th>OUTPUT</th>
<th>INPUTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td></td>
<td>$z_1$</td>
</tr>
<tr>
<td>$q_2$</td>
<td></td>
<td>$z_2$</td>
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<tr>
<td>...</td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>$q_{n-1}$</td>
<td></td>
<td>$z_m$</td>
</tr>
<tr>
<td>$q_n$</td>
<td>$q$</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
q_1 \\
q_2 \\
... \\
q_{n-1} \\
q_n
\end{bmatrix} = \begin{bmatrix}
-z_1 \\
-z_2 \\
... \\
-z_m \\
+q
\end{bmatrix}
\]

Outputs: + net additions to the stock of a good
Inputs: − reductions in the stock of a good
Aggregation

- Consider an industry with two firms
  - Let $q_i^f$ be net output for firm $f$ of good $i, \ f = 1,2$
  - Let $q_i$ be net output for whole industry of good $i$

- How is total related to quantities for individual firms?
  - Just add up
  - $q_i = q_i^1 + q_i^2$

- Example 1: both firms produce $i$ as output
  - $q_i^1 = 100, q_i^2 = 100$
  - $q_i = 200$

- Example 2: both firms use $i$ as input
  - $q_i^1 = -100, q_i^2 = -100$
  - $q_i = -200$

- Example 3: firm 1 produces $i$ that is used by firm 2 as input
  - $q_i^1 = 100, q_i^2 = -100$
  - $q_i = 0$
Net output: summary

- Sign convention is common sense
- If $i$ is an output…
  - addition to overall supply of $i$
  - so sign is positive
- If $i$ is an inputs
  - net reduction in overall supply of $i$
  - so sign is negative
- If $i$ is a pure intermediate good
  - no change in overall supply of $i$
  - so assign it a zero in aggregate
Overview...

A production function with many outputs, many inputs...
Rewriting the production function…

- Reconsider single-output firm example given earlier
  - goods 1,...,m are inputs
  - good m+1 is output
  - n = m + 1

- Conventional way of writing feasibility condition:
  - $q \leq \phi(z_1, z_2, \ldots, z_m)$
  - where $\phi$ is the production function

- Express this in net-output notation and rearrange:
  - $q_n \leq \phi(-q_1, -q_2, \ldots, -q_{n-1})$
  - $q_n - \phi(-q_1, -q_2, \ldots, -q_{n-1}) \leq 0$

- Rewrite this relationship as
  - $\Phi(q_1, q_2, \ldots, q_{n-1}, q_n) \leq 0$
  - where $\Phi$ is the implicit production function

- Properties of $\Phi$ are implied by those of $\phi$…
The production function $\Phi$

- Recall equivalence for single output firm:
  - $q_n - \phi(-q_1, -q_2, \ldots, -q_{n-1}) \leq 0$
  - $\Phi(q_1, q_2, \ldots, q_{n-1}, q_n) \leq 0$

- So, for this case:
  - $\Phi$ is increasing in $q_1, q_2, \ldots, q_n$
  - if $\phi$ is homogeneous of degree 1, $\Phi$ is homogeneous of degree 0
  - if $\phi$ is differentiable so is $\Phi$
  - for any $i, j = 1, 2, \ldots, n-1$ $\text{MRTS}_{ij} = \Phi_j(q)/\Phi_i(q)$

- It makes sense to generalise these…
The production function $\Phi$ (more)

- For a vector $q$ of net outputs
  - $q$ is feasible if $\Phi(q) \leq 0$
  - $q$ is technically efficient if $\Phi(q) = 0$
  - $q$ is infeasible if $\Phi(q) > 0$

- For all feasible $q$:
  - $\Phi(q)$ is increasing in $q_1, q_2, \ldots, q_n$
  - if there is CRTS then $\Phi$ is homogeneous of degree 0
  - if $\phi$ is differentiable so is $\Phi$
  - for any two inputs $i, j$, $\text{MRTS}_{ij} = \Phi_j(q)/\Phi_i(q)$
  - for any two outputs $i, j$, the marginal rate of transformation of $i$ into $j$ is $\text{MRT}_{ij} = \Phi_j(q)/\Phi_i(q)$

- Illustrate the last concept using the *transformation curve*…
Firm’s transformation curve

