Another Graphical Proof of Arrow's Impossibility Theorem

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Abstract: Arrow's (1951) Impossibility Theorem is the idea that, given several well-known assumptions, the social orderings of particular alternatives that are meant to reflect *individuals'* preferences must match the preferences of an arbitrary individual (the dictator). A social-choice rule other than dictatorship is impossible. Following from Fountain (2000), the author presents another graphical proof of the theorem that is intended to be more accessible to students and teachers of economics. The principal strength of this approach is that the patterns of agreements and conflicts over all possible combinations of two individuals' rankings of alternatives are transparent; appreciating these patterns is the key to intuitively understanding Arrow's theorem. A self-test for readers (or a classroom exercise for students) is included.

Key words: Arrow's Impossibility Theorem, graphical proof, independence of irrelevant alternatives, Pareto
JEL codes: A20, D71

Arrow's (1951) so-called Impossibility Theorem is the idea that, given several well-known assumptions,[1] the social orderings of particular alternatives that are meant to reflect *individuals'* preferences must match the preferences of an arbitrary individual (the dictator). A social-choice rule other than dictatorship (not in the pejorative sense of the word) is impossible. Although most economists (and many students) know this result, probably only a minority appreciate the intuition of the theorem's derivation, largely because formal (and general) proofs tend not to engage participants actively in discovering insights for themselves. In particular, Fountain (2000, 89) identified the lack of graphical demonstrations,[2] which he sought to remedy with "A Simple Graphical Proof of Arrow's Impossibility Theorem" that "students (and instructors) of intermediate microeconomics should have no problem in following."

Fountain performed a valuable service to students and teachers alike; nonetheless in this article, I offer another simple graphical proof that is intended to be more constructive and therefore more accessible to readers. Integral to the proof is the graphical representation of interpersonal agreements and conflicts introduced in the next section. Using this apparatus, I explain Arrow's theorem and then present the proof.

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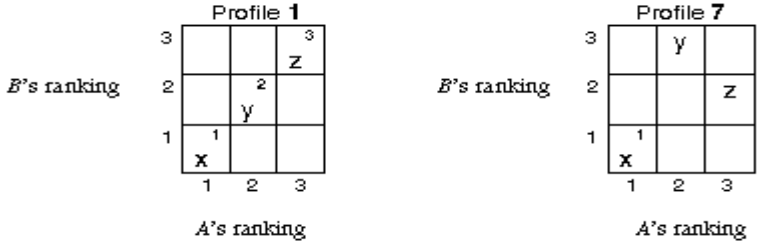
A GRAPHICAL REPRESENTATION

The underlying framework (the simplest possible) consists of two individuals, *A* and *B*, who each have individual preferences over three alternatives, *x*, *y*, and *z*. Ruling out equal rankings (i.e., indifference), six rankings of *x*, *y*, and *z* are possible for each individual: *xyz*, *xzy*, *yxz*, *yzx*, *zxy*, and *zyx* (where the position of a letter signifies its ranking relative to the other alternatives in the string). Six possible rankings for *A* and (the same) six for *B* equate to 36 pairs of rankings (i.e., 6×6), comprising all possible combinations of the two individuals' rankings of *x*, *y*, and *z*. Fountain (2000) refers to these pairs as "preference profiles" (e.g., one profile is *A*'s *xyz* and *B*'s *xyz*, another is *A*'s *xyz* and *B*'s *xzy*, etc.). An alternative to Fountain's representation of the profiles that is intended to more clearly reveal the patterns of agreements and conflicts between the individuals' rankings of alternatives is now explained. Appreciating these patterns is the key to intuitively understanding Arrow's theorem.

Thus, each of the 36 preference profiles is represented in Figure 1 by a grid, where the horizontal axis represents *A*'s ranking of the three alternatives (first, second or third),[3] and the vertical axis represents *B*'s ranking. For example (as reproduced below), for Profile 1, $x = (1,1)$, $y = (2,2)$, and $z = (3,3)$, and for Profile 7, $x = (1,1)$, $y = (2,3)$, and $z = (3,2)$ (where the first coordinate is *A*'s ranking of the alternative and the second is *B*'s). The 36 grids are arranged in Figure 1 in a convenient order for the proof presented later in the article.

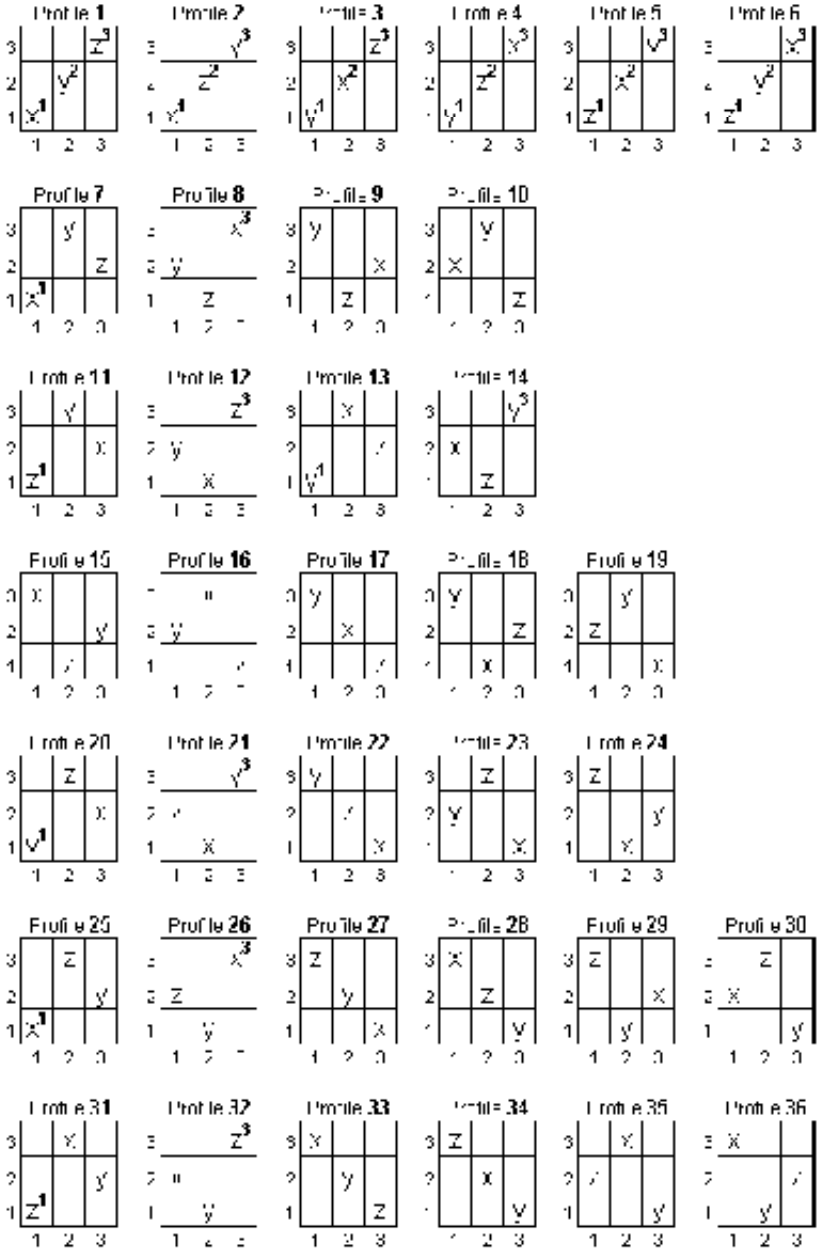


The grids enable pairwise agreements and conflicts between *A* and *B* over their rankings of *x*, *y*, and *z* in each preference profile to be visually identified. When an alternative is positioned to the south-west of another (i.e., $_**$), it is preferred over the second by *both* individuals; that is, there is a preference agreement. When two alternatives are aligned northwest-southeast of each other ($*_*$), the individuals disagree in their rankings; that is, there is a preference conflict.

