Exercise #3 - Per unit taxes vs. Ad valorem taxes

3. A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used: In one case, a *per unit* tax is levied, where an amount t is paid per unit bought or sold. In the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

   (a) Show that, with a per unit tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax. As a guidance, let us use the following steps:

      1. *Consumers:* Let p^c be the competitive equilibrium price when the *consumer* pays the tax. Note that when the consumer pays the tax, he pays $p^c + t$ whereas the producer receives p^c. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the consumer pays the tax.

         - If the per unit tax t is levied on the consumer, then he pays $p+t$ for every unit of the good, and the demand at market price p becomes $x(p+t)$. The equilibrium market price p^c is determinded from equalizing demand and supply:

$$x(p^c + t) = q(p^c).$$

2. *Producers:* Let p^p be the competitive equilibrium price when the *producer* pays the tax. Note that when the producer pays the tax, he receives $p^p - t$ whereas the consumer pays p^p. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the producer pays the tax.

   - On the other hand, if the per unit tax t is levied on the producer, then he collects $p - t$ from every unit of the good sold, and the supply at market price p becomes $q(p-t)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q(p^p - t).$$

(b) Show that if an equilibrium price p solves your equality in part (a), then $p + t$ solves the equality in (b). Show that, as a consequence, equilibrium amounts are independent of whether consumers or producers pay the tax.

- It is easy to see that p solves the first equation if and only if $p + t$ solves the second one. Therefore, $p^p = p^c + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is

$$x(p^p) = x(p^c + t).$$

(c) Show that the result in part (b) is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? [*Hint:* Use the same steps as above, first for the consumer and then for the producer, but taking into account that now the tax increases the price to $(1+\tau)p$. Then, construct the excess demand function for the case of the consumer and the producer.]

- If the ad valorem tax τ is levied on the consumer, then he pays $(1+\tau)p$ for every unit of the good, and the demand at market price p becomes $x((1+\tau)p)$. The equilibrium market price p^c is determined from equalizing demand and supply:

$$x((1+\tau)p^c) = q(p^c).$$

  On the other hand, if the ad valorem tax τ is levied on the producer, he receives $(1+\tau)p$ for every unit of the good sold, and the supply at market price p becomes $q((1-\tau)p)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q((1-\tau)p^p).$$

Consider the excess demand function for this case:

$$z(p) = x(p) - q((1-\tau)p) \tag{1}$$

Since the demand curve $x(\cdot)$ is non-increasing and the supply curve $q(\cdot)$ is non-decreasing, $z(p)$ must be non-increasing. From (1) we have

$$\begin{aligned} z((1+\tau)p^c) &= x((1+\tau)p^c) - q((1-\tau)[(1+\tau)p^c]) = \\ &= x((1+\tau)p^c) - q((1-\tau^2)p^c) \geq \\ &\geq x((1+\tau)p^c) - q(p^c) = 0, \end{aligned}$$

where the inequality takes into account that $q(\cdot)$ is non-decreasing.

- Therefore, $z((1+\tau)p^c) \geq 0$ and $z(p^p) = 0$. Since $z(\cdot)$ is non-increasing, this implies that $(1+\tau)p^c \leq p^p$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way, it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax on producers).

(d) Are there any special cases in which the collection method is irrelevant with an ad valorem tax? [*Hint:* Think about cases in which the tax introduces the same wedge on consumers and producers (inelasticity). Then prove your statement by using the above argument on excess demand functions.]

- If the supply function $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1+\tau)p^c < p^p$. On the other hand, when the supply is perfectly inelastic, i.e., $q(p) = \overline{q}$ =constant, then yield

$$x((1+\tau)p^c) = \overline{q} = x(p^p),$$

and therefore $p^p = (1+\tau)p^c$. Here both taxes result in the same cost to consumers. However, producers still bear a higher burden when the tax is levied directly on them:

$$(1-\tau)p^p = (1-\tau)(1+\tau)p^c < p^c.$$

these prices are depicted in the next figure, where $x(p)$ reflects the demand function with no taxes and $x((1+\tau)p)$ represents the demand function with the ad valorem tax. While the inelastic supply curve guarantees that sales are unaffected by the tax (remaining at $\overline{q}$ units), the price that the producer receives drops from p^p to $(1+\tau)p^p$. Therefore, the two taxes are still not fully equivalent.