- Goods 1 and 2 are outputs
- Feasible outputs
- Technically efficient outputs
- MRT at $q^o$

\[ \Phi(q) \leq 0 \]

\[ \Phi_1(q^o)/\Phi_2(q^o) \]

\[ \Phi(q) = 0 \]
An example with five goods

- Goods 1 and 2 are outputs
- Goods 3, 4, 5 are inputs
- A linear technology
  - fixed proportions of each input needed for the production of each output:
  - \( q_1 a_{1i} + q_2 a_{2i} \leq -q_i \)
  - where \( a_{ji} \) is a constant \( i = 3,4,5, j = 1,2 \)
  - given the sign convention \(-q_i > 0\)
- Take the case where inputs are fixed at some arbitrary values…
The three input constraints

points satisfying
\[ q_1 a_{13} + q_2 a_{23} \leq -q_3 \]

points satisfying
\[ q_1 a_{14} + q_2 a_{24} \leq -q_4 \]

points satisfying
\[ q_1 a_{15} + q_2 a_{25} \leq -q_5 \]

- Draw the feasible set for the two outputs:
  - input Constraint 3
  - Add Constraint 4
  - Add Constraint 5

Intersection is the feasible set for the two outputs
The resulting feasible set

The transformation curve

how this responds to changes in available inputs
Changing quantities of inputs

The feasible set for the two consumption goods as before:
- Suppose there were more of input 3
- Suppose there were less of input 4

points satisfying
\[ q_1 a_{13} + q_2 a_{23} \leq -q_3 - dq_3 \]

points satisfying
\[ q_1 a_{14} + q_2 a_{24} \leq -q_4 + dq_4 \]
Overview...

*Integrated approach to optimisation*

- The Multi-Output Firm
  - Net outputs
  - Production possibilities
  - Profit maximisation
Profits

- The basic concept is (of course) the same
  - Revenue – Costs
- But we use the concept of net output
  - this simplifies the expression
  - exploits symmetry of inputs and outputs
- Consider an “accounting” presentation…
Accounting with net outputs

- Suppose goods 1,...,m are inputs and goods m+1 to n are outputs

\[
\begin{align*}
\sum_{i=m+1}^{n} p_i q_i & \quad \text{Revenue} \\
\sum_{i=1}^{m} p_i [-q_i] & \quad - \text{Costs} \\
\sum_{i=1}^{n} p_i q_i & = \text{Profits}
\end{align*}
\]

- Cost of inputs (goods 1,...,m)
- Revenue from outputs (goods m+1,...,n)
- Subtract cost from revenue to get profits
Iso-profit lines...

Net-output vectors yielding a given $\Pi_0$.

Iso-profit lines for higher profit levels.

$p_1 q_1 + p_2 q_2 = \text{constant}$

$p_1 q_1 + p_2 q_2 = \Pi_0$

use this to represent profit-maximisation
Profit maximisation: multi-product firm (1)

- Feasible outputs
- Isoprofit line
- Maximise profits
- Profit-maximising output
- MRTS at profit-maximising output

Here \( q_1^* > 0 \) and \( q_2^* > 0 \)

- \( q^* \) is technically efficient

- Slope at \( q^* \) equals price ratio
Profit maximisation: multi-product firm (2)

- Feasible outputs
- Isoprofit line
- Maximise profits
- Profit-maximising output
- MRTS at profit-maximising output

Here $q_1^* > 0$ but $q_2^* = 0$

- $q^*$ is technically efficient
- Slope at $q^* \leq$ price ratio

Increasing profit
Maximising profits

- Problem is to choose \( q \) so as to maximise

\[
\sum_{i=1}^{n} p_i q_i \quad \text{subject to} \quad \Phi(q) \leq 0
\]

- Lagrangean is

\[
\sum_{i=1}^{n} p_i q_i - \lambda \Phi(q)
\]

- FOC for an interior maximum is
  \[ p_i - \lambda \Phi_i(q) = 0 \]
Maximised profits

- Introduce the *profit function*
  - the solution function for the profit maximisation problem

\[
\Pi(p) = \max_{\{\Phi(q) \leq 0\}} \sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} p_i q_i^*
\]

- Works like other solution functions:
  - non-decreasing
  - homogeneous of degree 1
  - continuous
  - convex

- Take derivative with respect to \( p_i \):
  - \( \Pi_i(p) = q_i^* \)
  - write \( q_i^* \) as net supply function
  - \( q_i^* = q_i(p) \)
Summary

- Three key concepts
- Net output
  - simplifies analysis
  - key to modelling multi-output firm
  - easy to rewrite production function in terms of net outputs
- Transformation curve
  - summarises tradeoffs between outputs
- Profit function
  - counterpart of cost function