For example, $_x{}^y$ in Profile 1 identifies that both *A* and *B* prefer *x* to *y* in that profile. Indeed, in Profile 1 all three alternatives are stacked on the 45° diagonal ($_x y^z$), signifying that *A* and *B* agree that *x* is first, *y* is second, and *z* is third, hence the alternatives are superscripted 1, 2, and 3 in Figure 1. Profiles 2 to 6 also exhibit unanimity between *A* and *B* (but, naturally, in each profile the three alternatives are uniquely ranked).

Similarly, Profile 7 is characterized by agreements $_x{}^y$ (*x* is preferred to *y* by *A* and *B*) and $_x{}^z$ (*x* is preferred to *z* by both), which ensures that *x* is unanimously ranked first (superscripted 1). (Equivalently, *x* is explicitly ranked first by both

FIGURE 1
36 Preference Profiles for Two Individuals and Alternatives *x, y,* and *z*



individuals.) As can be seen in Figure 1, 11 other profiles have an alternative that is unanimously ranked (superscripted) first or third. However, alternatives unanimously ranked second but without agreement over the first and third places (e.g., Profile 17: $^{y}x_{z}$), that is, without any *pairwise* preference agreements, are not superscripted and, in this respect, are irrelevant in the proof.

As mentioned, when two alternatives are aligned northwest-southeast of each other, there is a preference conflict, for example, $^{y}_{z}$ in Profile 7, where *A* prefers *y* to *z* but *B* prefers *z* to *y*. Appreciating the "direction" of the preference conflicts between the individuals is particularly important; clearly, conflict $^{y}_{z}$ (as above), for example, is not the same as its inverse, $^{z}_{y}$ (*A* prefers *z* to *y*, and *B* prefers *y* to *z*). Accordingly, there are six types of preference conflict across the 36 profiles: $^{y}_{z}$, $^{y}_{x}$, $^{x}_{z}$, $^{z}_{x}$, $^{z}_{y}$, and $^{x}_{y}$ (listed in the order they are encountered in the proof). Each type appears in nine profiles, either on its own (e.g., Profile 7: $^{y}_{z}$) or with another conflict (e.g., Profile 9: $^{y}_{z}$ and $^{y}_{x}$) or with two others (e.g., Profile 17: $^{y}_{z}$, $^{y}_{x}$, and $^{x}_{z}$).[4] Resolving these conflicts is, in the present context, the essence of welfare economics in general and social-choice rules in particular.

WELFARE ECONOMICS, SOCIAL-CHOICE RULES, AND ARROW'S IMPOSSIBILITY THEOREM

"The central objective of the study of welfare economics is to provide a framework which permits meaningful statements to be made about whether some [alternatives] are socially preferable to others" (Boadway and Bruce 1984, 137). Fundamental to this framework is the doctrine of *individualism*: that the social orderings of alternatives be based exclusively on individuals' preferences. This requires in the present setting that for each of the 36 preference profiles enumerated in Figure 1, *A* and *B*'s individual rankings of *x*, *y*, and *z*—which, as discussed above, can be in agreement or conflict, depending on the profile—are to be aggregated. Such aggregation, especially the resolution of preference conflicts, is achieved by way of social-choice rules (Sen 1970).[5]

The essence of Arrow's theorem is that, given the assumptions explained below, then dictatorship by either *A* or *B* is the *only* social-choice rule capable of socially ordering the alternatives for *all 36* preference profiles (and in the process, resolving preference conflicts). *Dictatorship* is a value-free term in this context; here it simply means that one individual's rankings of *x*, *y*, and *z* prevail in all 36 profiles regardless of the other individual's preferences. As discussed in the following section, the dictator (either *A* or *B*) is effectively chosen when the first preference conflict encountered is resolved in her/his favor. No other social-choice rule that satisfies the assumptions will produce a complete ranking of the alternatives for all profiles. Therefore if a dictatorship is disqualified *a priori*, then specifying a social-choice rule is impossible—hence Arrow's Impossibility Theorem.

Note that the emphasis here is on a social-choice rule capable of ranking the alternatives for all 36 profiles. This assumption, which is often known as the *unrestricted domain*, requires for consistency that the rule be universally applicable so that it applies not just to a particular profile that may exist at a point in time but to the full set of profiles that are theoretically possible. (In other words,

the domain from which an actual profile is drawn must be unrestricted.) On the contrary, if a rule is capable of socially ordering only a subset of possible profiles, and not the full set, then it does not qualify as a *bona fide* social-choice rule.

Implicit in the present framework is the assumption that the only information available concerning individuals' preferences (utilities) is ordinally measurable and interpersonally noncomparable. That is, the social-choice rule (or the social planner implementing it) has access only to each individual's rankings of the alternatives (as per the preference profiles introduced in the previous section). By how much each individual prefers one alternative over another or the strength of one individual's preferences relative to the other's is not known.

Also assumed are the *Pareto principle* and the *independence of irrelevant alternatives* (*IIA*).[6] The former assumption embodies the unanimity characteristic discussed in the previous section; for example, if both *A* and *B* prefer *x* to *y* (preference agreement $_{x}^{y}$), then the social-choice rule must rank *x* ahead of *y* (*xy*). Thus the Pareto principle generates complete social orderings for Profiles 1 to 6 (as evidenced by the superscripts 1, 2, and 3 in Figure 1), as well as many partial social orderings (including, but not limited to, the alternatives superscripted 1 or 3 in other profiles in Figure 1).

However, to completely socially order the remaining profiles (Profiles 7 to 36), stronger value judgments than the Pareto principle are required because these profiles include at least one preference conflict. Resolving these conflicts is the key function performed by social-choice rules. The independence of irrelevant alternatives assumption requires that any value judgment applied to a *particular* conflict in a particular profile must hold for *all* profiles with the *same* type of conflict. This assumption (more fully explained later) and the others discussed in the previous paragraphs ensure that the resulting social orderings of all 36 profiles are identical to the rankings of *one* of the individuals. This startling result (Arrow's Impossibility Theorem) is now proved using the graphical apparatus introduced earlier.

THE PROOF

The objective is to specify a social-choice rule capable of socially ordering alternatives *x*, *y*, and *z* in all 36 preference profiles. This amounts to assigning superscripts 1, 2, or 3 to the alternatives in each profile in Figure 1. Thanks to the Pareto principle, Profiles 1 to 6 are already completely ordered (superscripted). The task, therefore, is to socially order Profiles 7 to 36 (most of which are partially ordered by Pareto)—one at a time in the order they appear in Figure 1, beginning with Profile 7. The choice of Profile 7 is arbitrary; nonetheless, as shown in the following section, regardless of with which profile we begin, the same result (a dictatorship) will emerge.

In Profile 7, as can clearly be seen in Figure 1, although *A* and *B* agree that *x* comes first, (and therefore by the Pareto principle, it must head Profile 7's social ordering), they disagree in their second and third placings of *y* and *z*: *A*'s ranking is *yz* whereas *B*'s is *zy* (i.e., preference conflict $_{z}^{y}$). (Readers are encouraged to consult closely the profiles in the figures as they are referred to.) The only way

of resolving this conflict (assuming, as we have, that a tie is unacceptable) is for one individual's preference to supersede the other's. The social-choice rule must impose the value judgment in Profile 7 that either *A* gets her way (*yz*) or *B* gets his (*zy*). Recall, the only information available is the individuals' rankings of alternatives and not the strengths of their preferences. Suppose (arbitrarily) that *A* is favored so that her preference *yz* prevails and *y* is *socially* ranked ahead of *z*, and hence Profile 7's social ordering is determined as *xyz* (given *x* is first via Pareto). The key issue henceforth is the application of the independence of irrelevant alternatives (IIA) assumption introduced in the previous section.

According to the IIA assumption, any value judgment imposed by the social-choice rule to resolve a particular preference conflict must hold for *all* other profiles with the *same* type of conflict.[7] As noted earlier, across the 36 profiles there are six types of conflict—$^{y}_{z}$, $^{y}_{x}$, $^{x}_{z}$, $^{z}_{x}$, $^{z}_{y}$, and $^{x}_{y}$—each appearing in 9 profiles; therefore the social-choice rule effectively has six generic conflicts to resolve. Hence the social ordering *yz* imposed above for Profile 7's conflict $^{y}_{z}$ (the first of the types in the list above) must apply to the 8 other profiles with conflict $^{y}_{z}$:[8] Profiles 8, 9, 10, 16, 17, 18, 22, and 33 (see Figure 1).