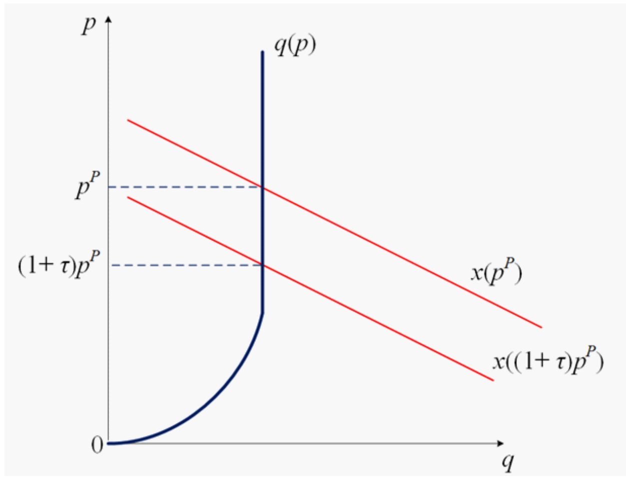

Figure 6.1. Introducing a tax.

- The intuition behind these results is simple: with a tax, there is always a wedge between the "consumer price"and the "producer price." Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax on consumers.

Exercise #4 - Distribution of tax burden

4. Consider a competitive market in which the government will be impossing an ad valorem tax of τ. Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve $q(p) = \alpha p^{\gamma}$, where $\alpha > 0$ and $\gamma > 0$. Denote $\kappa = (1+\tau)$. Assume that a partial equilibrium analysis is valid.

   (a) Evaluate how the equilibrium price is affected by a marginal increase in the tax, i.e., a marginal increase in κ.

   - To compute the change in the price received by producers, we can use the results from Example 6.2

$$\begin{aligned} p^{*'}(0) &= -\frac{x'(p_*)}{x'(p_*) - q'(p_*)} = -\frac{A\varepsilon p_*^{\varepsilon-1}}{A\varepsilon p_*^{\varepsilon-1} - \alpha\gamma p_*^{\gamma-1}} = -\frac{A\varepsilon p_*^{\varepsilon}}{A\varepsilon p_*^{\varepsilon} - \alpha\gamma p_*^{\gamma}} = \\ &= -\frac{\varepsilon x(p^*)}{\varepsilon x(p^*) - \gamma q(p^*)} = -\frac{\varepsilon}{\varepsilon - \gamma}. \end{aligned}$$

(We have multiplied both the numerator and the denominator by p^* and used the fact that p^* is an equilibrium price, which entails $x(p^*) = q(p^*)$.) The price paid by consumers is $(p^*) + t$, and its derivative with respect to t at $t = 0$ is

$$p'(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}.$$

(b) Describe the incidence of the tax when $\gamma = 0$.

- From the above expression,

$$p'(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}.$$

we can see that when $\gamma = 0$ (supply is perfectly inelastic) or $\varepsilon \to -\infty$ (demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax. That is, producers bear the full effect of the tax while consumers are essentially unaffected.

(c) What is the tax incidence when, instead, $\epsilon = 0$?

- On the other hand, when $\varepsilon = 0$ (demand is perfectly inelastic) or $\gamma \to \infty$ (supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax. That is, consumers bear now the full burden of the tax.

(d) What happens when each of these elastiticities approaches ∞ in absolute value?

- As suggested above, when $\varepsilon \to -\infty$ (demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax. In contrast, when $\gamma \to \infty$ (supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax.

Exercise #6 - Linear and Leontief Preferences

6. Consider an economy in which preferences are

$$\text{Consumer 1:} \qquad U^1 = x_1^1 + x_2^1$$
$$\text{Consumer 2:} \qquad U^2 = \min\{x_1^2, x_2^2\}$$

(a) Given the endowments $\omega^1 = (1, 2)$ and $\omega^2 = (3, 1)$, find the set of Pareto efficient allocations and the contract curve.