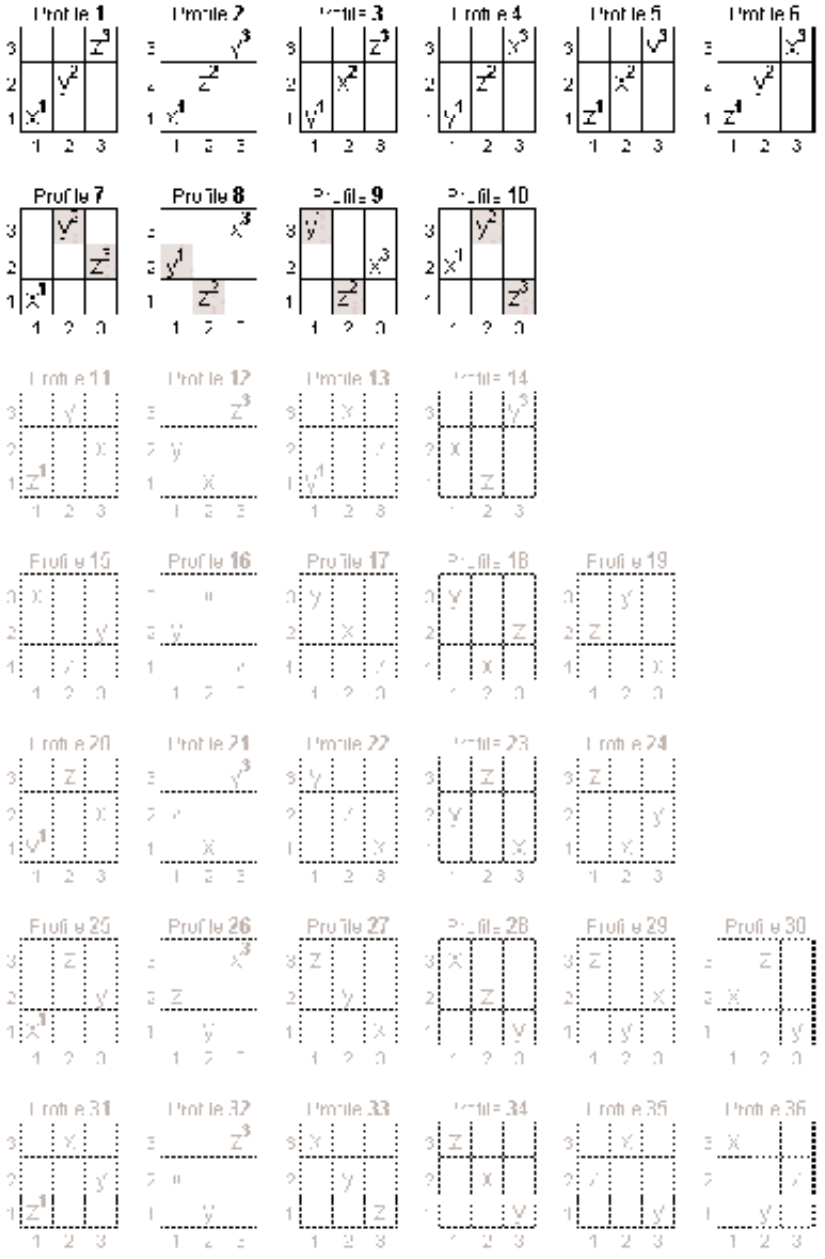
Moreover, implicitly this social ordering (*yz* for conflict $^{y}_{z}$) is independent of the ranking of the third alternative, *x*. *yz* applies to conflict $^{y}_{z}$ no matter whether in the nine profiles with this type of conflict *x* is unanimously ranked first (e.g., Profile 7) or second (Profile 17) or third (Profile 8); that is, *x* is irrelevant to the social ordering of *y* vis-à-vis *z* in these profiles. In other words, Arrow's (1951) IIA assumption requires that, if value judgment *yz* is imposed on conflict $^{y}_{z}$, then it must apply to all conflicts of this type, *independently of the irrelevant alternative, x*.[9]

Therefore, to reiterate because of the IIA assumption the social ordering *yz* imposed on Profile 7 must also be imposed on the eight other profiles (listed in the preceeding paragraph) with conflict $^{y}_{z}$, including Profiles 8, 9, and 10. Profiles 7 to 10 are highlighted in the second row of Figure 2, where conflict $^{y}_{z}$ is shaded and the social orderings from applying the IIA assumption in conjunction with the Pareto principle (and transitivity; see note 6) are in plain superscripts (as opposed to bold ones which denote unanimity). For Profile 8, *yzx* arises from combining Profile 7's value judgment *yz*, which also applies to Profile 8 because of the IIA assumption, with *x*'s unanimous third placing (via Pareto). Profile 9's *yzx* arises from, first, *yz* (Profile 7's value judgment, again via the IIA assumption) and, second, *z*'s unanimous ranking ahead of *x* (preference agreement $^{x}_{z}$ via Pareto)—so that (by transitivity) *y* must also be socially ranked ahead of *x* (i.e., combining *yz* with *zx* gives *yx*—and ultimately *yzx*). Similarly, Profile 10's *xyz* is from combining *xy* (agreement $^{y}_{x}$ via Pareto) with *yz* (the initial value judgment via the IIA assumption).

The five other profiles with the same conflict ($^{y}_{z}$) and therefore social ordering *yz*—Profiles 16, 17, 18, 22, 33—are not highlighted and superscripted in Figure 2 because their rankings of *x* (the irrelevant alternative) relative to *y* and *z* are as yet unspecified by the social-choice rule that I am in the process of developing—I am working through the profiles in the order they appear in the figures.

The next step is to recognize the *supplementary* value judgments—supplementary to *yz* for conflict $^{y}_{z}$ (as above)—implied by the social orderings derived

FIGURE 2
Social Orderings for Profiles 1 to 10 (with Conflict Type $^{y}_{z}$ Shaded)



immediately above. An inevitable byproduct of *yzx* for Profile 9 is the partial social ordering (value judgment) *yx* for that profile's other conflict, $^{y}_{x}$ (lightly shaded in Figure 3). Likewise, for Profile 10, a byproduct of its social ordering *xyz* is value judgment *xz* for its other conflict, $^{x}_{z}$ (heavily shaded).[10] Both of these supplementary value judgments are consequences of the initial value judgment (*yz* for conflict $^{y}_{z}$ at Profile 7)—that, because of the IIA assumption, extended to Profiles 9 and 10 (and six others), thereby implicitly resolving conflicts $^{y}_{x}$ and $^{x}_{z}$ with *yx* and *xz* (see note 10). Moreover, because of the IIA assumption, both supplementary value judgments must also apply to the eight other profiles with each conflict type. (The five other occurrences of conflicts $^{y}_{x}$ and $^{x}_{z}$ that are relevant to this stage of the social ordering exercise are also lightly and heavily shaded respectively in Figure 3.)

Thus, as well as the first of the six conflict types identified earlier ($^{y}_{z}$), the second and third types ($^{y}_{x}$ and $^{x}_{z}$) have been resolved in *A*'s favor (*yx* and *xz*), from which emerges the social orderings for Profiles 11 to 19 in Figure 3. The orderings for Profiles 11 to 14 are obvious (Figure 3), given *yx* and *xz* and unanimity over *z*'s and *y*'s first and third places, respectively. The orderings for Profiles 15 and 16 arise from *xz* and (Pareto) preference agreements $_{z}^{y}$ and $_{y}^{x}$, respectively: combining *xz* with *zy* gives *xzy*, and *yx* with *xz* gives *yxz*. Analogously, Profile 18's *yxz* and 19's *zyx* follow from combining *yx* with their respective agreements $_{x}^{z}$ and $_{z}^{y}$. Profile 17's *yxz* is from *yx* with *xz*.[11] Notice from Figure 3 that the social orderings for Profiles 1 to 19 all conform to *A*'s preferences,[12] thus far confirming the dictatorship at the heart of Arrow's Impossibility Theorem.

In the next stage of the social ordering exercise, an inevitable byproduct of Profile 19's *zyx* is value judgment *zx* for that profile's other conflict, $^{z}_{x}$ (shaded)—the fourth of the conflict types identified earlier. Profiles 20 to 24 (and three others) also have this type of conflict (also shaded). Therefore by the IIA assumption, *zx* must also apply to them, from which follows their social orderings in Figure 4. For Profiles 20 and 21, they are obvious given *A* and *B*'s unanimity over *y*'s first and third places, respectively. Profile 22's *yzx* is from combining *yz* (the initial value judgment at Profile 7) with *zx*. Profile 23 and 24's orderings follow from their respective agreements $_{y}^{z}$ and $_{x}^{y}$ combined with *zx*: *yz* with *zx* gives *yzx*, and *zx* with *xy* gives *zxy*.

Next, it follows from Profile 24 that when the conflict is $^{z}_{y}$ (shaded in Figure 5), the fifth conflict type, the social ordering (value judgment) is *zy*, which by the IIA assumption must also apply to the eight other profiles with this conflict type, in particular Profiles 25 to 30. Their social orderings, by analogous arguments to the above, are as in Figure 5. Finally, it follows from Profile 30 that when the conflict is $^{x}_{y}$ (shaded in Figure 6)—the last of the six conflict types that must be resolved—the social ordering is *xy*, which, again by the IIA assumption, holds for the eight other profiles with this conflict type, including Profiles 31 to 36.[13] Profiles 31 to 36 are ordered as in Figure 6 and the social ordering exercise is complete.

The social-choice rule is now fully specified: Apply the Pareto principle for preference agreements between *A* and *B* and then resolve the preference conflicts with *yz* for $^{y}_{z}$, *yx* for $^{y}_{x}$, *xz* for $^{x}_{z}$, *zx* for $^{z}_{x}$, *zy* for $^{z}_{y}$, and *xy* for $^{x}_{y}$. These value judgments are all in *A*'s favor, and from Figure 6 it is easily verified that all 36

FIGURE 3
Social Orderings for Profiles 1 to 19
(with Conflict Types $^{y}_{z}$ Lightly Shaded and $^{x}_{z}$ Heavily Shaded)

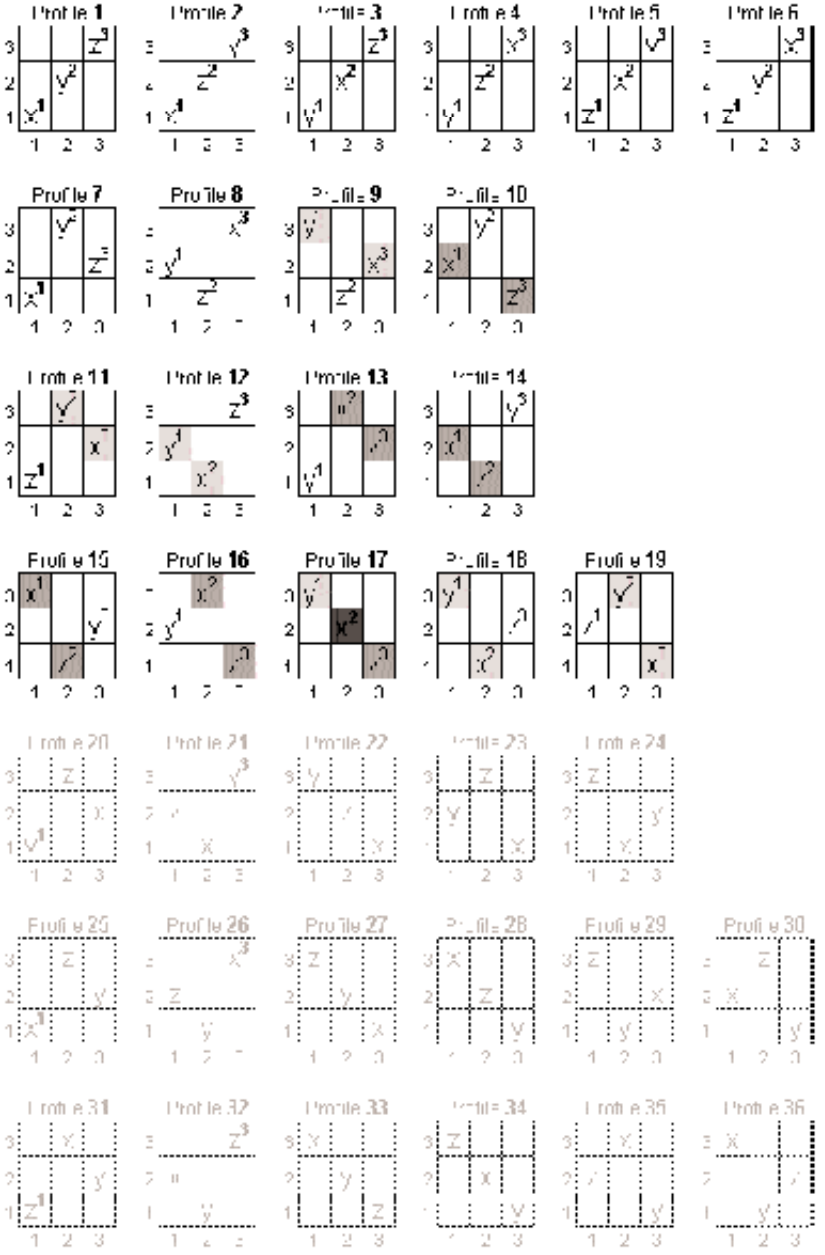


FIGURE 4
Social Orderings for Profiles 1 to 24 (with Conflict Type z_x Shaded)