- For consumer 1, the indifference curves are found be solving for x_2^1, i.e., $x_2^1 = U^1 - x_1^1$, and thus are depicted as straight lines with a slope of -1. For consumer 2, the indifference curves are right angles with corners ("kinks") at consumption bundles with equal quantities of the two goods, $x_1^2 = x_2^2$. Figure 6.2 illustrates the Edgeworth box. Recall that there are 4 units of good 1, but only 3 units of good 2, explaining the rectangular shape of the Edgeworth box. The Pareto efficient allocations (PEAs) are at the corners of consumer 2's indifference curves.

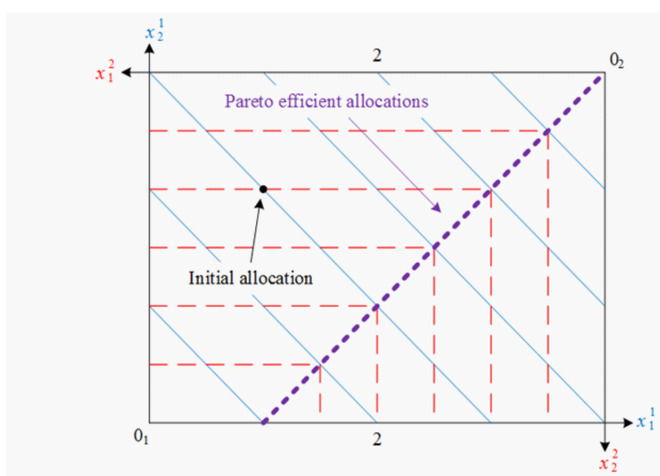

Figure 6.2. Edgeworth box and PEAs.

(b) Which allocations are competitive equilibria?

- The only equilibrium must be on the indifference curve of consumer 1 through the endowment point ω. This is shown by point e in figure 6.3, where consumer 1 is as well off as in his initial endowment. The budget line must therefore overlap his indifference curve. Any other price ratio will lead consumer 1 to choose a corner allocation (either spending all his income on good 1 alone if $p_1 < p_2$, or on good 2 alone if $p_2 < p_1$). In contrast, consumer 2 wish to consume at the corner of an indifference curve. Point e therefore must be the unique equilibrium (unique WEA) which is $((x_1^1, x_1^2), (x_2^1, x_2^2)) = ((2,1),(2,2))$.[25] As depicted in the figure WEA $\in$ PEA.

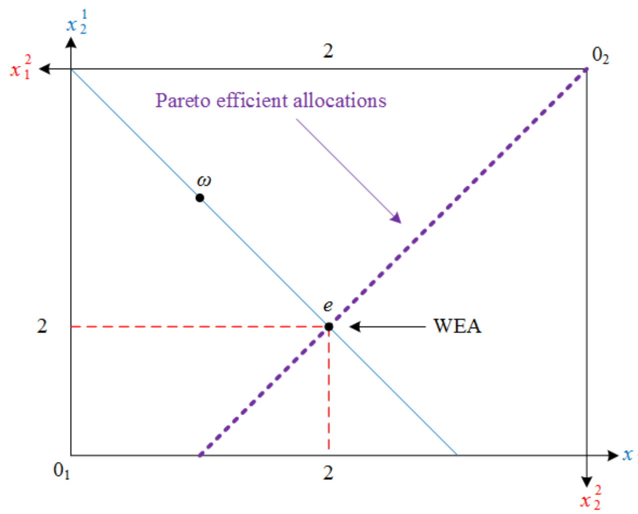

Figure 6.3. PEAs and WEA.

Exercise #7 - Finding Offer Curves for Different Preferences

7. Consider a two-good economy, where every person has the endowment $\omega = (0, 20)$. For each of the following preferences, solve the individuals UMP in order to find his demand curve. The use the endowment to identify his offer curve.

   (a) Cobb-Douglas type: $\alpha \log(x_1) + (1-\alpha)\log(x_2)$, where $\alpha \in (0,1)$.