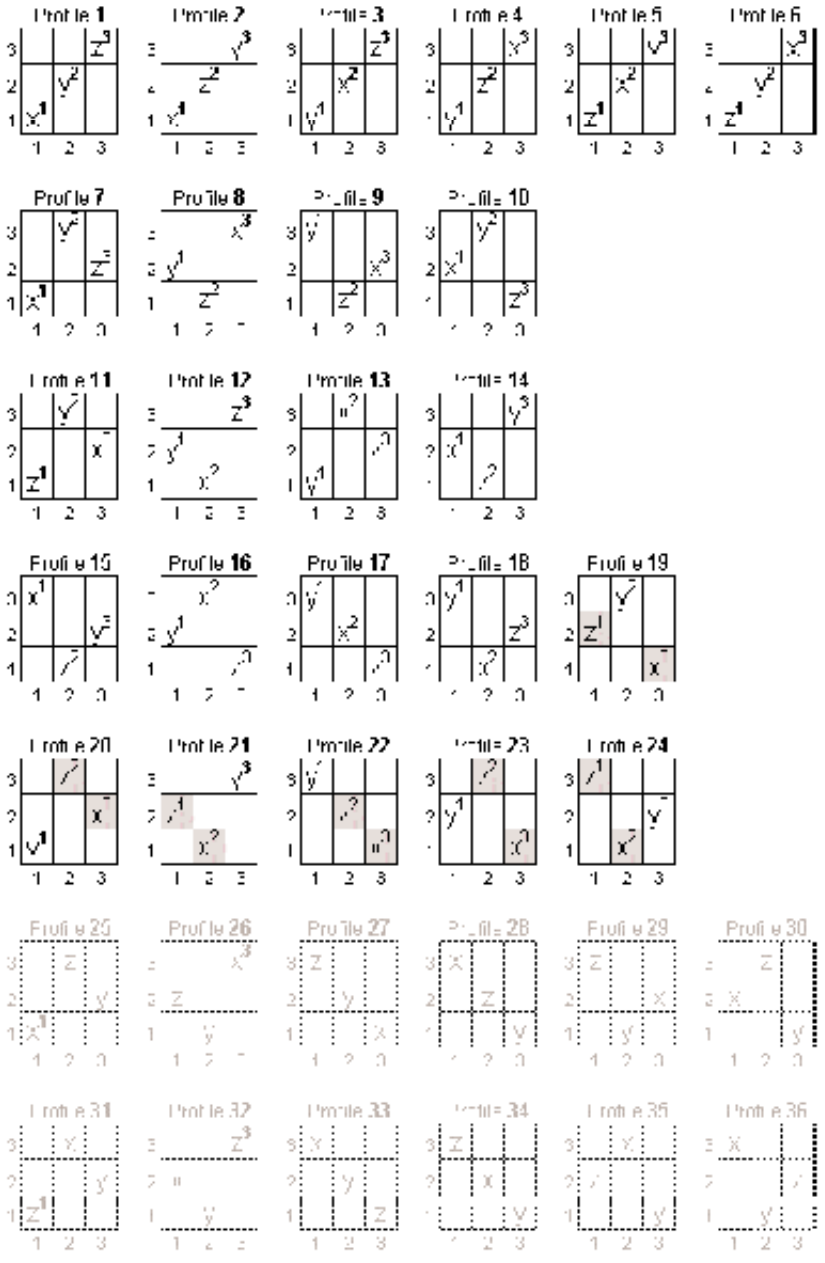


FIGURE 5
Social Orderings for Profiles 1 to 30 (with Conflict Type $^{z}_{y}$ Shaded)

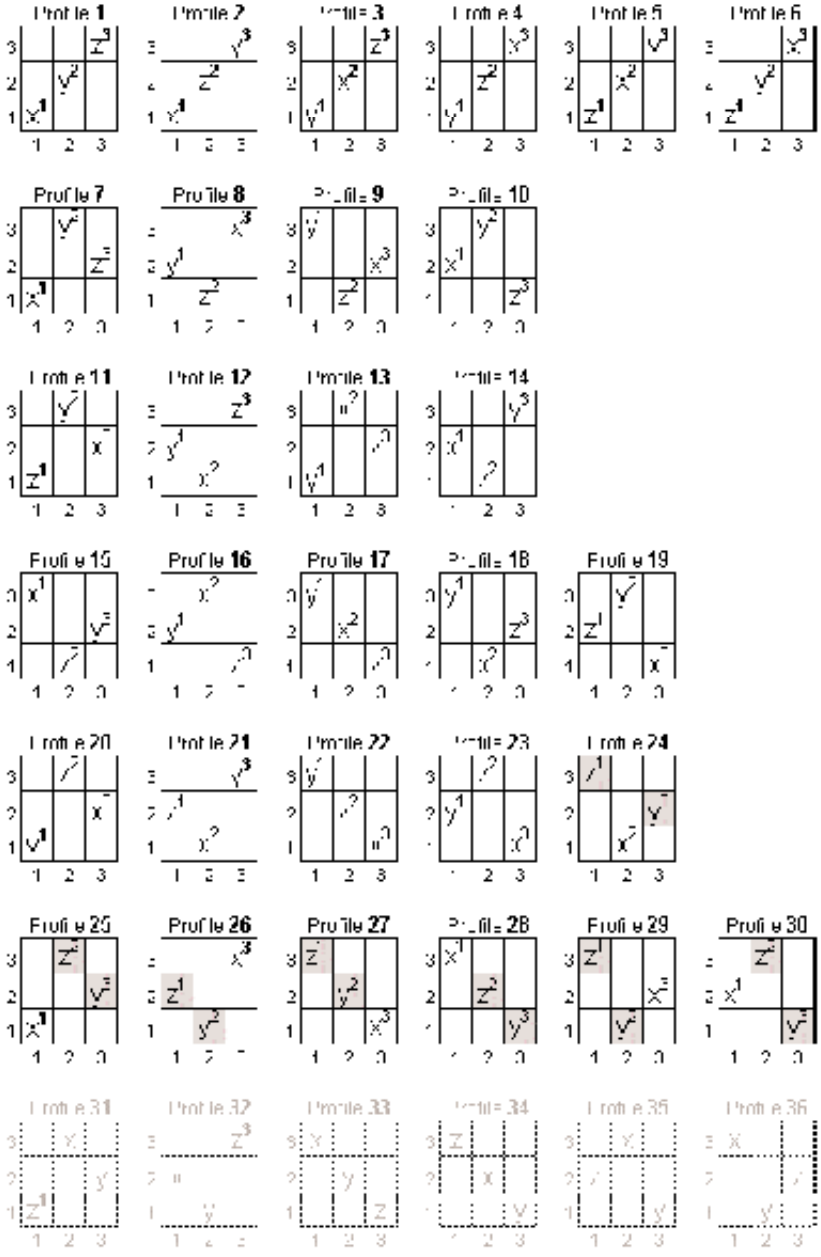
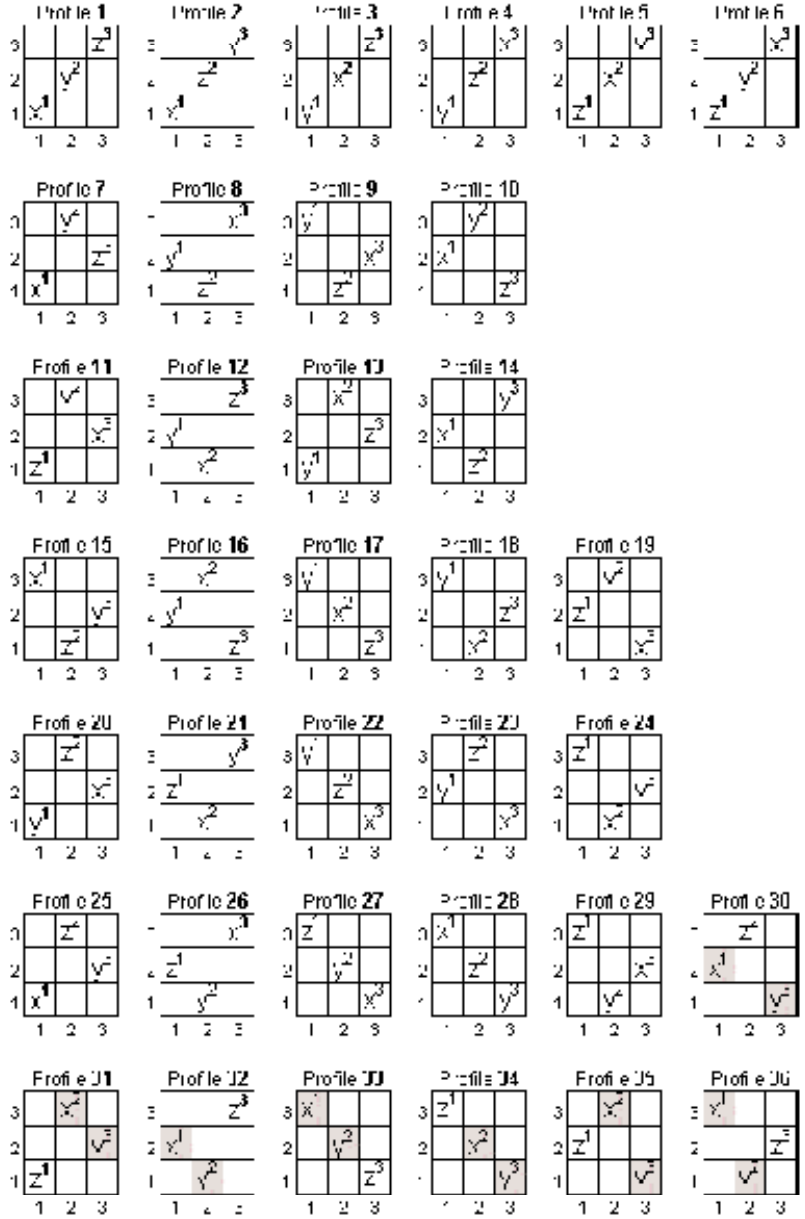


FIGURE 6
Social Orderings for Profiles 1 to 36 (with Conflict Type $^{x}_{y}$ Shaded)—
***A*'s Dictatorship across the 36 Profiles is Universal!**



social orderings match her preferences. *A*'s "privileged" position is a consequence of the initial value judgment at Profile 7 favoring *A* over *B* (*yz* for $^{y}_{z}$). At that moment, the die was cast because of the independence of irrelevant alternatives assumption and *A* was destined to dictate the social orderings for all 36 profiles. Alternatively, had *B* been favored at Profile 7 instead of *A* (*zy* for $^{y}_{z}$), then the social ordering exercise would proceed in an identical fashion except the six value judgments and the resulting 36 social orderings would conform to *B*'s preferences.

This result, however, is not sufficient to prove Arrow's theorem that dictatorship is the *only* social-choice rule capable of ordering the alternatives for all possible preference profiles. We have proven that *one* set of social orderings corresponds to a dictatorship; it remains, therefore, to show that no other set of orderings is possible. In particular, might not a *non*dictatorial social ordering of the 36 profiles arise if a different profile and a different value judgment were initially imposed (one, say, that resulted in neither individual's ranking being adopted for the profile)? In other words, could *A*'s dictatorship be an artifact of the initial value judgment adopted (*yz* at Profile 7)? The answer is *no*. To prove this, we must consider the full range of alternatives to Profile 7 (excluding Profiles 1 to 6, which are ordered by Pareto) and show that, regardless of the value judgment initially imposed, dictatorship is the only social-choice rule capable of ordering all 36 profiles.

GENERALIZING THE DICTATORSHIP RESULT

To this end, Profiles 7 to 36 can be classified according to the number of conflicts in each (discernible from the pairs of alternatives that are aligned northwest-southeast: $*_{*}$). Thus there are three types of profile: 12 profiles with just *one* conflict each—with the pattern $_{*}{}^{*}{}_{*}$ or $*_{*}{}^{*}$ (e.g., Profiles 7 and 8); 12 profiles with *two* conflicts each—$^{*}{}_{*}{}^{*}$ or $*{}^{*}{}_{*}$ (e.g., Profiles 9 and 10); and six profiles with *three* conflicts each—$^{*}{}_{*}{}^{*}{}_{*}$ (e.g., Profile 17). Completing the proof of Arrow's theorem centers on showing that, for each type of profile, the initial value judgment (whatever it is) leads inexorably to either a dictatorship or the rule is incapable of ordering all 36 profiles and therefore is disqualified as a *bona fide* social-choice rule.

This result is seen most easily for profiles with one conflict each. No matter which of the 12 profiles of this type is initially confronted—and, accordingly, no matter which of the six conflict types that are possible is resolved in either *A*'s or *B*'s favor—the initial value judgment leads to a repeat of the steps detailed above, albeit in a different order. The end result is a dictatorship.

For example, if in Profile 7's stead (in which, in the previous section's demonstration, the initial value judgment was imposed), Profile 11 were initially confronted and value judgment *yx* imposed for conflict $^{y}_{x}$ (in *A*'s favor, see Figure 3). Then, because of the IIA assumption, *yx* would also apply to the eight other profiles with this conflict type ($^{y}_{x}$), including Profile 18. An inevitable byproduct of Profile 18's resulting social ordering (*yxz*) is value judgment *yz* for that profile's other conflict, $^{y}_{z}$, which, because of the IIA assumption, applies to the other eight profiles with this conflict type, including Profile 7. With Profile 7 socially ordered *xyz* as in the previous section, the same ordering process as

detailed there unfolds. Again *A* is the dictator. (A similar example that begins with conflict $^{x}_{y}$ in Profile 32 appears in Figure A1 in the appendix. It and two other appendix worksheets analogous to the examples discussed here are offered for readers to complete themselves.)

For profiles with two or three conflicts, the demonstration is more complicated. For the 12 profiles with two conflicts each, one or two value judgments are required to arrive at a complete social ordering, from which three orderings are possible (e.g., for Profile 15: *xzy*, *zyx,* or *zxy*). One corresponds to *A*'s ranking (e.g., for Profile 15 *xzy*), another to *B*'s (*zyx*), and the third matches neither individual's ranking (*zxy*). When the social ordering is either *A*'s or *B*'s, the dictatorship proof proceeds as in the previous section, albeit from a different starting profile; again a dictatorship emerges. However if the ordering matches neither individual's ranking (e.g., *zxy* for Profile 15), that is, a dictatorship across all 36 profiles is immediately precluded, then an intransitivity eventuates in the orderings being derived, thereby preventing this particular rule from socially ordering the full set of profiles. As discussed earlier, if it does not apply universally (i.e., to an "unrestricted domain"), it is not a *bona fide* social-choice rule.

For example, if the initial value judgment for Profile 15 were *xz* (*A*'s preference, resulting in social ordering *xzy*; see Figure 3 again) then, because of the IIA assumption, *xz* must be imposed on the eight other profiles with conflict $^{x}_{z}$, including Profile 16. From Profile 16's resulting social ordering (*yxz*), *yz* is implied for conflict $^{y}_{z}$, which then orders Profile 7 (and seven others) as before, and the proof proceeds as before with *A* as the dictator. On the other hand, if the initial value judgment for Profile 15 were *zx* (neither individual's preference, resulting in *zxy*; see Figure 3), then *zx* for conflict $^{x}_{z}$ and *xy* for $^{x}_{y}$ must be imposed on the other profiles. Notice that Profile 36 also has these conflict types ($^{x}_{z}$ and $^{x}_{y}$), and therefore the same orderings (*zx* and *xy*) must apply to it. By transitivity, *zx* and *xy* gives *zy*, but this contradicts *yz* (via Pareto) from Profile 36's preference agreement $_{y}^{z}$ (which, not surprisingly, is the opposite of Profile 15's $_{z}^{y}$). In general, therefore, when a value judgment that corresponds to neither individual's ranking is initially imposed (thereby immediately ruling out a dictatorship), it is impossible to socially order all 36 preference profiles because of an inevitable intransitivity.

An analogous result holds for the six profiles with three preference conflicts each (e.g., Profile 17: $^{y}x_{z}$). From the three value judgments necessary to socially order such profiles, six orderings are possible (*xyz*, *xzy*, *yzx*, *zyx*, *yxz*, *zxy*), of which one corresponds to *A*'s ranking and another to *B*'s. Only these two lead to a full set of social orderings for all 36 preference profiles—each confirming the by-now familiar dictatorship result. The other four orderings, corresponding to neither individual's preference, are incapable of socially ordering all profiles.

For example (see Figure 1), if *yxz*, corresponding to *A*'s ranking, were imposed for Profile 17 then, because of the IIA assumption, *yz* applies for conflict $^{y}_{z}$, which orders Profile 7 as before and *A* again emerges as the dictator. Alternatively, if *xyz* were imposed for Profile 17 (neither individual's ranking) then *xy* for conflict $^{y}_{x}$ and *yz* for $^{y}_{z}$ must also apply to other profiles, including Profile 9. By transitivity, *xy* and *yz* gives *xz*, but this contradicts *zx* implied by Profile 9's preference agreement $_{z}^{x}$ (via Pareto). Therefore, as above, when neither individ-

ual's ranking is initially imposed (and hence a dictatorship is immediately precluded), it is impossible to socially order all 36 preference profiles.

This completes the proof of Arrow's Impossibility Theorem. No matter which of the three profile types is initially confronted, and no matter which value judgment is imposed, dictatorship is the only social-choice rule capable of socially ordering all profiles.

CONCLUSION

In this article, I sought to constructively demonstrate, in the simplest setting possible, that provided individuals' rankings of alternatives is the only information available, and assuming the Pareto principle and the independence of irrelevant alternatives, then the only social-choice rule capable of ordering all possible preferences profiles is a dictatorship. This is Arrow's Impossibility Theorem. The principal strength of the graphical approach developed here is the transparency of the patterns of preference agreements and conflicts between two individuals. Appreciating these patterns is essential for intuitively understanding Arrow's theorem. The more obvious the agreements, the easier it is to apply the Pareto principle; the more obvious the conflicts, especially their direction (e.g., $^{x}_{y}$ versus $^{y}_{x}$), the easier it is to apply the independence of irrelevant alternatives assumption.