   - Setting up the Lagrangian and normalizing the price of good 2, so $p_2 = 1$ and $p_1 = p$, we obtain

$$\mathcal{L} = \alpha \log(x_1) + (1-\alpha)\log(x_2) + \lambda\left[20 - px_1 - x_2\right]$$

[25] The WEA simultaneously satisfies $x_2^1 = 3 - x_1^1$ for consumer 1, $x_1^2 = x_2^2$ for consumer 2 (points at the kink of his indifference curve), and the feasibility conditions $x_1^1 + x_2^1 = 4$ and $x_1^2 + x_2^2 = 3$.

which yields first-order conditions

$$\begin{aligned} \frac{\alpha}{x_1} - \lambda p &= 0 \\ \frac{1-\alpha}{x_2} - \lambda &= 0 \\ 20 - px_1 - x_2 &= 0 \end{aligned}$$

Subtracting the first two equations from the third one, we find $\lambda = \frac{1}{20}$, and so the demands will be

$$x_1 = \frac{20\alpha}{p} \qquad \text{and} \qquad x_2 = 20(1-\alpha)$$

and the offer curve will simply be a horizontal straight line at $x_2^h = 20(1-\alpha)$. Since the offer curve depicts the relationship between the demand of good 2 and good 1, the offer curve in this case is just $x_2 = 20(1-\alpha)$, i.e., a horizontal straight line with height $20(1-\alpha)$ in the Edgeworth box.

(b) Perfect substitutes: $ax_1 + x_2$

- In this case, the consumer demands units of one of the good alone (when the slope of his indifference curve and budget line differs) or any bundle on his budget line (if their slopes coincide). In particular, since the $MRS_{1,2} = \frac{a}{1} = a$, and the price ration is $\frac{p_1}{p_2} = p$, the consumer only demands good 2 if $p > a$, i.e., $x = (0, 20)$; only good 1 if $p < a$, yielding a demand $x = \left(\frac{20}{a}, 0\right)$; and any point on the budget line $px_1 + x_2 = 20$ if $p = a$; as depicted in figures 6.4a and 6.4b.

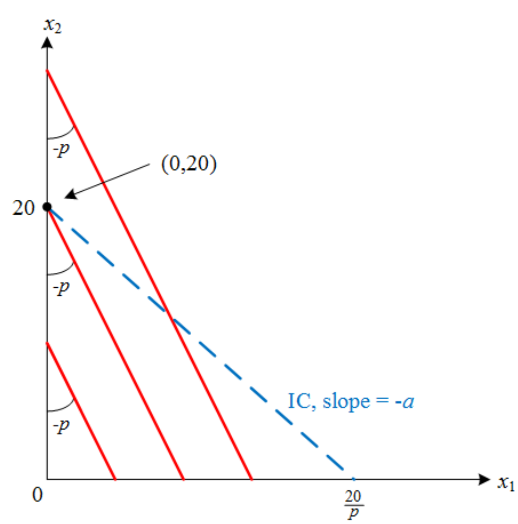

Figure 6.4a. Demand when $p > a$.

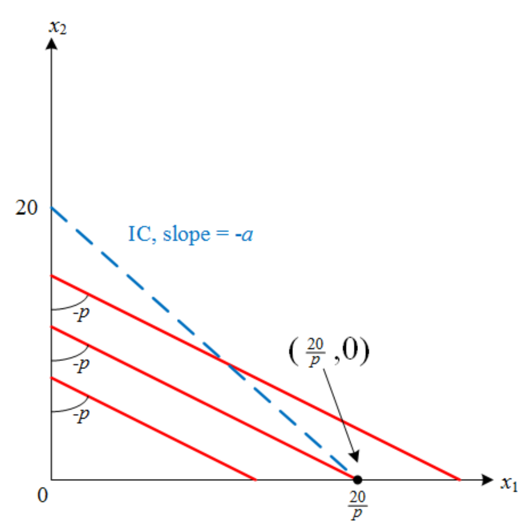

Figure 6.4b. Demand when $p < a$.

(c) Perfect complements: $\min\{ax_1, x_2\}$.

- Demand will be at the kink of the indifference curve, i.e., $ax_1 = x_2$, which together with the budget constraint $px_1 + x_2 = 20$ yields $px_1 + ax_1 = 20$, or $x_1 = \frac{20}{p+a}$. Hence, the demand for good 2 is $x_2 = ax_1 = a\frac{20}{p+a}$. That is, the offer curve satisfies $x_2 = ax_1$, thus being a straight line from the origin $(0, 0)$ and with a positive slope $a > 0$.