The IIA assumption is very important in Arrow's theorem. Because of it, the dictator is effectively chosen when the first preference conflict encountered is resolved with a value judgment in her favor (after having exhausted the possible, but incomplete, orderings enabled by the Pareto principle). Indeed, the dictatorship result is often overturned by relaxing the IIA assumption, such that opposite social orderings are allowed for identical types of preference conflict. Also, the range of social-choice rules that are possible is expanded if more information concerning individuals' preferences than just their rankings of alternatives is available. See, for example, Boadway and Bruce (1984, chap. 5) for a discussion of these extensions.

I hope the proof presented here is accessible to anyone not pathologically averse to logical thinking. As a self-test for readers or a classroom exercise for students, social ordering exercises analogous to the ones presented here could be completed by anyone with an interest in Arrow's theorem. For this purpose, three new figures with different arrangements of the profiles are presented in the appendix that readers are invited to socially order using the logic demonstrated in the article. Answers are available from the author on request. Happy social dictating!

APPENDIX
Self-Test or Classroom Exercise for Students

As in the proof presented here, the objective in each of Figures A1 to A3 (which could be photocopied for students) is to assign superscripts 1, 2, or 3 to the alternatives that do not already have one. Start at the top of each of the figures (after Profiles 1 to 6), where one or more value judgments (in *A*'s favor) have been imposed on an arbitrarily selected preference conflict between *A* and *B* (shaded). Using the logic outlined in the article, socially order each of the profiles in the sequence they appear. One hint is provided near

Figure A1
36 Preference Profiles for Two Individuals and Alternatives *x*, *y*, and *z*
(with Conflict Type $^{x}_{y}$ at Profile 32 Shaded)

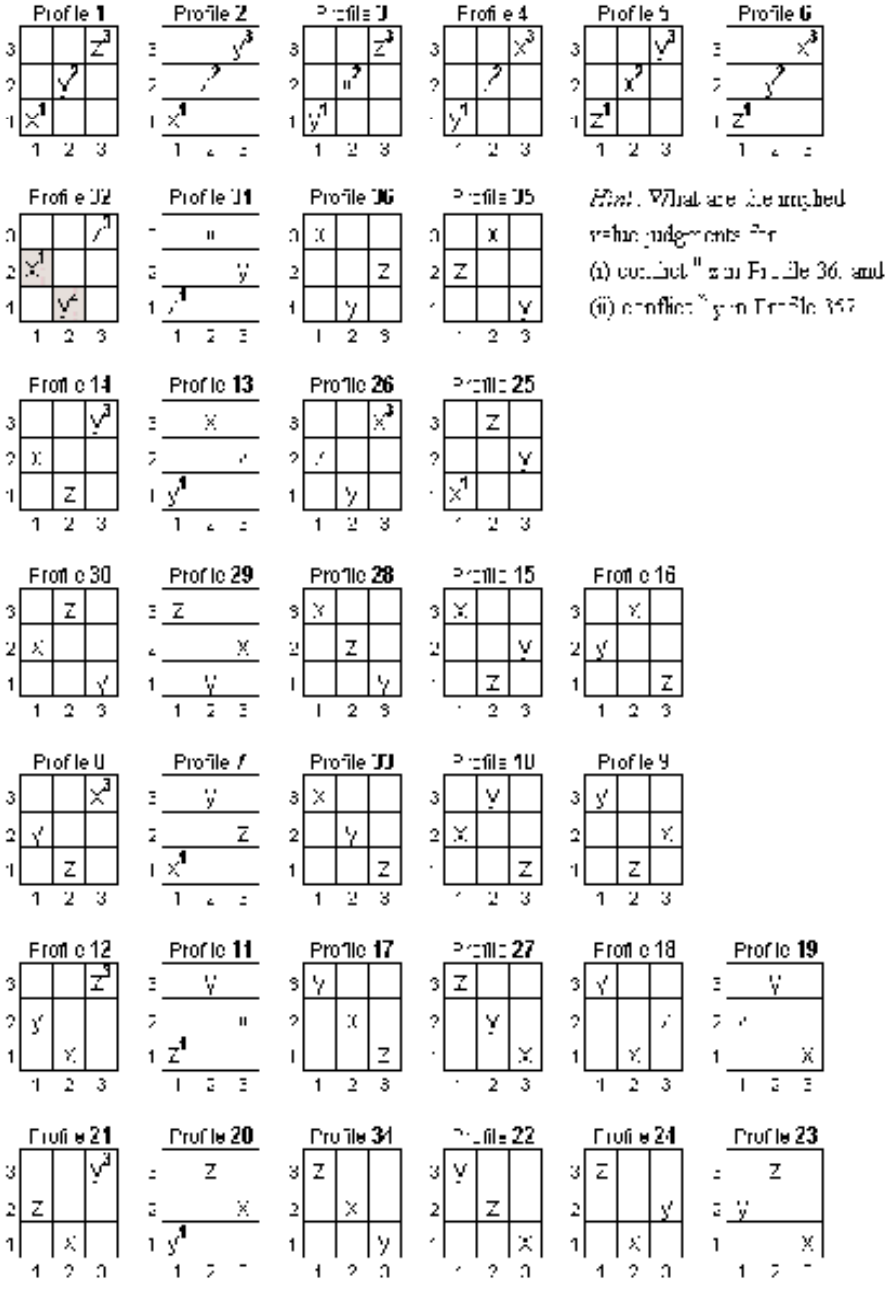


Figure A2
36 Preference Profiles for Two Individuals and Alternatives *x, y,* and *z*
(with Conflict Types $^{x}_{y}$ and $^{x}_{z}$ at Profile 36 Shaded)