(d) Consider now an economy where all individuals have the Cobb-Douglas preferences of part (a). There are two individuals: consumer A with $\alpha = \frac{1}{2}$ and endowment $\omega = (10, 0)$, and consumer B with $\alpha = \frac{3}{4}$ and $\omega = (0, 20)$. Find the WEA.

- If a person with preferences of $\alpha \log(x_1) + (1-\alpha)\log(x_2)$ had an income of 10 units of commodity 1 (as opposed to 20 in part (a)) then, by analogy with part (a), demand would be

$$x^1 = \begin{bmatrix} 10\alpha \\ 10p(1-\alpha) \end{bmatrix}$$

and the offer curve will simply be a vertical straight line at $x_1^h = 10\alpha$. From our demands in part (a) and the equation above, we have $x_1^1 = 10(\frac{1}{2}) = 5$, $x_2^2 = 20(1 - \frac{3}{4}) = 5$. Given that there are 10 units in total of commodity 1 and 20 units in total of commodity 2 the materials balance condition then means that the equilibrium allocation must be

$$x^1 = \begin{bmatrix} 5 \\ 15 \end{bmatrix} \quad \text{and} \quad x^2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Solving for p from our equilibrium we find that the equilibrium price ratio must be 3.

Exercise #8 - Barter Economies

8. Consider the following indirect utility functions for consumers A and B

$$\begin{aligned} v^A(\mathbf{p}, m) &= \ln m - \frac{1}{2}\ln p_1 - \frac{1}{2}\ln p_2 \\ v^B(\mathbf{p}, m) &= \left(\frac{1}{p_1} + \frac{1}{p_2}\right) m \end{aligned}$$

Initial endowments coincide across consumers, $\mathbf{e}^A = \mathbf{e}^B = (5.8, 2.1)$. Assuming good 1 is the numeraire, $p_1 = 1$, find the equilibrium price vector $\mathbf{p}^*$.

- By Walras' law we know that if the market for good 1 clears, $z_1(\mathbf{p}) = 0$ then so does the market of good 2, $z_2(\mathbf{p}) = 0$. Let us then take the market of good 1, where $z_1(\mathbf{p}) = 0$ implies

$$e_1^A + e_1^B = x_1^A(\mathbf{p}, m) + x_1^B(\mathbf{p}, m)$$

where $e_1^A + e_1^B = 5.8 + 5.8$. The Walrasian demand functions can be recovered from the indirect utility function using Roy's identity, as follows

$$x_1^A(\mathbf{p}, m^A) = -\frac{\frac{\partial v^A(\mathbf{p}, m^A)}{\partial p_1}}{\frac{\partial v^A(\mathbf{p}, m^A)}{\partial m^A}} = -\frac{-\frac{1}{2p_1}}{\frac{1}{m^A}} = \frac{m^A}{2p_1}$$

for consumer A, and similarily for consumer B,

$$x_1^B(\mathbf{p}, m^B) = -\frac{\frac{\partial v^B(\mathbf{p}, m^B)}{\partial p_1}}{\frac{\partial v^B(\mathbf{p}, m^B)}{\partial m^B}} = -\frac{-\frac{m^B}{2p_1^2}}{\frac{1}{p_1} + \frac{1}{p_2}} = \frac{\frac{m^B}{2p_1^2}}{\frac{1}{p_1} + \frac{1}{p_2}}$$

In addition, since their initial endowments coincide $m^A = m^B = m$. In particular, the market value of their endowments, m, is

$$m = p_1 e_1^A + p_2 e_2^A = 5.8 + 2.1p_2$$

since good 1 is the numeraire, i.e., $p_1 = 1$. Plugging $m = 5.8 + 2.1p_2$ into the Walrasian demands found above, and using $p_1 = 1$, yields

$$x_1^A(\mathbf{p}, m^A) = \frac{5.8 + 2.1p_2}{2} \text{ and } x_1^B(\mathbf{p}, m^B) = \frac{5.8 + 2.1p_2}{1 + \frac{1}{p_2}}$$