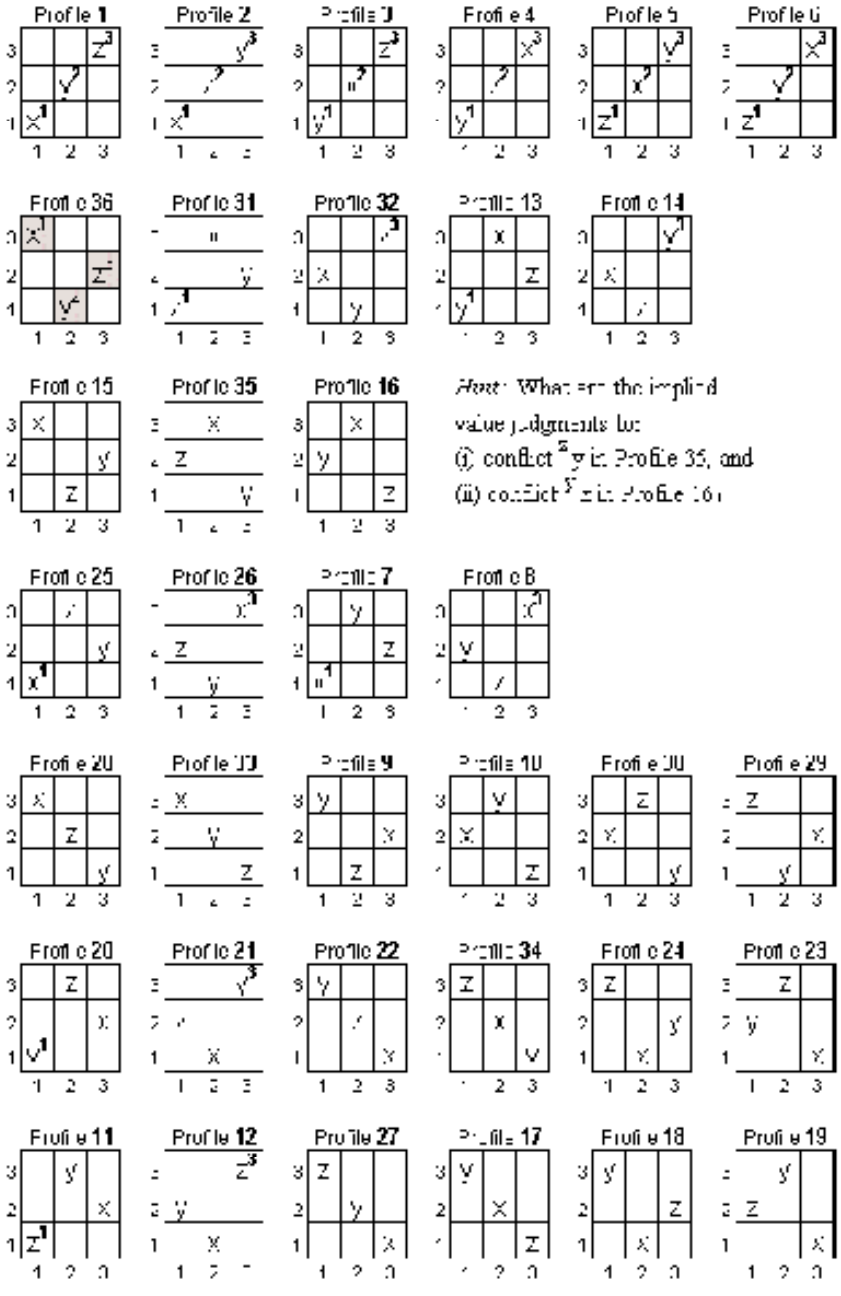
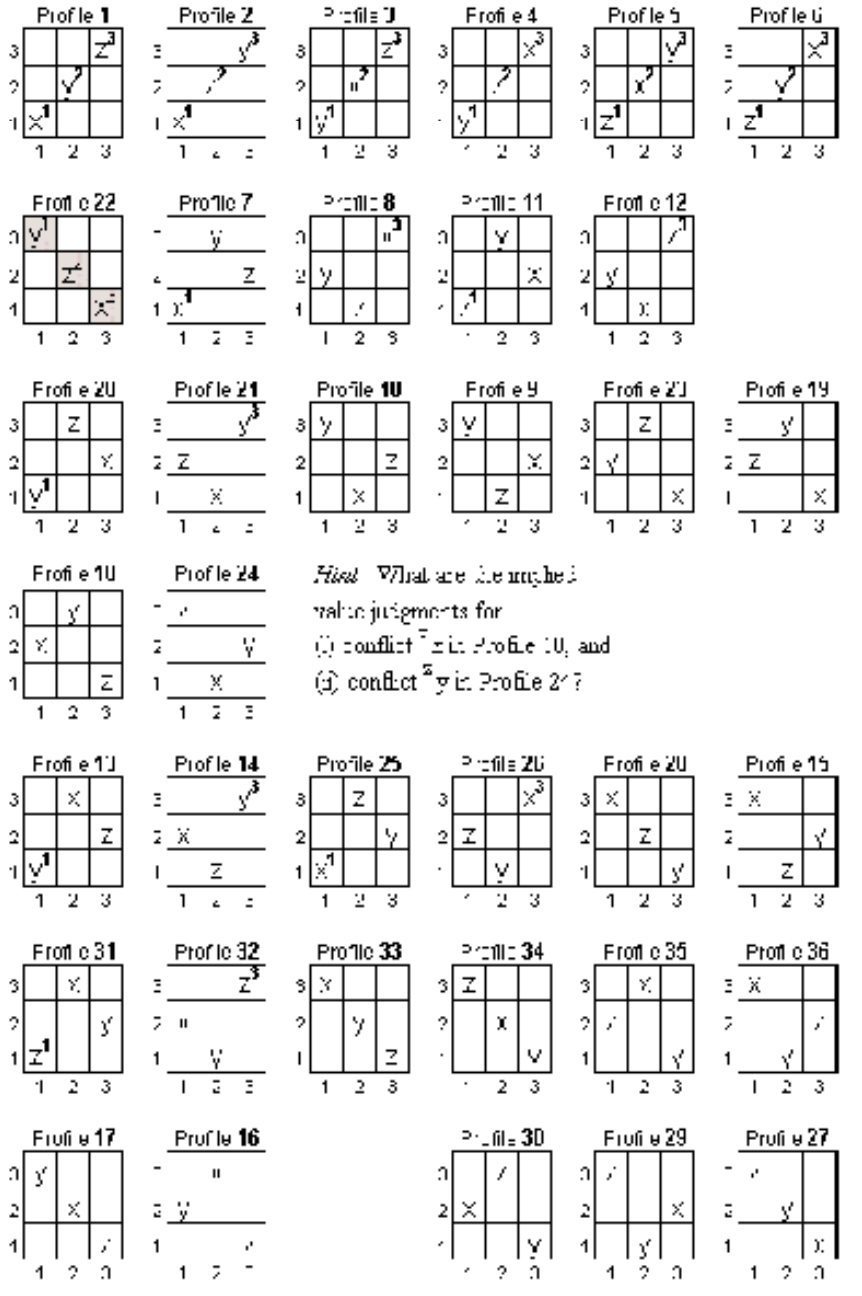


Figure A3
36 Preference Profiles for Two Individuals and Alternatives *x*, *y*, and *z*
(with Conflict Types $^{y}_{z}$ and $^{z}_{x}$ at Profile 22 Shaded)



the beginning of each worksheet to make sure that you are on the right track. (Note also that alternative sequences of the profiles are possible, all originating from the designated profile where the first value judgment is imposed (see note 13). As should be obvious from reading the article, the expected result is that individual *A* (whose rankings are on the horizontal axis) is revealed as the inevitable dictator, thereby confirming Arrow's Impossibility Theorem.

NOTES

1. The assumptions are unrestricted domain, the Pareto principle, the independence of irrelevant alternatives, and individual utilities that are ordinally measurable and interpersonally noncomparable. Each of these is explained below.
2. Although Fountain does not acknowledge it, Blackorby, Donaldson, and Weymark (1984) have a graphical proof that appears in textbooks such as Boadway and Bruce (1984) and Jehle and Reny (2001).
3. The negative of these places could be interpreted as ordinal utilities.
4. Equivalently, there are three types of profile: profiles with one, two and three conflicts (as well as Profiles 1 to 6 with none). This trichotomy is exploited in the penultimate section of the article.
5. Social welfare functions are specialized forms of social-choice rules.
6. Uncontroversially, the transitivity property is also assumed in the individual and social orderings; for example, if *x* is preferred to *y*, and *y* is preferred to *z*, then *x* is also preferred to *z* (i.e., combining *xy* with *yz* gives *xz*—and ultimately *xyz*).
7. Implicitly, this assumption applies to (Pareto) agreements between individuals as well.
8. Note, as emphasized earlier, conflict $^{y}_{z}$ (*A* prefers *yz* and *B* prefers *zy*), for example, should not be confused with $^{z}_{y}$ (*A* prefers *zy* and *B* prefers *yz*).
9. See Arrow's (1951, 27–28) discussion of the "reasonableness of this assumption."
10. Profile 9's *yx* arises, by transitivity, from combining Profile 7's *yz* (via the IIA assumption) with *zx* (via Pareto from $_{z}^{x}$); likewise, Profile 10's *xz* arises from *xy* (via Pareto from $_{x}^{y}$) and Profile 7's *yz* (via the IIA assumption).
11. Profile 17's overlap of value judgments *yx* and *xz* is represented in Figure 3 by *x*'s "double-shading" (i.e., light and heavy combined).
12. This is easily discerned from the grids in Figure 3 (and subsequent figures) by the sequencing from left to right of the superscripts 1, 2, and 3 in each profile.
13. The two other profiles with conflict type $^{x}_{y}$, Profiles 15 and 28, have already been ordered *xy* as byproducts of earlier value judgments (*xz* and *zy*). Likewise, other partial orderings implied at this final stage (e.g., *xz* for conflict $^{x}_{z}$ and *zy* for $^{z}_{y}$ in Profiles 36 and 35 respectively) confirm value judgments that were imposed earlier (such as the two above). This illustrates the point that in the process of socially ordering the 36 profiles, alternative sequences with the same beginning (value judgment *yz* at Profile 7) and outcome (dictatorship) are possible.

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