Therefore, the initial market clearing condition for good 1, $e_1^A + e_1^B = x_1^A(\mathbf{p}, m) + x_1^B(\mathbf{p}, m)$ becomes

$$5.8 + 5.8 = \frac{5.8 + 2.1p_2}{2} + \frac{5.8 + 2.1p_2}{1 + \frac{1}{p_2}}$$

where, solving for p_2, yields an equilibrium price of $p_2^* = 2$. Since good 1 acted as the numeraire, this result implies that the equilibrium price of good 2 needs to be double that of good 1, i.e., the equilibrium price ratio is $\frac{p_2^*}{p_1^*} = 1.98$.

Exercise #9 - Pure Exchange Economy

9. Consider a pure-exchange economy with two individuals, A and B, each with utility function $u^i(x^i, y^i)$ where $i = \{A, B\}$, whose initial endowments are $e^A = (10, 0)$ and $e^B = (0, 10)$, that is, individual A (B) owns all units of good x (y, respectively).

   (a) Assuming that utility functions are $u^i(x^i, y^i) = \min\{x^i, y^i\}$ for all individuals $i = \{A, B\}$, find the set of PEAs and the set of WEAs.

   - *PEAs.* Since the utility functions are not differentiable we cannot follow the property of $MRS^A_{x,y} = MRS^B_{x,y}$ across consumers. Figure 6.5 helps us identify the set of PEAs. Points away from the 45^o-line, satisfying $y^A = x^A$, such as N, cannot be pareto efficient since we can still find other points, such as M, where consumer 2 is make better off while consumer 1 reaches the same utility level as under N. Once we are at points on the 45^o-line, such as M, we cannot find other points making at least once consumer better off (and keep the other consumer at least as well off). Hence, the set of PEAs is

$$\{(x^A, y^A), (x^B, y^B) : y^A = x^A \text{ and } y^B = x^B\}$$

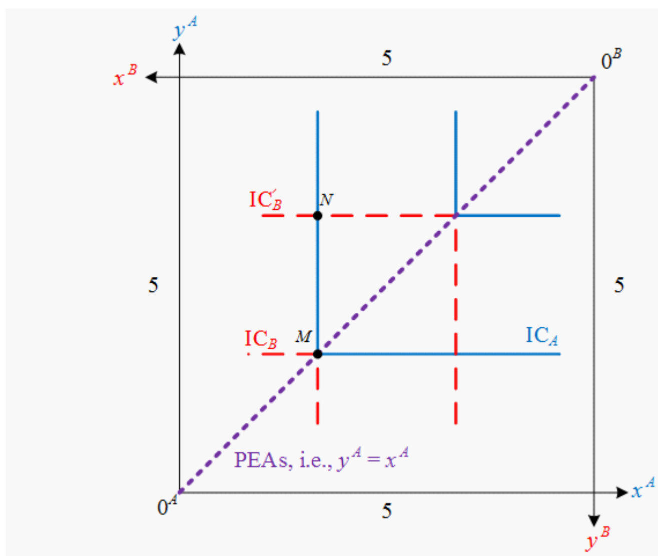

Figure 6.5. Edgeworth box and PEAs.

   - *WEAs.* Using good 2 as the numeraire, i.e., $p_2 = 1$, the price ratio becomes $\frac{p_1}{p_2} = p_1$. The budget line of both consumers therefore has a slope $-p_1$ and crosses the point representing the initial endowment e in figure 6.6 (where e

lies at the lower right-hand corner)

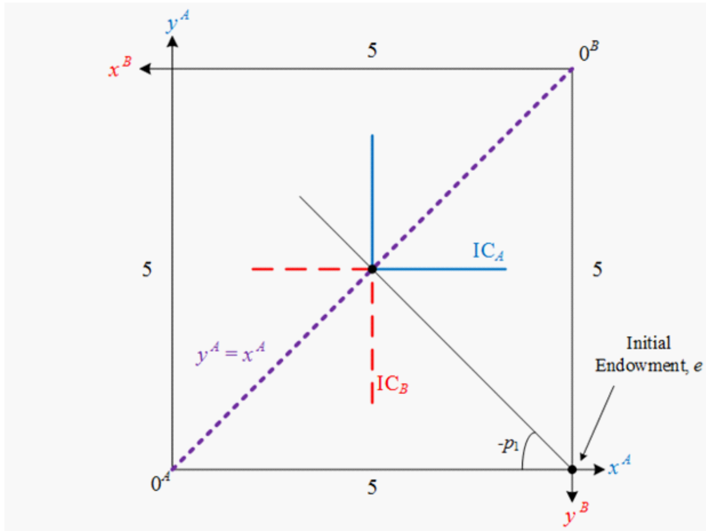

Figure 6.6. Edgeworth box and WEA.

(b) Assuming utility functions of $u^A(x^A, y^A) = x^A y^A$ and $u^B(x^B, y^B) = \min\{x^B, y^B\}$, find the set of PEAs and WEAs.

- *PEAs.* By the same argument as in question (a), the set of PEAs satisfies $y^A = x^A$, as depicted in figure 6.7. Point N cannot be efficient as we can still find other feasible points, such as M, where at least one consumer is made strictly better off (in this case consumer A). At points on the 45^o-line, however, we can no longer find alternatives that would constitute a Pareto improvement.

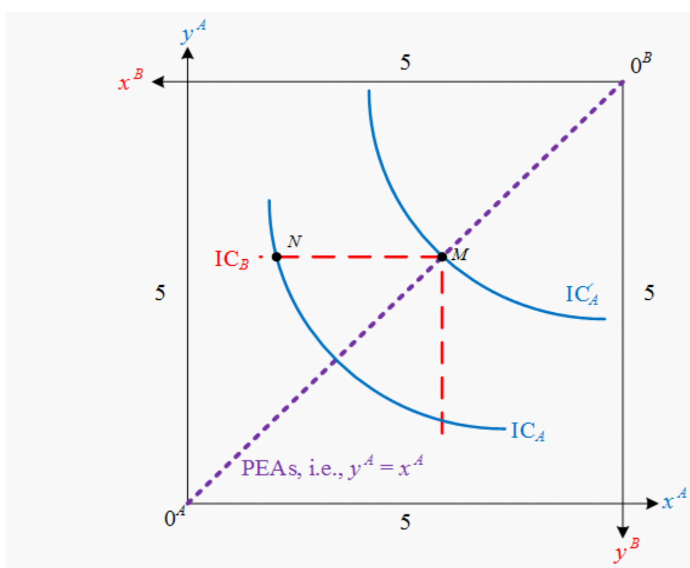

Figure 6.7. Edgeworth box and PEAs.

- *WEAs.* Using good y as the numeraire, $p_y = 1$, so that the price vector becomes $\mathbf{p} = (p_x, 1)$. Hence, *Consumer A's UMP is*

$$\max_{x^A, y^A} \quad x^A y^A$$
$$\text{subject to } p_x x^A + y^A = 10p_x$$

Taking first-order conditions

$$\begin{aligned} y^A - \lambda^A p_x &= 0 \\ x^A - \lambda^A &= 0 \\ p_x x^A + y^A &= 10p_x \end{aligned}$$

Combining the first two FOCs and rearranging, we have

$$p_x x^A = y^A$$

and substituting this equation into the third FOC yields

$$p_x x^A + p_x x^A = 10p_x \implies x^A = 5$$

and substituting this back into $p_x x^A = y^A$

$$y^A = 5p_x$$

Consumer B's UMP is not differentiable, but in equilibrium his Walrasian demands satisfy $x^B = y^B$. Substituting this into his budget constraint yields

$$p_x x^B + x^B = 10 \implies x^B = y^B = \frac{10}{p_x + 1}$$

Furthermore, the feasibility condition for good x entails

$$5 + \frac{10}{p_x + 1} = 10 + 0, \text{ or } p_x = 1$$

Therefore, the market of good x will clear at an equilibrium price of $p_x = 1$, i.e., $z_x(p_x, 1) = 0$ when $p_x = 1$. Since market y clears when market x does (by Walras' law), $z_y(p_x, 1)$ must also be zero when $p_x = 1$. Summarizing, the equilibrium price $p_x = 1$ yields a WEA

$$\{(5,5),(5,5)\}$$